Spectral Theory for Operator Matrices Related to Models in Mechanics

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Abstract

We derive various properties of the operator matrix $\mathcal{A} = \left[\begin{smallmatrix} 0 & I \\ -A_0 & -D \end{smallmatrix}\right]$, where A_0 is a uniformly positive operator and $A_0^{-1/2} D A_0^{-1/2}$ is a bounded non-negative operator in a Hilbert space H. Such operator matrices are associated with second order problems of the form $\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0$ which are used as models for transverse motions of thin beams in the presence of damping.

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1 Introduction

A linear equation describing transverse motions of a thin beam can be written in the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2} + C_d \frac{\partial^3 u}{\partial r^2 \partial t}\right] = 0, \qquad r \in (0,1), t > 0,$$

where $u(r,t)$ is the transverse displacement of the beam at time t and position r. The existence and behaviour of solutions u depend also on boundary and initial conditions. In the example above we are interested in solutions having finite energy, i.e. solutions such that $\|u(\cdot,t)\|^2 + \|u''(\cdot,t)\|^2 < \infty$ for all $t > 0$ where $\|\cdot\|$ denotes the usual norm in the Hilbert space $L^2(0,1)$. Identifying the function $u(\cdot,t)$ with an element $z(t) \in L^2(0,1)$ by $z(t)(r) = u(r,t)$ we obtain from the partial differential equation above a second order equation in $L^2(0,1)$ of the form

$$\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0, \tag{1}$$

where $A_0 = E\frac{\partial^4}{\partial r^4}$, $D = \frac{\partial^2}{\partial r^2}C_d\frac{\partial^2}{\partial r^2}$ acting in $L^2(0,1)$ with appropriate domains encoding the boundary conditions under consideration.

In this paper we study second order equations of type (1) in an abstract Hilbert space H where the stiffness operator A_0 is a possibly unbounded positive operator on H and is assumed to be boundedly invertible, and D, the damping operator, is an unbounded operator on H. This second order equation is equivalent to the standard first-order equation $\dot{x}(t) = \mathcal{A}x(t)$ where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(A_0^{1/2}) \times H \to \mathcal{D}(A_0^{1/2}) \times H$ is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in \mathcal{D}(A_0^{1/2}) \times \mathcal{D}(A_0^{1/2}) \mid A_0 z + Dw \in H \right\}.$$

This operator matrix has been studied in the literature for more than 20 years. Interest in this particular model is motivated by various problems such as stabilization, see for example [7], [29], [30], [32], solvability of Riccati equations [14], minimum-phase property [20] and compensator problems with partial observations [15]. It is well-known that $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{D}(A_0^{1/2}) \times H$, where $\mathcal{D}(A_0^{1/2})$ is equipped with the norm $x \mapsto \|A_0^{1/2}x\|_H$, and thus the spectrum of $\mathcal{A}$ is located in the closed left half plane. This goes back to [4] and [28], see also [5], [9]. Several authors have proved independently of each other that the condition

$$\inf_{z \in \mathcal{D}(A_0^{1/2}) \setminus \{0\}} \frac{\langle A_0^{-1/2}Dz, A_0^{1/2}z\rangle_H}{\|z\|_H^2} > 0$$

is sufficient for exponential stability of the C_0-semigroup generated by $\mathcal{A}$, see for example [4], [5], [6], [9], [16], [17], [34] and [35]. Other properties of the C_0-semigroup such as analyticity have been studied in [4], [5], [9], [10], [11], [12], [13], [18], [19] and [25].

Most of the papers require that the damping operator D is comparable with A^ρ for some $\rho \in [1/2, 1]$. In [25] the damping operator D is of the form

$$D = \alpha A_0 + B, \tag{2}$$

where $\alpha > 0$ is a constant, A_0^{-1} is compact and B is symmetric and A_0-compact. If $-1/\alpha \notin \sigma_p(\mathcal{A})$, then it is shown in [25] that $\mathcal{A}$ generates an analytic semigroup. In [22] we extend this result to the case that A_0^{-1} is compact in H and $0 \notin \sigma_{ess}(A_0^{-1}D)$. Moreover, it was shown in [22] that $\mathcal{A}$ is a definitizable operator.

In this paper we also assume that A_0^{-1} is a compact operator. We show that the condition $\sigma_{ess}(A_0^{-1}D) = \{0\}$ is equivalent to the definitizabilty of $\mathcal{A}$. It turns out, that in the case that $\mathcal{A}$ has a compact resolvent, in contrast to the situation considered in [25] and [22], the operator $\mathcal{A}$ is no longer definitizable.

