Spectral properties of singular Sturm-Liouville operators with indefinite weight sgn x

Illya Karabash and Carsten Trunk

(MS received ;)

We consider a singular Sturm-Liouville expression with the indefinite weight sgn x. To this expression there is naturally a self-adjoint operator in some Krein space associated. We characterize the local definitizability of this operator in a neighbourhood of ∞. Moreover, in this situation, the point ∞ is a regular critical point. We construct an operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ with non-real spectrum accumulating to a real point. The obtained results are applied to several classes of Sturm-Liouville operators.

1. Introduction

We consider the singular Sturm-Liouville differential expression

$$a(y)(x) = (\operatorname{sgn} x)(-y''(x) + q(x)y(x)), \qquad x \in \mathbb{R}, \tag{1.1}$$

with the signum function as indefinite weight and a real potential $q \in L^1_{loc}(\mathbb{R})$. We assume that (1.1) is in the limit point case at both $-\infty$ and $+\infty$. This differential expression is naturally connected with a self-adjoint operator A in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$ (see e.g. [12]), where the indefinite inner product $[\cdot,\cdot]$ is defined by

$$[f, g] = \int_R f\overline{g} \ \operatorname{sgn} x \ dx, \qquad f, g \in L^2(\mathbb{R}).$$

The operator $J : f(x) \mapsto (\operatorname{sgn} x)f(x)$ is a fundamental symmetry in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$. Let us define the operator $L := JA$. Then $L = -d^2/dx^2 + q$ is a self-adjoint Sturm-Liouville operator in the Hilbert space $L^2(\mathbb{R})$. It was shown in [12] that if L is a non-negative operator in the Hilbert space sense then A is a definitizable operator with ∞ as a regular critical point.

In general, the operator A may be not definitizable (in Section 3 we give a criterion). However, under certain assumptions, A is still locally definitizable over an appropriate subset of $\mathbb{C}$. It seems that the first result of such type was obtained in [5] for the operator $y \mapsto \frac{1}{w}[(py')' + qy]$ with w as indefinite weight function. Note that in [5] w may have many turning points, but rather strong assumptions on the spectra of certain associated self-adjoint operators are supposed.

As a main result we show the equivalence of the semi-boundedness from below of the operator L and the local definitizability of the operator A in a neighbourhood of ∞. Moreover, we give a precise description of the domain of definitizability of A. If L is semi-bounded from below, we show the existence of a decomposition $A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b$ such that the operator $\mathcal{A}_\infty$ is similar to a self-adjoint operator in the Hilbert space sense and $\mathcal{A}_b$ is a bounded operator, that is, the point ∞ is a regular critical point. Hence, the non-real spectrum of A remains bounded. But, in contrast to the case of a non-negative operator L, now the non-real spectrum may accumulate to the real axis. We prove in Section 4 the existence of an even continuous potential q with a sequence of non-real eigenvalues of A accumulating to a real point. This potential q can be chosen in such a way that A is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$.

Finally, in Section 5, we discuss the spectrum and the sets of definitizability of A for various classes of potentials q.

Differential operators with indefinite weights appears in many areas of physics and applied mathematics (see [4, 21, 28, 43] and references therein). Under certain assumptions such operators are definitizable; this case was studied extensively (see [8, 12, 13, 14, 15, 18, 19, 20, 32, 35, 36, 42, 44, 47] and references therein). In [5, 6, 7, 29, 31, 33, 34] certain classes of differential operators that contain definitizable as well as not definitizable operators were considered.

Notation: Let T be a linear operator in a Hilbert space $\mathfrak{H}$. In what follows $\mathrm{dom}(T)$, $\ker(T)$, $\mathrm{ran}(T)$ are the domain, kernel, range of T, respectively. We denote the resolvent set by $\rho(T)$; $\sigma(T) := \mathbb{C} \setminus \rho(T)$ stands for the spectrum of T. By $\sigma_p(T)$ the set of eigenvalues of T is indicated. The discrete spectrum $\sigma_{disc}(T)$ is the set of isolated eigenvalues of finite algebraic multiplicity; the essential spectrum is $\sigma_{ess}(T) := \sigma(T) \setminus \sigma_{disc}(T)$. We denote the indicator function of a set S by $\chi_S(\cdot)$.

2. Sturm-Liouville operators with the indefinite weight sgn x

2.1. Differential operators

We consider the differential expression

$$\ell(y)(x) = -y''(x) + q(x)y(x), \quad x \in \mathbb{R} \tag{2.1}$$

with a real potential $q \in L^1_{loc}(\mathbb{R})$. Throughout this paper it is assumed that we have limit point case at both $-\infty$ and $+\infty$. We set

$$a(y)(x) = (\operatorname{sgn} x)\left(-y''(x) + q(x)y(x)\right), \quad x \in \mathbb{R}.$$

Let $\mathfrak{D}$ be the set of all $f \in L^2(\mathbb{R})$ such that f and f' are absolutely continuous with $\ell(f) \in L^2(\mathbb{R})$. On $\mathfrak{D}$ we define the operators A and L as follows:

$$\mathrm{dom}(A) = \mathrm{dom}(L) = \mathfrak{D}, \qquad Ay = a(y), \qquad Ly = \ell(y).$$

We equip $L^2(\mathbb{R})$ with the indefinite inner product

$$[f, g] := \int_{\mathbb{R}} (\operatorname{sgn} x) f(x)\overline{g(x)}dx, \quad f, g \in L^2(\mathbb{R}). \tag{2.2}$$

Then $(L^2(\mathbb{R}), [\cdot,\cdot])$ is a Krein space (for the definition of a Krein space and basic notions therein we refer to [2]). A fundamental symmetry J in $(L^2(\mathbb{R}), [\cdot,\cdot])$ is given by

$$(Jf)(x) = (\operatorname{sgn} x) f(x), \quad f \in L^2(\mathbb{R}).$$

Obviously,

$$A = JL$$

holds.

Since the differential expressions $a(\cdot)$ and $\ell(\cdot)$ are in the limit point case both at $+\infty$ and $-\infty$, the operator L is self-adjoint in the Hilbert space $L^2(\mathbb{R})$. As $A = JL$, the operator A is self-adjoint in the Krein space $L^2(\mathbb{R}, [\cdot,\cdot])$.

Definition 2.1. *We shall say that A is* the operator associated with *the differential expression $a(\cdot)$.*

2.2. Titchmarsh-Weyl coefficients

In the following we denote by $\mathbb{C}_\pm$ the set $\{z \in \mathbb{C} : \pm \mathrm{Im}\, z > 0\}$. Let $c_\lambda(x)$ and $s_\lambda(x)$ denote the fundamental solutions of the equation

$$-y''(x) + q(x)y(x) = \lambda y(x), \qquad x \in \mathbb{R}, \tag{2.3}$$

which satisfy the following conditions

$$c_\lambda(0) = s'_\lambda(0) = 1; \qquad c'_\lambda(0) = s_\lambda(0) = 0.$$

Since the equation (2.3) is limit-point at $+\infty$, the Titchmarsh-Weyl theory (see, for example, [40]) states that there exists a unique holomorphic function $m_+(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, such that the function $s_\lambda(\cdot) - m_+(\lambda)c_\lambda(\cdot)$ belongs to $L^2(\mathbb{R}_+)$. Similarly, the limit point case at $-\infty$ yields the fact that there exists a unique holomorphic function $m_-(\lambda)$, $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$, such that $s_\lambda(\cdot) + m_-(\lambda)c_\lambda(\cdot) \in L^2(\mathbb{R}_-)$. The function m_+ (m_-) is called *the Titchmarsh-Weyl m-coefficient for* (2.3) on $\mathbb{R}_+$ (on $\mathbb{R}_-$, respectively).

We put

$$M_\pm(\lambda) := \pm m_\pm(\pm\lambda)\ .$$

Definition 2.2. *The function $M_+(\cdot)$ ($M_-(\cdot)$) is said to be* the Titchmarsh-Weyl coefficient of the differential expression $a(\cdot)$ *on $\mathbb{R}_+$ (on $\mathbb{R}_-$).*

It is easy to see that for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$ the functions

$$\psi_\lambda^\pm(x) := \begin{cases} s_{\pm\lambda}(x) - M_\pm(\lambda)c_{\pm\lambda}(x), & x \in \mathbb{R}_\pm \\ 0, & x \in \mathbb{R}_\mp \end{cases} \tag{2.4}$$

belongs to $L^2(\mathbb{R})$. Moreover, the following formula (see [40]) for the norms of $\psi_\lambda^\pm$ in $L^2(\mathbb{R})$ holds true

$$\|\psi_\lambda^\pm(x)\|^2 = \frac{\mathrm{Im}\, M_\pm(\lambda)}{\mathrm{Im}\, \lambda}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.5}$$

A holomorphic function $G : \mathbb{C}_+ \cup \mathbb{C}_- \to \mathbb{C}$ is called *Nevanlinna function* or *of class (R)*, see e.g. [27], if $G(\overline{\lambda}) = \overline{G(\lambda)}$ and $\mathrm{Im}\, \lambda \cdot \mathrm{Im}\, G(\lambda) \geq 0$ for $\lambda \in \mathbb{C}_+ \cup \mathbb{C}_-$. It follows easily from (2.5) that the functions M_+ and M_- (as well as $m_\pm$) belong to the class (R). Moreover, the functions $M_\pm$ have the following asymptotic behavior

$$M_\pm(\lambda) = \pm \frac{i}{\sqrt{\pm\lambda}} + O\left(\frac{1}{|\lambda|}\right), \quad (\lambda \to \infty,\ 0 < \delta < \arg\lambda < \pi - \delta) \tag{2.6}$$

for $\delta \in (0, \frac{\pi}{2})$, see [17]. Here and below $\sqrt{z}$ is the branch of the multifunction on the complex plane $\mathbb{C}$ with the cut along $\mathbb{R}_+$, singled out by the condition $\sqrt{-1} = i$.

