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COMPACT AND FINITE RANK PERTURBATIONS OF LINEAR RELATIONS IN HILBERT SPACES

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Abstract. For closed linear operators or relations A and B acting between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ the concepts of compact and finite rank perturbations are introduced with the help of the orthogonal projections P_A and P_B in $\mathcal{H}\oplus\mathcal{K}$ onto A and B. Various equivalent characterizations for such perturbations are proved and it is shown that these notions are a natural generalization of the usual concepts of compact and finite rank perturbations.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and assume first that A and B are bounded linear operators defined on $\mathcal{H}$ with values in $\mathcal{K}$. Then A is said to be a compact (finite rank) perturbation of B if the operator $A-B$ is compact (resp. finite dimensional). If A and B are unbounded closed operators these notions in general make no sense since the domains $\mathrm{dom}\,A$ and $\mathrm{dom}\,B$ may not coincide and hence $A-B$ can only be defined on the (possibly trivial) subspace $\mathrm{dom}\,A \cap \mathrm{dom}\,B$ of $\mathcal{H}$. However, if in the special case $\mathcal{H}=\mathcal{K}$ the operators A and B have a common point in their resolvent sets, then a natural generalization of the above notions of compact and finite rank perturbations is done via the resolvent difference of A and B. Namely, A is said to be a compact (finite rank) perturbation of B if

$$(A-\lambda)^{-1}-(B-\lambda)^{-1}, \qquad \lambda\in\rho(A)\cap\rho(B), \tag{1.1}$$

is a compact (resp. finite rank) operator. Such types of compact and finite rank perturbations play an important role in pure and applied linear functional analysis and have been studied extensively for a long time, see, e.g., [6].

The main objective of this paper is to introduce the notions of compact and finite rank perturbations of closed linear operators and, more generally, closed linear relations A and B acting between $\mathcal{H}$ and $\mathcal{K}$, and to give various equivalent characterizations. The key idea here is to use the orthogonal projections P_A and P_B in $\mathcal{H}\oplus\mathcal{K}$ onto the closed subspaces A and B of $\mathcal{H}\oplus\mathcal{K}$. We shall say that A is a compact (finite rank) perturbation of B if P_A-P_B is a compact (resp. finite dimensional) operator. It is shown in Theorem 3.1 that A is a finite rank perturbation of B if and only if A and B are both finite dimensional extensions of their common part $A\cap B$. Furthermore, it is verified in Theorem 4.2 that the linear relation A is a compact perturbation of the linear relation B if and only if for every $\varepsilon>0$ there exists a closed linear relation F from $\mathcal{H}$ in $\mathcal{K}$ such that P_B-P_F is a finite rank

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operator and $\|P_A - P_F\| < \varepsilon$. This characterization of compact perturbations is very convenient and useful and will be applied in a subsequent note.

The paper is organized as follows. In Section 2 we first recall some basic definitions (cf. [1, 3]) and decompositions of linear relations in Hilbert spaces. The orthogonal projection P_A in $\mathcal{H} \oplus \mathcal{K}$ onto a closed linear relation A from $\mathcal{H}$ in $\mathcal{K}$ is expressed in terms of the operator part $A_{\rm op}$ of A in Proposition 2.1. Moreover, we rewrite this representation of P_A in different forms to obtain some generalizations of the results in [10] and [4, 7, 9]. Sections 3 and 4 are devoted to the concepts of finite rank and compact perturbations of closed linear relations. Here we introduce the corresponding notions and prove various equivalent formulations. Moreover, we show that theses notions are natural generalizations of the usual ones for bounded and, resp., unbounded operators.

2. The orthogonal projection onto a closed linear relation

Let throughout this paper $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. We study linear relations from $\mathcal{H}$ in $\mathcal{K}$, that is, linear subspaces of $\mathcal{H} \times \mathcal{K}$. The set of all closed linear relations from $\mathcal{H}$ in $\mathcal{K}$ will be denoted by $\widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$. If $\mathcal{K} = \mathcal{H}$ we write $\widetilde{\mathcal{C}}(\mathcal{H})$. For a linear relation A we write dom A, ran A, ker A and mul A for the domain, range, kernel and multivalued part of A, respectively. The elements in a linear relation A will usually be written as column vectors $\binom{x}{x'}$, where $x \in \operatorname{dom} A$ and $x' \in \operatorname{ran} A$. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [1, 3, 5]. The set of all densely defined closed linear operators from $\mathcal{H}$ to $\mathcal{K}$ will be denoted by $\mathcal{C}(\mathcal{H},\mathcal{K})$, we write $\mathcal{C}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$. For the set of everywhere defined bounded linear operators from $\mathcal{H}$ in $\mathcal{K}$ we write $\mathcal{L}(\mathcal{H},\mathcal{K})$ and $\mathcal{L}(\mathcal{H})$ if $\mathcal{H}$ and $\mathcal{K}$ coincide. Linear operators are identified as linear relations via their graphs, so that the inclusions $\mathcal{L}(\mathcal{H},\mathcal{K}) \subset \mathcal{C}(\mathcal{H},\mathcal{K}) \subset \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ hold.