Throughout this paper we assume that all Hilbert spaces are infinite dimensional.

2 Spectrum of operators in Krein spaces

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space. We briefly recall that a complex linear space $\mathcal{H}$ with a hermitian nondegenerate sesquilinear form $[\cdot,\cdot]$ is called a *Krein space* if there exists a decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\cdot,\cdot]$ such that $(\mathcal{H}_\pm, \pm[\cdot,\cdot])$ are Hilbert spaces. In the following all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. For the basic theory of Krein spaces and operators acting therein we refer to [8] and [2].

Let A be a closed operator in $\mathcal{H}$. We define the extended spectrum $\sigma_e(A)$ of A by $\sigma_e(A) := \sigma(A)$ if A is bounded and $\sigma_e(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. The resolvent set of A is denoted by $\rho(A)$ and we set $\rho_e(A) := (\mathbb{C} \cup \{\infty\}) \setminus \sigma_e(A)$. A point $\lambda_0 \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{ap}(A)$ of A if there exists a sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(A - \lambda_0 I)x_n\| \to 0$ if $n \to \infty$. For a self-adjoint operator A in $\mathcal{H}$ all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. [8, Corollary VI.6.2]).

The indefiniteness of the scalar product on $\mathcal{H}$ leads to the definition of several subsets of the spectrum of an operator. The following definition was given in [24], [27] and [3].

Definition 2.1 *For a self-adjoint operator A in $\mathcal{H}$ with domain $\mathcal{D}(A)$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of* type π_+ (type π_-) of A *if $\lambda_0 \in \sigma_{ap}(A)$ and if there exists a linear submanifold $\mathcal{H}_0 \subset \mathcal{H}$ with* $\operatorname{codim} \mathcal{H}_0 < \infty$ *such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0 I)x_n\| \to 0$ as $n \to \infty$ we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0 \big).$$

A point $\lambda_0 \in \sigma(A)$ is called a spectral point of positive type (negative) type of A *if $\lambda_0 \in \sigma_{\pi_+}(A)$ (resp. $\lambda \in \sigma_{\pi_-}(A)$) and $\mathcal{H}_0 = \mathcal{H}$. The point ∞ is said to* *be of* positive (negative) type of A *if* A *is unbounded and for every sequence* $(x_n) \subset \mathcal{D}(A)$ *with* $\lim_{n\to\infty} \|x_n\| = 0$ *and* $\|Ax_n\| = 1$ *we have*

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0\big).$$

We denote the set of all points of $\sigma_e(A)$ *of positive (negative) type by* $\sigma_{++}(A)$ *(resp.* $\sigma_{--}(A)$*).*

Similarly, one introduces the notion that ∞ is a spectral point of type π_+ (type π_-) of A (cf. [3]). We denote the set of all points of $\sigma_e(A)$ of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$). It is not difficult to see that the sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. Moreover the non-real spectrum of A cannot accumulate to $\sigma_{++}(A) \cup \sigma_{--}(A)$. More properties of these sets can be found in [3].

Recall that a self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *definitizable* if $\rho(A) \neq \emptyset$ and if there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the spectrum of A is real or its non-real part consists of a finite number of points, cf. [26].

The theorems below will be used in the proof of our main result in Section 3 below. It shows that a real spectral point which is not an eigenvalue and which is adjacent to the spectrum of type π_+/type π_- and to the resolvent set of a definitizable operator has to be a spectral point of postive/negative type.

Theorem 2.2 *Let* A *be a definitizable operator and let* (a, b) *be an open interval such that*

$$(a, b) \subset \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A) \cup \rho(A) \quad \textit{and} \quad b \notin \sigma_p(A).$$

If there exists $c > b$ *with* $(b, c) \subset \rho(A)$*, then*

$$b \in \sigma_{++}(A) \cup \sigma_{--}(A) \cup \rho(A).$$

Proof:
As A is a definitizable operator there exists $a_0 \in (a, b)$ and $c_0 \in (b, c)$ such that $[a_0, c_0] \setminus \{b\}$ is either a subset of $\sigma_{++}(A) \cup \rho(A)$ or a subset of $\sigma_{--}(A) \cup \rho(A)$. Let $[a_0, c_0] \setminus \{b\} \subset \sigma_{++}(A) \cup \rho(A)$. We denote by E thespectral function of A (cf. [26]). Then $E([a_0, c_0])$ is defined and $E([a_0, c_0])\mathcal{H}$ is a Krein space. By [23, Theorem 4.8 and Remark 4.9] the operator

$$A' := A|E([a_0, c_0])\mathcal{H}.$$

is definitizable and the restriction of E to $E([a_0, c_0])$ is the spectral function of A'. By Proposition [3, Proposition 25], for every interval $\delta \subset [a_0, c_0]$ with