2.3. The non-real spectrum of A

In the following we identify functions $f \in L^2(\mathbb{R})$ with elements $\binom{f_+}{f_-}$, where $f_\pm := f\restriction_{\mathbb{R}_\pm} \in L^2(\mathbb{R}_\pm)$. Similarly we write $q_\pm := q\restriction_{\mathbb{R}_\pm} \in L^1_{\mathrm{loc}}(\mathbb{R}_\pm)$. Note that the differential expressions

$$-\frac{d^2}{dx^2} + q_+ \qquad \text{and} \qquad \frac{d^2}{dx^2} - q_-$$

in $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ are both regular at the endpoint 0 and in the limit point case at the singular endpoint $+\infty$ and $-\infty$, respectively. Therefore the operators

$$A^+_{\min} f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A^-_{\min} f_- = f_-'' - q_- f_-$$

defined on

$$\operatorname{dom} A^\pm_{\min} = \left\{ f_\pm \in \mathcal{D}^\pm_{\max} : f_\pm(0) = f_\pm'(0) = 0 \right\},$$

with

$$\mathcal{D}^+_{\max} = \left\{ f_+ \in L^2(\mathbb{R}_+) : f_+, f_+' \text{ absolutely continuous, } -f_+'' + q_+ f_+ \in L^2(\mathbb{R}_+) \right\},$$
$$\mathcal{D}^-_{\max} = \left\{ f_- \in L^2(\mathbb{R}_-) : f_-, f_-' \text{ absolutely continuous, } f_-'' - q_- f_- \in L^2(\mathbb{R}_-) \right\},$$

are closed symmetric operators in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [45, 46], with deficiency indices $(1, 1)$. The adjoint operators $(A^\pm_{\min})^*$ in the Hilbert space $L^2(\mathbb{R}_\pm)$ are the usual maximal operators defined on $\mathcal{D}^\pm_{\max}$.

We introduce the operators

$$A^+_0 f_+ = -f_+'' + q_+ f_+ \quad \text{and} \quad A^-_0 f_- = f_-'' - q_- f_-$$

defined on

$$\operatorname{dom} A^\pm_0 = \left\{ f_\pm \in \mathcal{D}^\pm_{\max} : f_\pm'(0) = 0 \right\},$$

Evidently, $A^\pm_0$ are self-adjoint extensions of $A^\pm_{\min}$ in the Hilbert spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$, respectively, cf. [45, 46]. In the following we consider $\operatorname{dom} A^\pm_{\min}$ as subsets of $L^2(\mathbb{R})$. Then above considerations imply the following lemma.

Lemma 2.3. *Let* $\operatorname{dom} A_{\min} := \operatorname{dom} A^+_{\min} \oplus \operatorname{dom} A^-_{\min}$ *and let the operator* $A_{\min}$ *be defined on* $\operatorname{dom} A_{\min}$*,*

$$A_{\min} := \begin{pmatrix} A^+_{\min} & 0 \\ 0 & A^-_{\min} \end{pmatrix},$$

with respect to the decomposition $L^2(\mathbb{R}) = L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_-)$. *Then* $A_{\min}$ *is a closed symmetric operator in the Hilbert space* $L^2(\mathbb{R})$ *with deficiency indices* $(2,2)$. *Moreover, we have*

$$A_{\min} = A \upharpoonright_{\operatorname{dom} A_{\min}}, \qquad A = A_{\min}^* \upharpoonright_{\mathfrak{D}},$$

where

$$\mathfrak{D} = \operatorname{dom}(A) = \\ = \left\{ f = \left(\begin{smallmatrix} f_+ \\ f_- \end{smallmatrix}\right) \in \operatorname{dom}(A_{\min}^+)^* \oplus \operatorname{dom}(A_{\min}^-)^* : f_+(0) = f_-(0), f'_+(0) = f'_-(0) \right\}.$$

In the following proposition we collect some spectral properties of A.

Proposition 2.4. *Let* A *be the operator associated with the differential expression* $a(\cdot)$. *Then:*

(i) $\{\lambda \in \mathbb{C} \setminus \mathbb{R} \;:\; M_+(\lambda) = M_-(\lambda)\} = \sigma_p(A) \setminus \mathbb{R}$;

(ii) $\{\lambda \in \mathbb{C} \setminus \mathbb{R} \;:\; M_+(\lambda) \neq M_-(\lambda)\} = \rho(A) \setminus \mathbb{R}$;

(iii) $\rho(A) \neq \emptyset$.

(iv) *The essential spectrum* $\sigma_{ess}(A)$ *of* A *is real and*

$$\sigma_{ess}(A) = \sigma_{ess}(A_0^+) \cup \sigma_{ess}(A_0^-).$$

The sets $\sigma_p(A) \cap \mathbb{C}_\pm$ *are at most countable with possible limit points belonging to* $\sigma_{ess}(A) \cup \{\infty\}$.

For a proof of Proposition 2.4 we refer to [34, Proposition 2.5] and [30, 31]. We mention only that the statements **(iii)** and **(iv)** follow from the first and second statement and (2.6).

3. Criterions for definitizability

3.1. Definitizable and locally definitizable operators

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let A be a closed operator in $\mathcal{H}$. We define the extended spectrum $\sigma_e(A)$ of A by $\sigma_e(A) := \sigma(A)$ if A is bounded and $\sigma_e(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. We set $\rho_e(A) := \overline{\mathbb{C}} \setminus \sigma_e(A)$. A point $\lambda_0 \in \mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{ap}(A)$ of A if there exists a sequence $(x_n) \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, and $\|(A - \lambda_0)x_n\| \to 0$ if $n \to \infty$. For a self-adjoint operator A in $\mathcal{H}$ all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. [9, Corollary VI.6.2]).

First we recall the notions of spectral points of positive and negative type.

The following definition was given in [37], [39] (for bounded self-adjoint operators).

Definition 3.1. *For a self-adjoint operator* A *in* $\mathcal{H}$ *a point* $\lambda_0 \in \sigma(A)$ *is called a spectral point of* positive (negative) type of A *if* $\lambda_0 \in \sigma_{ap}(A)$ *and for every sequence* $(x_n) \subset \operatorname{dom}(A)$ *with* $\|x_n\| = 1$ *and* $\|(A - \lambda_0)x_n\| \to 0$ *for* $n \to \infty$, *we have*

$$\liminf_{n\to\infty} \,[x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty}\,[x_n, x_n] < 0).$$

The point ∞ is said to be of positive (negative) type of A *if A is unbounded and for every sequence $(x_n) \subset \mathrm{dom}(A)$ with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have*

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad (\textit{resp.} \ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0).$$

We denote the set of all points of $\sigma_e(A)$ of positive (negative) type by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$). We shall say that an open subset δ of $\overline{\mathbb{R}}$ $(= \mathbb{R} \cup \infty)$ is of positive type (negative type) *with respect to A if*

$$\delta \cap \sigma_e(A) \subset \sigma_{++}(A) \quad (\textit{resp.} \ \delta \cap \sigma_e(A) \subset \sigma_{--}(A)).$$

An open set δ of $\overline{\mathbb{R}}$ is called of definite type *if δ is of positive or negative type with respect to A.*

The sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. The non-real spectrum of A cannot accumulate at a point belonging to an open set of definite type.

Recall, that a self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called definitizable if $\rho(A) \neq \emptyset$ and there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the non-real part of the spectrum of A consists of no more than a finite number of points. Moreover, A has a spectral function E defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not coincide with the points of some finite set which is contained in $\{t \in \mathbb{R} : p(t) = 0\} \cup \{\infty\}$ (see [38]).

A self-adjoint operator in a Krein space is definitizable if and only if it is definitizable over $\overline{\mathbb{C}}$ in the sense of the following definition (see e.g. [24, Definition 4.4]), which localizes the notion of definitizability.

Definition 3.2. *Let Ω be a domain in $\overline{\mathbb{C}}$ such that*

$$\Omega \quad \textit{is symmetric with respect to } \mathbb{R}, \quad \Omega \cap \overline{\mathbb{R}} \neq \emptyset, \tag{3.1}$$

$$\textit{and the domains} \quad \Omega \cap \mathbb{C}^+, \ \Omega \cap \mathbb{C}^- \quad \textit{are simply connected.} \tag{3.2}$$

Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ such that $\sigma(A) \cap (\Omega \setminus \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of A. The operator A is called definitizable over Ω, *if the following holds.*

(i) *For every closed subset Δ of $\Omega \cap \overline{\mathbb{R}}$ there exist an open neighbourhood $\mathcal{U}$ of Δ in $\overline{\mathbb{C}}$ and numbers $m \geq 1$, $M > 0$ such that*

$$\|(A - \lambda)^{-1}\| \leq M(|\lambda| + 1)^{2m-2} |\mathrm{Im}\, \lambda|^{-m} \tag{3.3}$$

for all $\lambda \in \mathcal{U} \setminus \overline{\mathbb{R}}$.

(ii) *Every point $\lambda \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_λ in $\overline{\mathbb{R}}$ such that both components of $I_\lambda \setminus \{\lambda\}$ are of definite type (cf. Definition* 3.1*) with respect to A.*

A self-adjoint operator definitizable over Ω where Ω is as in Definition 3.2 possesses a local spectral function E. For the construction and the properties of this spectral function we refer to [24] (see also [23]). We mention only that $E(\Delta)$ is defined and is a self-adjoint projection in $(\mathcal{H}, [\cdot,\cdot])$ for every union Δ of a finite number of connected subsets Δ_i, $i = 1, \dots, n$, of $\Omega \cap \overline{\mathbb{R}}$, $\overline{\Delta_i} \subset \Omega \cap \overline{\mathbb{R}}$, such that the endpoints of Δ_i belong to intervals of definite type. A real point $\lambda \in \sigma(A) \cap \Omega$ belongs to $\sigma_{++}(A)$ if and only if there exists a bounded open interval $\Delta \subset \Omega$, $\lambda \in \Delta$, such that $E(\Delta)\mathcal{H}$ is a Hilbert space (cf. [3]). A point $t \in \overline{\mathbb{R}} \cap \Omega$ is called a *critical point* of A if there is no open subset $\Delta \subset \Omega$ of definite type with $t \in \Delta$. The set of critical points of A is denoted by $c(A)$. A critical point t is called *regular* if there exists an open deleted neighbourhood $\delta_0 \subset \Omega$ of t such that the set of the projections $E(\delta)$ where δ runs through all intervals δ with $\overline{\delta} \subset \delta_0$ is bounded. The set of regular critical points of A is denoted by $c_r(A)$. The elements of $c_s(A) := c(A) \setminus c_r(A)$ are called *singular* critical points.

We will make use of the following perturbation result, see [6].

Theorem 3.3. *Let T_1 and T_2 be self-adjoint operators in the Krein space $\mathcal{H}$, let $\rho(T_1) \cap \rho(T_2) \cap \Omega \neq \emptyset$ and assume that*

$$(T_1 - \lambda_0 I)^{-1} - (T_2 - \lambda_0 I)^{-1}$$

is a finite rank operator for some $\lambda_0 \in \rho(T_1) \cap \rho(T_2)$. Then T_1 is definitizable over Ω if and only if T_2 is definitizable over Ω.

Moreover, if T_1 is definitizable over Ω and $\Delta \subset \Omega \cap \overline{\mathbb{R}}$ is an open interval with end point $\eta \in \Omega \cap \overline{\mathbb{R}}$ and Δ is of positive type (negative type) with respect to T_1, then there exist open interval Δ', $\Delta' \subset \Delta$, with endpoint η such that Δ' is of positive type (resp. negative type) with respect to T_2.

3.2. Definitizability of A

In this section we will give conditions which ensures the definitizability of the operator A from Definition 2.1. The following definition is needed below.

Definition 3.4. *We shall say that the sets S_1 and S_2 of real numbers are separated by a finite number of points if there exists a finite ordered set $\{\alpha_j\}_{j=1}^N$, $N \in \mathbb{N}$,*

$$-\infty = \alpha_0 < \alpha_1 \leq \cdots \leq \alpha_N < \alpha_{N+1} = +\infty,$$

such that one of the sets S_j, $j = 1, 2$, is a subset of $\bigcup_{k \text{ is even}} [\alpha_k, \alpha_{k+1}]$ and another one is a subset of $\bigcup_{k \text{ is odd}} [\alpha_k, \alpha_{k+1}]$.

The operator $A_0^+ \oplus A_0^-$, where $A_0^\pm$ are defined as in Section 2.3, is fundamentally reducible (cf. [22, Section 3]) in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$ (cf. (2.2)). Hence the following lemma is a easy consequence of Definitions 3.1 and 3.2.

Lemma 3.5. *Let $\lambda \in \mathbb{R}$. Then $\lambda \in \sigma_{++}(A_0^+ \oplus A_0^-)$ ($\lambda \in \sigma_{--}(A_0^+ \oplus A_0^-)$) if and only if $\lambda \in \sigma(A_0^+) \setminus \sigma(A_0^-)$ ($\lambda \in \sigma(A_0^-) \setminus \sigma(A_0^+)$, resp.). The operator $A_0^+ \oplus A_0^-$ is definitizable if and only if the sets $\sigma(A_0^+)$ and $\sigma(A_0^-)$ are separated by a finite number of points.*

It follows from Proposition 2.4 and $\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R}$ that $\rho(A) \cap \rho(A_0^+ \oplus A_0^-) \neq \emptyset$. Let $\lambda_0 \in \rho(A) \cap \rho(A_0^+ \oplus A_0^-)$. The operators $A_0^+ \oplus A_0^-$ and A are extensions of $A_{\min}$

and $\dim\left(\operatorname{dom}(A_0^+ \oplus A_0^-)/\operatorname{dom}(A_{\min})\right) = \dim\left(\operatorname{dom}(A)/\operatorname{dom}(A_{\min})\right) = 2$. This implies that

$$(A_0^+ \oplus A_0^- - \lambda_0 I)^{-1} - (A - \lambda_0 I)^{-1}$$

is an operator of rank 2. Then [25] and Lemma 3.5 imply the following theorem.

Theorem 3.6 ([30, 31]). *The operator A is definitizable if and only if the sets $\sigma(A_0^+)$ and $\sigma(A_0^-)$ are separated by a finite number of points.*

Example 3.7. *Let q be a constant potential, $q(x) \equiv c$, $c \in \mathbb{R}$. It is easy to calculate that $\sigma(A_0^+) = [c, +\infty)$ and $\sigma(A_0^-) = (-\infty, -c]$. Thus, Corollary 3.6 implies that the operator $(\operatorname{sgn} x)(-d^2/dx^2 + c)$ is definitizable in the Krein space $L^2(\mathbb{R}, \operatorname{sgn} x\, dx)$ if and only if $c \geq 0$.*

3.3. Local definitizability of A

In this subsection we consider Sturm-Liouville operators defined as in Section 2 and we prove that the operator A is a definitizable operator in a certain neighbourhood of ∞ (in the sense of the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$) if and only if the operator L is semi-bounded from below (in the sense of the Hilbert space $L^2(\mathbb{R})$).

Remark 3.8. *Clearly, $L \geq \eta_0 > -\infty$ whenever $q(x) \geq \eta_0 > -\infty$, $x \in \mathbb{R}$.*

The operator $A_0^+ \oplus A_0^-$ is a self-adjoint operator both in the Hilbert space $L^2(\mathbb{R})$ and in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$, cf. (2.2).

Lemma 3.9. *The following statements are equivalent:*

(i) *The operator L is semi-bounded from below.*

(ii) *There exists $R > 0$ such that the operator $A_0^+ \oplus A_0^-$ is definitizable over the domain $\{\lambda \in \overline{\mathbb{C}} \ : \ |\lambda| > R\}$.*

Proof. $(i) \Rightarrow (ii)$. Since $A_0^+ \oplus A_0^-$ is a self-adjoint operator in the Hilbert space $L^2(\mathbb{R})$, we see that

$$\sigma(A_0^+ \oplus A_0^-) \subset \mathbb{R} \quad \text{and (3.3) holds for all } \lambda \in \mathbb{C} \setminus \mathbb{R} \quad \text{with } m = 1. \tag{3.4}$$

Assume that $L \geq \eta_0$. The operator L is a self-adjoint extension of $A_{\min}^+ \oplus (-A_{\min}^-)$, hence the operator $A_{\min}^+$ is semi-bounded from below, $A_{\min}^+ \geq \eta_0$, and $A_{\min}^-$ is semi-bounded from above, $A_{\min}^- \leq -\eta_0$ The operators $A_0^\pm$ are self-adjoint extensions in $L^2(\mathbb{R}_\pm)$ of the symmetric operators $A_{\min}^\pm$ with deficiency indices (1,1). Hence the spectrum of A_0^+ (A_0^-) lies, with the possible exception of at most one normal eigenvalue, in $[\eta_0, \infty)$ (in $(-\infty, -\eta_0]$, respectively), see e.g. [1, Section VII.85].

Choose $R := \eta_0$. Lemma 3.5 implies that the set $(R, +\infty)$, with the possible exception of at most one eigenvalue, is of positive type and the set $(-\infty, -R)$, with the possible exception of at most one eigenvalue, is of negative type with respect to $A_0^+ \oplus A_0^-$. Thus, the operator $A_0^+ \oplus A_0^-$ is definitizable over $\{\lambda \in \overline{\mathbb{C}} \ : \ |\lambda| > R\}$.