$b \notin [a_0, c_0]$ the projection $E([a_0, c_0])$ is non-negative. Therefore, the subspace $\mathcal{S}$ of $E([a_0, c_0])\mathcal{H}$,

$$\mathcal{S} := \text{cls}\,\{E(\delta)\mathcal{H} : \delta \subset [a_0, c_0], \delta \text{ interval}, b \notin \overline{\delta}\}.$$

is non-negative ([26, Lemma I.5.3]). By ([26, Propositions II.5.1 and II.5.2]) the orthogonal subspace to $\mathcal{S}$ in $E([a_0, c_0])\mathcal{H}$ is the root subspace of A corresponding to b, hence, as $b \notin \sigma_p(A)$,

$$\mathcal{S} = E([a_0, c_0])\mathcal{H},$$

that is, $(E([a_0, c_0])\mathcal{H}, [\cdot, \cdot])$ is a Hilbert space. By [3, Proposition 25],

$$[a_0, c_0] \subset \sigma_{++}(A) \cup \rho(A).$$

A similar reasoning holds for $[a_0, c_0] \setminus \{b\} \subset \sigma_{--}(A) \cup \rho(A)$ and Theorem 2.2 is proved. □

Theorem 2.3 *Assume that the operator A is definitizable and the positive half-axis belongs to the resolvent set of A. Then*

$$\infty \in \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A) \cup \rho_e(A).$$

Proof:
By [3, Lemma 10] we have $\infty \in \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A) \cup \rho_e(A)$ if and only if $\infty \in \sigma_{++}(A) \cup \sigma_{--}(A) \cup \rho_e(A)$. This is equivalent to $\infty \in \sigma_{++}(A-1) \cup \sigma_{--}(A-1) \cup \rho_e(A-1)$. The latter holds if and only if

$$0 \in \sigma_{++}((A-1)^{-1}) \cup \sigma_{--}((A-1)^{-1}) \cup \rho_e((A-1)^{-1}),$$

see [1]. Here, $(A-1)^{-1}$ is a bounded definitizable operator. By the definitizability of A, some interval $(-\infty, -m)$ belongs to $\sigma_{++}(A) \cup \rho(A)$ or to $\sigma_{--}(A) \cup \rho(A)$. Then Theorem 2.3 follows from Theorem 2.2. □

3 Dampings which are small compared with A_0

Throughout the rest of this paper we make the following assumptions.

(A) The stiffness operator $A_0 : \mathcal{D}(A_0) \subset H \to H$ is a self-adjoint uniformly positive operator on a Hilbert space H. We define for $s > 0$ the space $H_s = \mathcal{D}(A_0^s)$ equipped with the norm $\|\cdot\|_{H_s} := \|A_0^s \cdot\|_H$ and H_{-s} is the completion of H with respect to the norm $\|z\|_{H_{-s}} = \|A_0^{-s} z\|_H$. Thus A_0 restricts (extends, respectively) to a bounded operator $A_0 : H_s \to H_{s-1}$. We use the same notation A_0 to denote this restriction/extension. We denote the inner product on H by $\langle \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle$, and the duality pairing on $H_{-s} \times H_s$ by $\langle \cdot, \cdot \rangle_{H_{-s} \times H_s}$. Note that for $(z', z) \in H \times H_s$, we have

$$\langle z', z \rangle_{H_{-s} \times H_s} = \langle z', z \rangle_H.$$

Moreover, for this section, we assume that the operator D is comparable with A_0 in the following sense.

(D1) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a bounded operator such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded self-adjoint operator in H and satisfies

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

The equation (1) is equivalent to the following standard first-order equation

$$\dot{x}(t) \;=\; \mathcal{A} x(t),$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$, is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + Dw \in H \right\}.$$

The operator $\mathcal{A}$ itself is not self-adjoint in the Hilbert space $H_{\frac{1}{2}} \times H$. It is easy to see (e.g. [34]) that $\mathcal{A}$ has a bounded inverse in $H_{\frac{1}{2}} \times H$ given by

$$\mathcal{A}^{-1} = \begin{bmatrix} -A_0^{-1} D & -A_0^{-1} \\ I & 0 \end{bmatrix}, \tag{3}$$

where $A_0^{-1} D$ is considered as an operator acting in $H_{\frac{1}{2}}$. This together with the fact that

$$J\mathcal{A}, \qquad \text{where } J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \tag{4}$$