$(i) \Leftarrow (ii)$ Obviously, the Sturm-Liouville operator A_0^+ (A_0^-) is not semi-bounded from above (below, resp.). That is,

$$\sup \sigma(A_0^+) = +\infty, \qquad \inf \sigma(A_0^-) = -\infty. \tag{3.5}$$

Assume that L is not semi-bounded from below. Then $A^+_{\min}$ or $-A^-_{\min}$ is not semi-bounded from below. Thus, $\inf \sigma(A^+_0) = -\infty$ or $\sup \sigma(A^-_0) = +\infty$.

Consider the case

$$\inf \sigma(A^+_0) = -\infty. \tag{3.6}$$

It follows from (3.6), (3.5) and Lemma 3.5 that

$$(-\infty, -r) \cap \sigma_{++}(A^+_0 \oplus A^-_0) \neq \emptyset \qquad \text{and} \qquad (-\infty, -r) \cap \sigma_{--}(A^+_0 \oplus A^-_0) \neq \emptyset$$

for all $r > 0$. Thus, by definition, the operator $A^+_0 \oplus A^-_0$ is not definitizable over $\{\lambda \in \overline{\mathbb{C}} \ : \ |\lambda| > r\}$ for arbitrary $r > 0$. The case $\sup \sigma(A^-_0) = +\infty$ can be considered in the same way. □

The following theorem is one of the main results.

Theorem 3.10. *The following assertions are equivalent:*

(i) *The operator L is semi-bounded from below.*

(ii) *There exists $R > 0$ such that the operator A is definitizable over the domain $\{\lambda \in \overline{\mathbb{C}} \ : \ |\lambda| > R\}$.*

Proof. It follows from Proposition 2.4 (iii) and $\sigma(A^+_0 \oplus A^-_0) \subset \mathbb{R}$ that $\rho(A) \cap \rho(A^+_0 \oplus A^-_0) \neq \emptyset$. Let $\lambda_0 \in \rho(A) \cap \rho(A^+_0 \oplus A^-_0)$. The operators $A^+_0 \oplus A^-_0$ and A are extensions of $A_{\min}$ and $\dim\big(\mathrm{dom}(A^+_0 \oplus A^-_0)/\,\mathrm{dom}(A_{\min})\big) = \dim\left(\mathrm{dom}(A)/\,\mathrm{dom}(A_{\min})\right) = 2$. This implies that

$$(A^+_0 \oplus A^-_0 - \lambda_0 I)^{-1} - (A - \lambda_0 I)^{-1} \tag{3.7}$$

is an operator of rank 2. Combining Lemma 3.9 and Theorem 3.3, Theorem 3.10 is proved. □

By Theorem 3.10, the semi-boundedness of L implies the definitizability of A over some domain. Now we give a precise description of the domain of definitizability of A in terms of the spectra of A^+_0 and A^-_0.

Let T be an operator such that $\sigma(T) \subset \mathbb{R}$. Let us introduce the sets $\sigma^{left}(T)$ and $\sigma^{right}(T)$ by the following way: a point $\lambda \in \overline{\mathbb{R}}$ $(= \mathbb{R} \cup \infty)$ is said to belong to $\sigma^{left}(T)$ $\ (\sigma^{right}(T))$ if there exists an increasing (resp. decreasing) sequence $\{\lambda_n\}_1^\infty \subset \sigma(T)$ such that $\lim_{n\to\infty} \lambda_n = \lambda$.

Note that

$$\sigma^{left}(T) \cup \sigma^{right}(T) \subset \sigma_{ess}(T) \cup \{\infty\}. \tag{3.8}$$

For differential operators $A^\pm_0$, equality holds in (3.8) since every point of $\sigma_{ess}(A^\pm_0)$ is an accumulation point of $\sigma(A^\pm_0)$.

We put

$$\mathcal{S}_A := \left(\sigma^{left}(A^+_0) \cap \sigma^{left}(A^-_0)\right) \cup \left(\sigma^{right}(A^+_0) \cap \sigma^{right}(A^-_0)\right). \tag{3.9}$$

Theorem 3.11. *Let Ω be a domain in $\overline{\mathbb{C}}$ such that* (3.1)-(3.2) *are fulfilled. Then the operator $A = (\mathrm{sgn}\, x)(-d^2/dx^2 + q)$ is definitizable over Ω if and only if $\Omega \subset \Omega_A$, where $\Omega_A := \overline{\mathbb{C}} \setminus \mathcal{S}_A$.*

Proof. Arguments from the proof of Theorem 3.10 show that it is enough to prove the theorem for the operator $A_0^+ \oplus A_0^-$.

Let $\lambda \in \mathcal{S}_A$ and let I_λ be an open connected neighbourhood of λ. Then (3.9) and Lemma 3.5 imply that one of the components of $I_\lambda \setminus \{\lambda\}$ is not of definite type. So if $A_0^+ \oplus A_0^-$ is definitizable over Ω, then $\lambda \notin \Omega$.

Conversely, if $\mathcal{S}_A \neq \overline{\mathbb{R}}$, then condition (ii) from Definition 3.2 is fulfilled for $\Omega_A = \overline{\mathbb{C}} \setminus \mathcal{S}_A$. Taking (3.4) into account, we see that $A_0^+ \oplus A_0^-$ is definitizable over Ω_A. □

Remark 3.12. *Note that $\Omega_A \cap \overline{\mathbb{R}} = \emptyset$ is equivalent to $\sigma_{ess}(A_0^+) = \sigma_{ess}(A_0^-) = \mathbb{R}$. In the converse case, (3.1)-(3.2) are fulfilled for Ω_A and it is the greatest domain over which the operator A is definitizable.*

The following statement is a simple consequence of Theorem 3.10, Theorem 3.11, and (3.8).

Corollary 3.13. *Assume that L is semi-bounded from below. Then the operator A is definitizable over the set $\overline{\mathbb{C}} \setminus (\sigma_{ess}(A_0^+) \cap \sigma_{ess}(A_0^-))$.*

3.4. Regularity of the critical point ∞

In the sequel we will use a result which follows easily from [12, Lemma 3.5 (iii)] and [12, Theorem 3.6 (i)].

Proposition 3.14. *If the operator $\widetilde{L} := -d^2/dx^2 + \widetilde{q}(x)$, for some real $\widetilde{q} \in L^1_{loc}(\mathbb{R})$, defined on $\mathfrak{D}$ is nonnegative in the Hilbert space $L^2(\mathbb{R})$, then the operator $\widetilde{A} := (\operatorname{sgn} x)\widetilde{L}$ is definitizable and ∞ is a regular critical point of $\widetilde{A}$.*

The following theorem can be considered as the main result of this note.

Theorem 3.15. *Assume that assertions (i), (ii) of Theorem 3.10 hold true. Then there exists a decomposition*

$$A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b \tag{3.10}$$

such that the operator $\mathcal{A}_\infty$ is similar to a self-adjoint operator in the Hilbert space sense and $\mathcal{A}_b$ is a bounded operator.

Remark 3.16. *The conclusion of Theorem 3.15 is equivalent to the regularity of critical point ∞ of the operator A.*

Proof of Theorem 3.15. Assume that A is an operator definitizable over $\{\lambda \in \overline{\mathbb{C}} : |\lambda| > R\}$, $R > 0$. By Theorem 3.10, this is equivalent to the fact that $L \geq \eta_0$ for certain $\eta_0 \in \mathbb{R}$.

Denote by E^A the spectral function of A. Choose $r > R$ such that $\sigma(A) \setminus \mathbb{R} \subset \{\lambda \in \mathbb{C} : |\lambda| \leq r\}$ and $E^A(\overline{\mathbb{R}} \setminus (-r, r))$ is defined. Then A decomposes,

$$A = \mathcal{A}_1 \dot{+} \mathcal{A}_0, \qquad \mathcal{A}_1 := A \upharpoonright \operatorname{dom}(A) \cap (E^A(\overline{\mathbb{R}} \setminus (-r,r))L^2(\mathbb{R})),$$
$$\mathcal{A}_0 := A \upharpoonright \operatorname{dom}(A) \cap ((I - E^A(\overline{\mathbb{R}} \setminus (-r,r)))L^2(\mathbb{R}))$$

and the following statements holds (cf. [22, Theorem 2.6]):

$\mathcal{A}_1$ is a definitizable operator in the Krein space $(E^A(\overline{\mathbb{R}} \setminus (-r,r))L^2(\mathbb{R}), [\cdot,\cdot])$;

$\mathcal{A}_0$ is a bounded operator and $\sigma(\mathcal{A}_0) \subset \{\lambda : |\lambda| \le r\}$.

Let us show that ∞ is not a singular critical point of $\mathcal{A}_1$.

Consider the operator $\mathcal{A}_2$ defined by $\mathcal{A}_2 = \mathcal{A}_1 \dot{+} 0$, where the direct sum is considered with respect to the decomposition

$$L^2(\mathbb{R}) = E^A(\overline{\mathbb{R}} \setminus (-r,r))L^2(\mathbb{R}) \dot{+} (I - E^A(\overline{\mathbb{R}} \setminus (-r,r)))L^2(\mathbb{R}),$$

and 0 is the zero operator in the subspace $\operatorname{ran}(I - E^A(\overline{\mathbb{R}} \setminus (-r,r)))$. Since $\mathcal{A}_0$ is a bounded operator, we have

$$\operatorname{dom}(\mathcal{A}_2) = \operatorname{dom} A.$$

It is easy to see that $\mathcal{A}_2$ is a definitizable operator in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$. Moreover, ∞ is not a singular critical point of $\mathcal{A}_2$ if and only if ∞ is not a singular critical point of A.

Now we prove that ∞ is not a singular critical point of $\mathcal{A}_2$. Let $\eta_1 < \eta_0$. Since $L \ge \eta_0$, we see that $L - \eta_1 I$ is a uniformly positive operator in the Hilbert space $L^2(\mathbb{R})$ (i.e., $L - \eta_1 I \ge \delta > 0$). Therefore $\widetilde{A} := J(L - \eta_1 I)$,

$$\widetilde{A}y(x) = (\operatorname{sgn} x)(-y''(x) + q(x)y(x) - \eta_1 y(x)), \qquad \operatorname{dom}(\widetilde{A}) = \operatorname{dom}(A),$$

is a definitizable nonnegative operator in the Krein space $(L^2(\mathbb{R}), [\cdot,\cdot])$. By Proposition 3.14, ∞ is not a singular critical point of $\widetilde{A}$. The Ćurgus criterion of the regularity of critical point ∞, see [11, Corollary 3.3], implies that ∞ is not a singular critical point of the operator $\mathcal{A}_2$. So ∞ is not a singular critical point of $\mathcal{A}_1$.

It follows from $L \ge \eta_0$ and Lemma 3.5 that for sufficiently large $r_1 > 0$ the set $(-\infty, -r_1]$ is of negative type and the set $[r_1, +\infty)$ is of positive type with respect to $A_0^+ \oplus A_0^-$. Combining this with Theorem 3.3, we obtain that there exists $r_2 \ge r_1$ such that $(-\infty, -r_2]$ is of negative type and the set $[r_2, +\infty)$ is of positive type with respect to the operator A. Evidently, we obtain the desired decomposition

$$A = \mathcal{A}_\infty \dot{+} \mathcal{A}_b, \qquad \mathcal{A}_\infty := A \upharpoonright \operatorname{dom}(A) \cap (E^A(\overline{\mathbb{R}} \setminus (-r_2,r_2))L^2(\mathbb{R})),$$
$$\mathcal{A}_b := A \upharpoonright \operatorname{dom}(A) \cap ((I - E^A(\overline{\mathbb{R}} \setminus (-r_2,r_2)))L^2(\mathbb{R})),$$

where $\mathcal{A}_b$ is a bounded operator and $\mathcal{A}_\infty$ is similar to a self-adjoint operator in the Hilbert space sense. □

4. Accumulation of non-real eigenvalues to a real point

By Proposition 2.4 (i), the non-real spectrum $\sigma(A) \setminus \mathbb{R}$ of A consists of eigenvalues.

Let $\mathcal{S}_A$ be the set defined by (3.9). The following proposition is a consequence of Theorems 3.11 and 3.10.

Proposition 4.1. *If λ is an accumulation point of $\sigma(A) \setminus \mathbb{R}$, then $\lambda \in \mathcal{S}_A$. In particular, if the operator $L = -d^2/dx^2 + q(x)$ is semi-bounded from below, then non-real spectrum of A is a bounded set.*

The goal of this subsection is to show that there exists a potential q continuous in $\mathbb{R}$ such that the set of non-real eigenvalues of the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q(x))$ *has* a real accumulation point.

It is well known (e.g. [40]) that M_+, the Titchmarsh-Weyl m-coefficient for (2.3) (see Subsection 2.2), admits the following integral representation

$$M_+(\lambda) = \int_{\mathbb{R}} \frac{d\Sigma_+(t)}{t-\lambda}, \qquad \lambda \in \mathbb{C} \setminus \mathbb{R},$$

where $\Sigma_+(\cdot)$ is a nondecreasing scalar functions such that $\int_{\mathbb{R}} (1+|t|)^{-1} d\Sigma_+(t) < \infty$. The function Σ_+ is called *a spectral function of* the boundary value problem

$$-y''(x) + q_+(x)y(x) = \lambda y(x), \quad y'(0) = 0, \qquad x \in [0, +\infty). \tag{4.1}$$

This means that the self-adjoint operator A_0^+ introduced in Subsection 2.3 is unitary equivalent to the operator of multiplication by the independent variable in the Hilbert space $L^2(\mathbb{R}, d\Sigma_+(t))$. This fact obviously implies

$$\sigma(A_0^+) = \operatorname{supp}(d\Sigma_+), \tag{4.2}$$

where $\operatorname{supp} d\tau$ denotes the *topological support* of a Borel measure $d\Sigma_+$ on $\mathbb{R}$ (i.e., $\operatorname{supp} d\Sigma_+$ is the smallest closed set $\Omega \subset \mathbb{R}$ such that $d\Sigma_+(\mathbb{R} \setminus \Omega) = 0$).

Lemma 4.2. *Assume that q is an even potential, $q(x) = q(-x)$, $x \in \mathbb{R}$. If $\varepsilon > 0$, then $i\varepsilon \in \sigma_p(A)$ if and only if* $\operatorname{Re} M_+(i\varepsilon) = 0$.

Proof. Since q is even, we get $m_+(\lambda) = m_-(\lambda)$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. So $\underline{M_-(i\varepsilon)} = -M_+(-i\varepsilon)$. Since M_+ is a Nevanlinna function, we see that $M_+(-i\varepsilon) = \overline{M_+(i\varepsilon)}$. Thus,

$$M_+(i\varepsilon) - M_-(i\varepsilon) = M_+(i\varepsilon) + \overline{M_+(i\varepsilon)} = 2 \operatorname{Re} M_+(i\varepsilon).$$

Proposition 2.4 completes the proof. □

The following lemma follows easily from the Gelfand–Levitan theorem (see e.g. [41, Subsection 26.5]).

Lemma 4.3. *Let $\Sigma(t)$, $t \in \mathbb{R}$, be a nondecreasing function such that*

$$\int_{-\infty}^{T_1 - 0} d\Sigma(t) = 0 \qquad \textit{and} \tag{4.3}$$

$$\int_{-\infty}^{s-0} d\Sigma(t) = \int_0^s \frac{1}{\pi\sqrt{t}} dt \ \left(= \frac{2}{\pi}\sqrt{s} \right) \qquad \textit{for all} \quad s > T_2. \tag{4.4}$$

with certain constants $T_1, T_2 \in \mathbb{R}$, $T_1 < T_2$. Then there exists a potential q_+ continuous in $[0, +\infty)$ such that $\Sigma(t)$ is a spectral function of the boundary value problem

$$-y''(x) + q_+(x)y(x) = \lambda y(x), \quad y'(0) = 0, \qquad x \in [0, +\infty).$$

Lemma 4.4. *There exist a nondecreasing function $\Sigma(t)$, $t \in \mathbb{R}$, with the following properties:*

(i) $\Sigma(t) = \Sigma_1(t) + \Sigma_2(t)$, *where*

$$\Sigma_1 \in AC_{loc}(\mathbb{R}), \qquad \Sigma_1^{'}(t) = \begin{cases} 0, & t \in (-\infty, 1), \\ \frac{1}{\pi\sqrt{t}}, & t \in (1, +\infty), \end{cases} \tag{4.5}$$

and the measure $d\Sigma_2$ *has the form*

$$d\Sigma_2(t) = \sum_{k=1}^{+\infty} h_k \delta(t - s_k),$$

$$h_k > 0, \quad s_k \in (-1, 1), \quad k \in \mathbb{N}; \qquad \sum_{k=1}^{+\infty} h_k < \infty, \tag{4.6}$$

(here $\delta(t)$ *is the Dirac delta-function).*

(ii) *Conditions* (4.3)-(4.4) *are valid for* Σ *with* $T_1 = -1$ *and* $T_2 = 1$.

(iii) *There exists a sequence* $\varepsilon_k > 0$, $k \in \mathbb{N}$, *such that* $\lim_{k\to\infty} \varepsilon_k = 0$ *and* $r(\varepsilon_k) = 0$, $k \in \mathbb{N}$, *where the function* $r(\varepsilon)$, $\varepsilon > 0$, *is defined by*

$$r(\varepsilon) := \operatorname{Re} \int_{\mathbb{R}} \frac{1}{t - i\varepsilon} d\Sigma(t) = \int_{\mathbb{R}} \frac{t}{t^2 + \varepsilon^2} d\Sigma(t).$$

Proof. Let $h_k = 2^{-k+1}/\pi$. Then

$$\sum_{k=1}^{\infty} h_k = 2/\pi. \tag{4.7}$$

Now, if $s_k \in (-1, 1)$ for all $k \in \mathbb{N}$, then Σ possesses property (ii). We should only choose $\{s_k\}_1^\infty \subset (-1, 1)$ such that statements (iii) holds true.