is a symmetric operator in the Hilbert space $H_{\frac{1}{2}} \times H$, imply the self-adjointness of $J\mathcal{A}$ in $H_{1/2} \times H$. Therefore, (compare also [33, Proof of Lemma 4.5])

$$\mathcal{A}^* = J\mathcal{A}J, \qquad \text{with } \mathcal{D}(\mathcal{A}^*) = J\mathcal{D}(\mathcal{A})$$

and

$$\mathrm{Re}\,\langle \mathcal{A}x, x\rangle \le 0 \quad \text{for } x\in\mathcal{D}(\mathcal{A}) \quad \text{and} \quad \mathrm{Re}\,\langle \mathcal{A}^*x, x\rangle \le 0 \quad \text{for } x\in\mathcal{D}(\mathcal{A}^*).$$

Hence, $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions on the state space $H_{\frac{1}{2}}\times H$. This fact is well-known.

For $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in H_{\frac{1}{2}}\times H$ we define an indefinite inner product on $H_{\frac{1}{2}}\times H$ by

$$\left[\begin{pmatrix}x_1\\y_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\end{pmatrix}\right] := \left\langle J\begin{pmatrix}x_1\\y_1\end{pmatrix}, \begin{pmatrix}x_2\\y_2\end{pmatrix}\right\rangle = \langle x_1, x_2\rangle_{H_{\frac{1}{2}}} - \langle y_1, y_2\rangle. \tag{5}$$

Then $(H_{\frac{1}{2}}\times H, [\cdot,\cdot])$ is a Krein space and $\mathcal{A}$ is a self-adjoint operator with respect to $[\cdot,\cdot]$. Therefore, we obtain the following proposition.

Proposition 3.1 *The operator $\mathcal{A}$ is self-adjoint in the Krein space $(H_{\frac{1}{2}}\times H, [\cdot,\cdot])$, its spectrum is contained in the closed left half plane and lies symmetric with respect to the real line. The operator $\mathcal{A}$ has a bounded inverse and is the generator of a strongly continuous semigroup of contractions on $H_{\frac{1}{2}}\times H$.*

In the following we will consider the case of an operator A_0 with a compact resolvent. Note that in this case the operator $\mathcal{A}^{-1}$ is, in general, not a compact operator in $H_{\frac{1}{2}}\times H$.

Lemma 3.2 *Assume that the operator A_0^{-1} is compact in H. Then*

$$\sigma_{ess}(\mathcal{A}) = \left\{\lambda\in\mathbb{C}\backslash\{0\} \mid \frac{1}{\lambda}\in\sigma_{ess}(-A_0^{-1}D)\right\} \subset (-\infty, 0). \tag{6}$$

Here $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. In particular, the following statements are equivalent.

(i) *The operator $\mathcal{A}^{-1}$ is a compact operator in $H_{\frac{1}{2}}\times H$.*

(ii) $\sigma_{ess}(A_0^{-1}D) = \{0\}$.

(iii) *The operator D is a compact operator acting from $H_{\frac{1}{2}}$ into $H_{-\frac{1}{2}}$.*

Proof:
Relation (6) follows from (3) (see e.g. [21]). Obviously, by (3), (i) implies (ii). The operator $A_0^{-1}D$ is a bounded self-adjoint operator in the Hilbert space $H_{\frac{1}{2}}$. Therefore, (ii) implies the compactness of $A_0^{-1}D$ in $H_{\frac{1}{2}}$ and, by (3), (i) follows. The operator $\mathcal{A}^{-1}$ considered as an operator acting from $H_{-\frac{1}{2}}$ into $H_{\frac{1}{2}}$ is unitary. This shows the equivalence of (ii) and (iii). □

Recall, that an at most countably infinite set $\mathcal{M}$ of elements of a Hilbert space is said to be a *Riesz basis* if there exists an isomorphic mapping $\mathcal{M}$ onto an orthonormal basis, cf. [31, Lecture VI]. The following theorem was proved in [22].

Theorem 3.3 *Assume that the operator A_0^{-1} is compact in H and that $0 \notin \sigma_{ess}(A_0^{-1}D)$, where $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. Then $\mathcal{A}$ is a definitizable operator and generates an analytic semigroup on $H_{1/2} \times H$. If, in addition, the set $\sigma_{ess}(A_0^{-1}D)$ is countably and has at most countable many accumulation points and if for all $\mu \in \sigma_{ess}(-A_0^{-1}D)$ we have $\frac{1}{\mu} \notin \sigma_p(\mathcal{A})$, then there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors and finitely many associated vectors of $\mathcal{A}$.*

The next theorem is the main result of this note. It shows that the the operator $\mathcal{A}$ is definitizable if and only if $0 \notin \sigma_{ess}(A_0^{-1}D)$.