Consider for $\varepsilon \geq 0$ the functions

$$r_0(\varepsilon) = \int_1^\infty \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t)$$

and

$$r_n(\varepsilon) := \int_1^\infty \frac{t}{t^2 + \varepsilon^2} d\Sigma_1(t) + \sum_{k=1}^{n} \frac{s_k h_k}{s_k^2 + \varepsilon^2}, \qquad n \in \mathbb{N}.$$

Let $s_k \neq 0$ for all $k \in \mathbb{N}$. Then r_n are well-defined and continuous on $[0, +\infty)$. Besides, $\lim_{n\to\infty} r_n(\varepsilon) = r(\varepsilon)$ for all $\varepsilon > 0$. It is easy to see that $\lim_{\varepsilon\to\infty} r_n(\varepsilon) = 0$, $n \in \mathbb{N}$. Since r_n are continuous on $[0, +\infty)$, we see that

$$\mathrm{SUP}_n := \sup_{\varepsilon\in[0,+\infty)} |r_n(\varepsilon)| < \infty, \qquad n \in \mathbb{N}.$$

Now we give a procedure to choose $s_k \in (-1, 1) \setminus \{0\}$.

Let s_1 be an arbitrary number in $(-1,0)$ such that

$$\left.\frac{s_1 h_1}{s_1^2+\varepsilon^2}\right|_{\varepsilon=|s_1|} = \frac{1}{\pi}\frac{1}{2s_1} < -\mathrm{SUP}_0 - 1,$$

$$\text{in other words,} \qquad -\frac{1}{2\pi(\mathrm{SUP}_0+1)} < s_1 < 0.$$

Then

$$r_1(|s_1|) = r_0(|s_1|) + \left.\frac{s_1 h_1}{s_1^2+\varepsilon^2}\right|_{\varepsilon=|s_1|} < r_0(|s_1|) - \sup_{\varepsilon\in[0,+\infty)} |r_0(\varepsilon)| - 1 < -1. \tag{4.8}$$

Let

$$\{s_k\}_2^\infty \in (-b_1, b_1)\setminus\{0\} \quad \text{with certain} \quad b_1 \in (0, |s_1|/2). \tag{4.9}$$

Let us show that we may choose a number b_1 such that (4.9) implies

$$r(|s_1|) < 0. \tag{4.10}$$

Indeed, (4.8) and (4.7) yield

$$r(|s_1|) = r_1(|s_1|) + \left[\sum_{k=2}^{\infty} \frac{s_k h_k}{s_k^2+\varepsilon^2}\right]_{\varepsilon=|s_1|} <$$

$$< -1 + \sum_{k=2}^{\infty} \frac{h_k |s_k|}{s_k^2+s_1^2} < -1 + \frac{b_1}{s_1^2}\sum_{k=2}^{\infty} h_k < -1 + \frac{2b_1}{\pi s_1^2}$$

and therefore (4.10) is valid whenever $0 < b_1 < \pi s_1^2/2$.

Similarly, there exist $s_2 \in (0, b_1)$ such that

$$\left.\frac{s_2 h_2}{s_2^2+\varepsilon^2}\right|_{\varepsilon=s_2} = \frac{1}{2\pi}\frac{1}{2s_2} > \mathrm{SUP}_1 + 1,$$

and therefore

$$r_2(s_2) > 1.$$

Further, there exist $b_2 \in (0, s_2/2)$ such that $\{s_k\}_3^\infty \subset (-b_2, b_2)\setminus\{0\}$ implies that $r(s_2) > 0$.

Continuing this process, we obtain a sequence $\{s_k\}_1^\infty \subset (-1,1)\setminus\{0\}$ with the following properties:

$$s_k \in (-1,0) \quad \text{if k is odd,} \qquad s_k \in (0,1) \quad \text{if k is even,}$$

$$|s_1| > \frac{|s_1|}{2} > |s_2| > \frac{|s_2|}{2} > |s_3| > \ldots > |s_k| > \frac{|s_k|}{2} > |s_{k+1}| > \ldots\,, \tag{4.11}$$

$$r(|s_k|) < 0 \quad \text{if k is odd,} \qquad r(|s_k|) > 0 \quad \text{if k is even.} \tag{4.12}$$

It is easy to show that r is continuous on $(0,+\infty)$. Combining this with (4.12), we see that there exists $\varepsilon_k \in (|s_{k-1}|, |s_k|)$ such that $r(\varepsilon_k) = 0$, $k \in \mathbb{N}$. Besides, (4.11) implies $\lim |s_k| = \lim \varepsilon_k = 0$. □

Theorem 4.5. *There exist an even potential $\widehat{q}$ continuous on $\mathbb{R}$ and a sequence $\{\varepsilon_k\}_1^\infty \subset \mathbb{R}_+$ such that*

(i) *the operator $\widehat{A}$ defined by the differential expression*

$$(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2}+\widehat{q}(x)\right) \tag{4.13}$$

on the natural domain $\mathfrak{D}$ (see Subsection 2.1) is a self-adjoint operator in the Krein space $L^2(\mathbb{R},[\cdot,\cdot])$;

(ii) $\{i\varepsilon_k\}_1^\infty \subset \sigma_p(\widehat{A})$, *i.e., $i\varepsilon_k$, $k\in\mathbb{N}$, are non-real eigenvalues of $\widehat{A}$;*

(iii) $\lim_{k\to\infty}\varepsilon_k = 0$*;*

(iv) *the operator $\widehat{A}$ is definitizable over the domain $\overline{\mathbb{C}}\setminus\{0\}$.*

Proof. **(i)** Let Σ and $\{\varepsilon_k\}_1^\infty$ be from Lemma 4.4. Then, by Lemma 4.3, Σ is a spectral function of the boundary value problem (4.1) with a certain potential $\widehat{q}_+$. Let us consider an even continuous potential $\widehat{q}(x)=\widehat{q}_+(|x|)$, $x\in\mathbb{R}$, and the corresponding operator $\widehat{A}=(\operatorname{sgn} x)\left(-\frac{d^2}{dx^2}+\widehat{q}(x)\right)$ defined as in Subsection 2.1.

It is well known that if equation (2.3) is in the limit-circle case at $+\infty$ then $M_+(\cdot)$ is a meromorphic function on $\mathbb{C}$ and the spectral function Σ_+ is a step function with jumps at the poles of $M_+(\cdot)$ only (see e.g. [10, Theorem 9.4.1]). As $\Sigma_+(t)=\Sigma(t)$, $t>0$, this condition does not hold for the function Σ since Σ satisfies (4.4). Indeed, (4.4) means that $\Sigma'(t)=\frac{1}{\pi\sqrt{t}}$ for $t>T_2=1$ and therefore Σ is not a step function. So (2.3) is limit-point at $+\infty$.

Since the potential $\widehat{q}$ is even, the same is true for $-\infty$. Thus, $\widehat{A}$ is a self-adjoint operator in the Krein space $L^2(\mathbb{R},[\cdot,\cdot])$, see Subsection 2.1.

(ii) and **(iii)** follow from Lemma 4.2 and statement (iii) of Lemma 4.4.

(iv) Let $\widehat{A}_0^\pm$ be the self-adjoint operators in the Hilbert spaces $L^2(\mathbb{R}_\pm)$ defined by the differential expression (4.13) in the same way as in Subsection 2.3 where q is replaced by $\widehat{q}$. By (4.2), $\sigma(\widehat{A}_0^+)=\{s_k\}_1^\infty\cup[1,+\infty)$. Since $\widehat{q}$ is even, one gets $\sigma(\widehat{A}_0^-)=\{-s_k\}_1^\infty\cup(-\infty,-1]$. It follows from $\{s_k\}_1^\infty\subset(-1,1)$ and $\lim_{k\to\infty}s_k=0$ that

$$\min\sigma_{ess}(\widehat{A}_0^+)=\max\sigma_{ess}(\widehat{A}_0^-)=0$$

and Theorem 3.13 concludes the proof. □

5. Some classes of Sturm-Liouville operators

As an illustration of the results from the previous sections, we discuss in this section various potentials $q\in L^1_{loc}(\mathbb{R})$ such that the differential operator $A=(\operatorname{sgn} x)(-d^2/dx^2+q)$ is definitizable over specific subsets of $\overline{\mathbb{C}}$. As before it is supposed that the differential expression (2.1) is in limit point case at $+\infty$ and at $-\infty$ (for instance, the letter holds if $\liminf_{|x|\to\infty}\frac{q(x)}{x^2}>-\infty$, see e.g., [47, Example 7.4.1]).

5.1. The case $q(x) \to -\infty$

In this subsection we assume that for some $X > 0$ the potential q has the following properties on the interval $(X, +\infty)$:

$$q',\ q'' \text{ exist and are continuous on } (X,+\infty), \qquad q(x) < 0, \qquad q'(x) < 0, \tag{5.1}$$

$$q''(x) \quad \text{is of fixed sign, i.e.,} \quad q''(x_1)q''(x_2) \geq 0 \quad \text{for all} \quad x_1, x_2 > X, \tag{5.2}$$

$$\lim_{x\to+\infty} q(x) = -\infty, \qquad \int_X^{+\infty} |q(x)|^{-1/2} dx = \infty, \quad \text{and} \quad \limsup_{x\to+\infty} \frac{|q'(x)|}{|q(x)|^p} < \infty, \tag{5.3}$$

where $p \in (0, 3/2)$ is a constant.

Then the well-known result of Titchmarsh (see e.g. [40, Theorems 3.4.1 and 3.4.2]) states that (2.1) is in the limit point case at $+\infty$ and $\sigma(A_0^+) = \mathbb{R}$. Hence the set $\mathcal{S}_A$ defined by (3.9) coincides with $\sigma_{ess}(A_0^-) \cup \infty$. By Theorem 3.11, there are two cases:

(i) Let $\sigma_{ess}(A_0^-) \neq \mathbb{R}$. Then the greatest domain over which A is definitizable is $\Omega_A := \mathbb{C} \setminus \sigma_{ess}(A_0^-)$ (note that $\infty \notin \Omega_A$).

(ii) Let $\sigma_{ess}(A_0^-) = \mathbb{R}$. Then $\Omega_A \cap \overline{\mathbb{R}} = \emptyset$ and there exists no domain Ω in $\overline{\mathbb{C}}$ such that A is definitizable over Ω. In particular, the letter holds if the analogues of assumptions (5.1)-(5.3) are fulfilled for $x \in (-\infty, 0]$.

Example 5.1. *Let us consider the operator* $A = (\operatorname{sgn} x)(-d^2/dx^2 - x)$. *By [45, Theorem 6.6] the differential expression* $-d^2/dx^2 - x$ *is in limit point case at* $+\infty$ *and* $-\infty$. *Assumptions* (5.1)-(5.3) *hold for* $x \in (0, +\infty)$, *hence* $\sigma_{ess}(A_0^+) = \sigma(A) = \mathbb{R}$. *On the other hand,* $\sigma_{ess}(A_0^-) = \emptyset$ *(see Subsection 5.2 and [40, Section 3.1]). Therefore the operator* A *is definitizable over* $\mathbb{C}$ *and there exists no domain* Ω *in* $\overline{\mathbb{C}}$ *with* $\infty \in \Omega$ *such that* A *is definitizable over* Ω. *By Proposition 4.1, the only possible accumulation point for non-real spectrum of* A *is the point* ∞.

5.2. The case $q(x) \to +\infty$

Let us assume that the following conditions holds with certain constants $X, c > 0$:

$$q(x) \geq c \quad \text{for} \quad x > X, \quad \text{and for any } \omega > 0, \quad \lim_{x\to+\infty} \int_x^{x+\omega} q(t)dt = +\infty. \tag{5.4}$$

Molčanov proved (see e.g., [40, Lemma 3.1.2] and [41, Subsection 24.5]) that (5.4) yields $\sigma_{ess}(A_0^+) = \emptyset$, i.e., the spectrum of the operator A_0^+ is discrete. Besides, (5.4) implies that A_0^+ is semi-bounded from below. It follows from the results of Subsection 3.3 that *the operator* A *is definitizable over* $\mathbb{C}$. More precisely,

(i) Let the operator A_0^- be semi-bounded from above. Then the operator A is definitizable, ∞ is a regular critical point of A (cf. [12]), and A admits decomposition (3.10).

(ii) Let A_0^- be not semi-bounded from above. Then A is definitizable over $\mathbb{C}$ and there exists no domain Ω in $\overline{\mathbb{C}}$ with $\infty \in \Omega$ such that A is definitizable over Ω. The only possible accumulation point for non-real spectrum of A is the point ∞.

Note that A_0^- is not semi-bounded from above if $\lim_{x\to-\infty} q(x) = -\infty$.

5.3. Summable potentials

We denote by $q_{neg}(x) := \min\{q(x), 0\}$, $x \in \mathbb{R}$.

Assumption 5.2. $\displaystyle\int_t^{t+1} |q_{neg}(x)|dx \to 0$ *as* $|t| \to \infty$.

If Assumption 5.2 is fulfilled then the differential expression $-d^2/dx^2 + q$ is in limit point case at $+\infty$ and $-\infty$, cf. [46, Satz 14.21]. By [45, Theorem 15.1], A_0^+ is semi-bounded from below, A_0^- is semi-bounded from above with

$$\sigma_{ess}(A_0^+) \subset [0, +\infty) \quad \text{and} \quad \sigma_{ess}(A_0^-) \subset (-\infty, 0].$$

This implies that the negative spectrums of the operators A_0^+ and $-A_0^-$ consist of eigenvalues,

$$\sigma(\pm A_0^\pm) \cap (-\infty, 0) = \{\pm\lambda_n^\pm\}_1^{N^\pm} \subset \sigma_p(\pm A_0^\pm),$$

where $0 \le N^\pm \le \infty$. Besides, $\lim_{n\to\infty} \lambda_n^\pm = 0$ if $N^\pm = \infty$. Then, by Theorem 3.13, A is definitizable over $\overline{\mathbb{C}} \setminus \{0\}$. Theorems 3.11 and 3.15 imply easily the following statement.

Theorem 5.3. *Let Assumption 5.2 be fulfilled. Then the operator* $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ *admits the decomposition* (3.10). *Moreover,*

(i) *If* $\min \sigma_{ess}(A_0^+) > 0$ *or* $\max \sigma_{ess}(A_0^-) < 0$, *then* A *is a definitizable operator and* ∞ *is a critical point of* A.

(ii) *If* $\min \sigma_{ess}(A_0^+) = \max \sigma_{ess}(A_0^-) = 0$ *and* $N^+ + N^- < \infty$, *then* A *is a definitizable operator,* 0 *and* ∞ *are critical points of* A .

(iii) *If* $\min \sigma_{ess}(A_0^+) = \max \sigma_{ess}(A_0^-) = 0$ *and* $N^+ + N^- = \infty$, *then the operator* A *is not definitizable. It is definitizable over* $\overline{\mathbb{C}} \setminus \{0\}$. *In particular,* 0 *is the only possible accumulation point of the non-real spectrum of* A.

We mention (cf. [5]) that Assumption 5.2, and therefore the statements of Theorem 5.3, hold true if $q \in L^1(\mathbb{R})$.

Remark 5.4. *By Theorem 3.15 (see also [12]) we have that if the operator* $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ *is definitizable, then* ∞ *is its regular critical point. In the case when* A *has a finite critical point, the question of the character of this critical point is difficult (see [13, 14, 18, 19, 33, 34, 32] and references therein). Let us mention one case. Assume that* q *is continuous in* $\mathbb{R}$ *and* $\int_{\mathbb{R}}(1 + x^2)|q(x)|dx < \infty$, *then* $\min \sigma_{ess}(A_0^+) = \max \sigma_{ess}(A_0^-) = 0$ *and* $N^+ < \infty$ *and* $N^- < \infty$ *(see [40]). Therefore Theorem 5.3 (as well as [12, Proposition 1.1]) implies that* $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ *is definitizable. It was shown (implicitly) in [18] that* 0 *is a regular critical point of* A.

In the following case, more detailed information may be obtained.

Corollary 5.5. *Suppose* $\lim_{x\to\infty} q(|x|) = 0$. *Then* $\min \sigma_{ess}(A_0^+) = \max \sigma_{ess}(A_0^-) = 0$ *and either the case (ii) or the case (iii) of Theorem 5.3 takes place. Moreover, the following holds.*

(i) *If* $\liminf_{x\to\infty} x^2 q(|x|) > -1/4$*, then* A *is a definitizable operator and* 0 *and* ∞ *are critical points of* A.

(ii) *If* $\limsup_{x\to\infty} x^2 q(|x|) < -1/4$*, then the operator* A *is not definitizable. It is definitizable over* $\overline{\mathbb{C}} \setminus \{0\}$.

Proof. The statement follows directly from [16, Corollary XIII.7.57], which was proved in [16] for infinitely differentiable q. Actually, this proof is valid for bounded potentials q. Finally, note that $\lim_{x\to\infty} q(|x|) = 0$ implies that q is bounded on $(-\infty, -X] \cup [X, +\infty)$ with X large enough. On the other hand, L^1 perturbations of potential q on any finite interval does not change $\sigma_{ess}(A_0^+)$, $\sigma_{ess}(A_0^-)$. Also such perturbations increase or decrease N^+, N^- on finite numbers only due to Sturm Comparison Theorem (see e.g., [47, Theorem 2.6.3]). This completes the proof. □

Example 5.6. *Let* $q(x) = -\frac{1}{1+|x|}$*. Then Corollary 5.5 yields that the operator* $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ *is not definitizable. It is definitizable over* $\overline{\mathbb{C}} \setminus \{0\}$.

It was shown above that under certain assumption on the potential q the operator $A = (\operatorname{sgn} x)(-d^2/dx^2 + q)$ is not definitizable, but it is definitizable over the domain $\overline{\mathbb{C}} \setminus \{\lambda_0\}$, where $\lambda_0 \in \overline{\mathbb{R}}$ ($\lambda_0 = \infty$ in Example 5.1 and $\lambda_0 = 0$ in Example 5.6). In this case, unusual spectral behavior may appear near points of the set $c(A) \cup \{\lambda_0\}$ only ($c(A)$ is the set of critical points, see Subsection 3.1). Indeed, a bounded spectral projection $E^A(\Delta)$ exists for any connected set $\Delta \subset \overline{\mathbb{R}} \setminus \{\lambda_0\}$ such that the endpoints of Δ do not belong to $c(A) \cup \{\lambda_0\}$. Note also that $c(A)$ is at most countable and that λ_0 is the only possible accumulation point of the non-real spectrum of A.

Acknowledgement

The first author acknowledges the hospitality and support of the Technische Universität Berlin and of the University of Zurich.