Theorem 3.4 *Assume that the operator A_0^{-1} is compact in H and $0 \in \sigma_{ess}(A_0^{-1}D)$, where $A_0^{-1}D$ is considered as an operator acting in $H_{\frac{1}{2}}$. Then $\mathcal{A}$ is not a definitizable operator in $H_{\frac{1}{2}} \times H$.*

Proof:
Assume that $\mathcal{A}$ is a definitizable operator. As $0 \in \sigma_{ess}(A_0^{-1}D)$, there exists a sequence $(x_n) \in H_{\frac{1}{2}}$ which converges weakly to zero in $H_{\frac{1}{2}}$ with $\|x_n\|_{H_{\frac{1}{2}}} = 1$ and $A_0^{-1}Dx_n \to 0$ as $n \to \infty$. Therefore, $(A_0^{\frac{1}{2}}x_n)$ converges weakly to zero in H, hence (x_n) converges to zero in H. The sequence $(\left(\begin{smallmatrix} x_n \\ 0 \end{smallmatrix}\right)) \subset H_{\frac{1}{2}} \times H$ satisfy

$$\|\left(\begin{smallmatrix} x_n \\ 0 \end{smallmatrix}\right)\|_{H_{\frac{1}{2}} \times H} = 1 \quad \text{and } \mathcal{A}^{-1}\left(\begin{smallmatrix} x_n \\ 0 \end{smallmatrix}\right) = \begin{pmatrix} -A_0^{-1}Dx_n \\ x_n \end{pmatrix} \to 0 \quad \text{as } n \to \infty.$$

Moreover, $(\left(\begin{smallmatrix} x_n \\ 0 \end{smallmatrix}\right))$ converges weakly to zero in $H_{\frac{1}{2}} \times H$ with

$$\limsup_{n\to\infty} [\left(\begin{smallmatrix} x_n \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} x_n \\ 0 \end{smallmatrix}\right)] = 1.$$

By [3, Lemma 14], 0 is not in $\sigma_{\pi_-}(\mathcal{A}^{-1})$.

We choose a sequence $(y_n) \in H$ which converges weakly to zero in H with $\|y_n\| = 1$. Therefore, $(A_0^{-1}y_n)$ converges to zero in H. The sequence $\left(\left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right)\right) \subset H_{\frac{1}{2}} \times H$ satisfy

$$\left\|\left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right)\right\|_{H_{\frac{1}{2}} \times H} = 1 \quad \text{and } \mathcal{A}^{-1}\left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right) = \begin{pmatrix} -A_0^{-1}y_n \\ 0 \end{pmatrix} \to 0 \quad \text{as } n \to \infty.$$

Moreover, $\left(\left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right)\right)$ converges weakly to zero in $H_{\frac{1}{2}} \times H$ with

$$\liminf_{n\to\infty} \left[\left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ y_n \end{smallmatrix}\right)\right] = -11.$$

By [3, Theorem 14], 0 is not in $\sigma_{\pi_+}(\mathcal{A}^{-1})$, hence $0 \notin \sigma_{\pi_+}(\mathcal{A}^{-1}) \cup \sigma_{\pi_-}(\mathcal{A}^{-1})$. It is easily seen (see e.g. [1]) that this is the case if and only if $\infty \notin \sigma_{\pi_+}(\mathcal{A}) \cup \sigma_{\pi_-}(\mathcal{A})$. But this is by Theorem 2.3 impossible. Hence $\mathcal{A}$ is not definitizable. □

Remark 3.5 *If B is a self-adjoint operator in H such that for some $0 \leq \theta < \frac{1}{2}$ we have*

$$k_1\|A_0^\theta x\| \leq \|Bx\| \leq k_2\|A_0^\theta x\| \quad \text{for } x \in \mathcal{D}(B) = \mathcal{D}(A_0^\theta),$$

then the semigroup generated by $\mathcal{A}$ is not analytic, cf. [11, Proposition 2.1]. However, it is shown in [11] that if B is a self-adjoint operator in H with $\mathcal{D}(A_0^{\frac{\theta}{2}}) = \mathcal{D}(B^{\frac{1}{2}})$ for some $\frac{1}{2} \leq \theta \leq 1$ and

$$k_1 A_0^\theta \leq B \leq k_2 A_0^\theta \quad x \in \mathcal{D}(B),$$

then the semigroup generated by $\mathcal{A}$ is analytic. Moreover [25] give sufficient conditions for the Riesz basis property.

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