References

[1] N.I. Akhiezer, I.M. Glazman, *Theory of linear operators in Hilbert space*, Dover, New York, 1993.

[2] T.Ya. Azizov, I.S. Iokhvidov, *Linear operators in spaces with an indefinite metric*, John Wiley and Sons, Chichester, New York, 1989.

[3] T.Ya. Azizov, P. Jonas, C. Trunk, *Spectral points of type* π_+ *and* π_- *of self-adjoint operators in Krein spaces*, J. Funct. Anal. **226** (2005) 114-137.

[4] R. Beals, *Indefinite Sturm-Liouville problems and Half-range completeness*, J. Differential Equations **56** (1985) 391-407.

[5] J. Behrndt, *On the spectral theory of singular indefinite Sturm-Liouville operators*, J. Math. Anal. Appl. **334** (2007) 1439-1449.

[6] J. Behrndt, *Finite rank perturbations of locally definitizable self-adjoint operators in Krein spaces*, to appear in J. Operator Theory.

[7] J. Behrndt, F. Philipp, C. Trunk, *Properties of the spectrum of type* π_+ *and type* π_- *of self-adjoint operators in Krein spaces*, Methods of Functional Analysis and Topology **12** (2006) 326-340.

[8] J. Behrndt, C. Trunk, *On the negative squares of indefinite Sturm-Liouville operators*, J. Differential Equations **238** (2007) 491-519.

[9] J. Bognar, *Indefinite inner product spaces*, Springer, Berlin, 1974.

[10] E.A. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[11] B. Ćurgus, *On the regularity of the critical point infinity of definitizable operators*, J. Integral Equations Operator Theory **8** (1985) 462-488.

[12] B. Ćurgus, H. Langer, *A Krein space approach to symmetric ordinary differential operators with an indefinite weight function*, J. Differential Equations **79** (1989) 31-61.

[13] B. Ćurgus, B. Najman, *Positive differential operators in Krein space* $L^2(\mathbb{R})$, Recent development in operator theory and its applications (Winnipeg, MB, 1994), Oper. Theory Adv. Appl., Birkhäuser, Basel, **87** (1996) 95–104.

[14] B. Ćurgus, B. Najman, *Positive differential operators in the Krein space* $L^2(\mathbb{R}^n)$, Oper. Theory Adv. Appl., Birkhäuser, Basel **106** (1998) 113-130.

[15] K. Daho, H. Langer, *Sturm-Liouville operators with an indefinite weight function*, Proc. Roy. Soc. Edinburgh Sect. A **87** (1977) 161-191.

[16] N. Dunford, J.T. Schwartz, *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, John Wiley & Sons, New York 1963.

[17] W.N. Everitt, *On a property of the m-coefficient of a second-order linear differential equation*, J. London Math. Soc. **4** (1971/72) no. 2, 443-457.

[18] M.M. Faddeev , R.G. Shterenberg, *On similarity of singular differential operators to a selfadjoint one*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **270**, Issled. po Linein. Oper. i Teor. Funkts. **28** (2000) 336–349 [russian]; English transl.: J. Math. Sciences **115** (2003), no. 2, 2279–2286.

[19] M.M. Faddeev, R.G. Shterenberg, *On similarity of differential operators to a selfadjoint one*, Math. Notes **72** (2002) 292-303.

[20] A. Fleige, *Spectral theory of indefinite Krein-Feller differential operators,* Mathematical Research **98** Akademie Verlag, Berlin 1996.

[21] W. Greenberg, C.V.M. van der Mee, V. Protopopescu, *Boundary value problems in abstract kinetic theory*, Oper. Theory Adv. Appl. **23** Birkhäuser, Basel 1987.

[22] P. Jonas: *On a class of selfadjoint operators in Krein space and their compact perturbations*, Integral Equations Operator Theory **11** (1988) 351-384.

[23] P. Jonas, *On a problem of the perturbation theory of selfadjoint operators in Krein spaces*, J. Operator Theory **25** (1991) 183-211.

[24] P. Jonas, *On locally definite operators in Krein spaces*, in: Spectral Theory and its Applications, Ion Colojoară Anniversary Volume, edited by A. Gheondea and M. Sabac (Theta, Bucharest, 2003) 95-127.

[25] P. Jonas, H. Langer, *Compact perturbations of definitizable operators*, J. Operator Theory **2** (1979) 63-77.

[26] P. Jonas, C. Trunk, *On a class of analytic operator functions and their linearizations*, Math. Nachrichten **243** (2002) 92-133.

[27] I.S. Kac, M.G. Krein, *R-functions - analytic functions mapping the upper halfplane into itself*, Supplement to the Russian edition of F.V. Atkinson, Discrete and continuous boundary problems, Mir, Moscow, 1968 (Russian); Engl. transl.: Amer. Math. Soc. Transl. Ser. 2 **103** (1974)

[28] H.G. Kaper, C.G. Lekkerkerker, J. Hejtmanek, *Spectral methods in linear transport theory*, Oper. Theory Adv. Appl. **5** Birkhäuser, Basel 1982. 1-18.

[29] I.M. Karabash, *J-selfadjoint ordinary differential operators similar to selfadjoint operators*, Methods of Functional Analysis and Topology **6** (2000) no. 2, 22-49.

[30] I.M. Karabash, *On similarity of differential operators to selfadjoint ones*, Candidate thesis, The Institute of Applied Mathematics and Mechanics NASU, Donetsk, 2005 [Russian].

[31] I.M. Karabash, *On eigenvalues in the essential spectrum of Sturm-Liouville operators with the indefinite weight sgn x*, Spectral and Evolution problems, Proc. of the Fifteenth Crimean Autumn Math. School-Symposium, Simferopol 15 (2005) 55-60.

[32] I.M. Karabash, A.S. Kostenko, *Indefinite Sturm-Liouville operators with the singular critical point zero*, arXiv:math.SP/0612173, http://arxiv.org/abs/math.SP/0612173 (Submitted to Proc. Roy. Soc. Edinburgh Sect. A)

[33] I.M. Karabash, M.M. Malamud, *The similarity of a J-self-adjoint Sturm-Liouville operator with finite-gap potential to a self-adjoint operator*, Doklady Mathematics **69** (2004), no. 2, 195-199.

[34] I.M. Karabash, M.M. Malamud *Indefinite Sturm-Liouville operators* $(\operatorname{sgn} x)(-\frac{d^2}{dx^2}+q)$ *with finite-zone potentials*, arXiv:math.SP/0610087, http://arxiv.org/abs/math.SP/0610087 (to appear in Operators and Matrices).

[35] Q. Kong, H. Wu, A. Zettl, M. Möller, *Indefinite Sturm-Liouville problems*, Proc. R. Soc. Edinb. Sect. A **133** (2003), no. 3, 639-652.

[36] A.S. Kostenko, *Similarity of some nonnegative operators to self-adjoint operators*, Math. Notes **80** (2006) 131-135.

[37] P. Lancaster, A. Markus, and V. Matsaev, *Definitizable operators and quasihyperbolic operator polynomials*, J. Funct. Anal. **131** (1995) 1-28.

[38] H. Langer: *Spectral Functions of Definitizable Operators in Krein Spaces*, in: Functional Analysis Proceedings of a Conference held at Dubrovnik, Yugoslavia, November 2-14, 1981, Lecture Notes in Mathematics **948** Springer Verlag Berlin-Heidelberg-New York (1982) 1-46.

[39] H. Langer, A. Markus and V. Matsaev, *Locally definite operators in indefinite inner product spaces*, Math. Ann. **308** (1997) 405-424.

[40] B.M. Levitan, I.S. Sargsjan, *Sturm-Liouville and Dirac operators*, Nauka, Moscow, 1988 [Russian]; Engl. transl.: Kluwer, Dordrecht 1991.

[41] M.A. Naimark, *Linear differential operators II*, F. Ungar Publ., New York, 1968.

[42] A.I. Parfyonov *On an embedding criterion for interpolation spaces and its applications to indefinite spectral problems*, Sib. Mat. Zurnal. **44** (2003), no.4, 810-819 (Russian).

[43] S.G. Pyatkov, *Operator Theory. Nonclassical Problems*, Utrecht, Boston, Köln, Tokyo, VSP 2002.

[44] H. Volkmer, *Sturm-Liouville problems with indefinite weights and Everitt's inequality*, Proc. Roy. Soc. Edinburg Sect. A **126** (1996) 1097-1112.

[45] J. Weidmann, *Spectral theory of ordinary differential operators*, Lecture Notes in Mathematics **1258**, Springer, 1987.

[46] J. Weidmann, *Lineare Operatoren in Hilberträumen Teil II*, Teubner, 2003 (German).

[47] A. Zettl, *Sturm-Liouville Theory*, AMS, 2005.

Illya Karabash
Department of Partial Differential Equations
Institute of Applied Mathematics and Mechanics of NAS of Ukraine
R. Luxemburg str. 74
Donetsk 83114
Ukraine
karabashi@yahoo.com, karabashi@mail.ru

Carsten Trunk
Institut für Mathematik
Technische Universität Berlin
Sekretariat MA 6-3
Straße des 17. Juni 136
D-10623 Berlin
Germany
trunk@math.tu-berlin.de