Let A be a linear relation from $\mathcal{H}$ in $\mathcal{K}$. Then the *adjoint relation* $A^* \in \widetilde{\mathcal{C}}(\mathcal{K},\mathcal{H})$ is defined by

$$A^* = \left\{ \begin{pmatrix} y \\ y' \end{pmatrix} : (x', y) = (x, y') \text{ for all } \begin{pmatrix} x \\ x' \end{pmatrix} \in A \right\}.$$

Note that this definition extends the usual definition of the adjoint of a densely defined operator. Let $A \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$. As mul $A = (\operatorname{dom} A^*)^\perp$ and mul $A^* = (\operatorname{dom} A)^\perp$ it is clear that A (A^*) is a densely defined closed operator if and only if dom A^* (dom A, respectively) is dense. Observe that the orthogonal complement of A in $\mathcal{H} \oplus \mathcal{K}$ is the relation $(-A^*)^{-1}$, that is, $\mathcal{H} \oplus \mathcal{K} = A \oplus (-A^*)^{-1}$. If A is a linear relation in $\mathcal{H}$, then A is said to be *symmetric* (*selfadjoint*) if $A \subset A^*$ ($A = A^*$, respectively).

Let $A \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$. In the following the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ will be decomposed in the form

$$\mathcal{H} = \operatorname{mul} A^* \oplus \mathcal{H}_1, \quad \text{where } \mathcal{H}_1 := (\operatorname{mul} A^*)^\perp = \overline{\operatorname{dom} A}, \tag{2.1}$$

and

$$\mathcal{K} = \mathcal{K}_1 \oplus \operatorname{mul} A, \quad \text{where } \mathcal{K}_1 := (\operatorname{mul} A)^\perp = \overline{\operatorname{dom} A^*}, \tag{2.2}$$

respectively. The *operator part* $A_{\rm op}$ of A is defined by

$$A_{\rm op} := A \cap \big(\mathcal{H}_1 \times \mathcal{K}_1\big).$$

It is easy to see that in fact $\operatorname{mul} A_{\rm op} = \{0\}$ holds, and hence it follows that $A_{\rm op}$ is a densely defined closed operator from $\mathcal{H}_1$ in $\mathcal{K}_1$, that is, $A_{\rm op} \in \mathcal{C}(\mathcal{H}_1, \mathcal{K}_1)$. Furthermore, $\operatorname{dom} A_{\rm op} = \operatorname{dom} A$ and

$$A = \left\{ \begin{pmatrix} x \\ A_{\rm op}x + z \end{pmatrix} : x \in \operatorname{dom} A_{\rm op},\ z \in \operatorname{mul} A \right\}. \tag{2.3}$$

Analogously the operator part $(A^*)_{\rm op}$ of the relation $A^* \in \widetilde{\mathcal{C}}(\mathcal{K}, \mathcal{H})$ is defined as

$$(A^*)_{\rm op} := A^* \cap \big(\mathcal{K}_1 \times \mathcal{H}_1\big) \in \mathcal{C}(\mathcal{K}_1, \mathcal{H}_1)$$

and it is straightforward to check that the adjoint $(A_{\rm op})^* \in \mathcal{C}(\mathcal{K}_1, \mathcal{H}_1)$ of the operator part $A_{\rm op}$ of A coincides with the operator part $(A^*)_{\rm op}$ of the adjoint relation A^*, that is, $(A_{\rm op})^* = (A^*)_{\rm op}$. In the sequel we simply write $A^*_{\rm op}$.

The next proposition will be useful for the considerations in Section 3 and Section 4.

Proposition 2.1. *Let $A \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ be a closed linear relation from $\mathcal{H}$ in $\mathcal{K}$ and let $\mathcal{H}_1$, $\mathcal{K}_1$, and the operators $A_{\rm op}$ and $A^*_{\rm op}$ be defined as above. Then the operator*

$$P_A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (I + A^*_{\rm op}A_{\rm op})^{-1} & A^*_{\rm op}(I + A_{\rm op}A^*_{\rm op})^{-1} & 0 \\ 0 & A_{\rm op}(I + A^*_{\rm op}A_{\rm op})^{-1} & A_{\rm op}A^*_{\rm op}(I + A_{\rm op}A^*_{\rm op})^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{2.4}$$

is the orthogonal projection in $\mathcal{H} \oplus \mathcal{K}$ onto the linear relation A with respect to the decomposition $\operatorname{mul} A^* \oplus \mathcal{H}_1 \oplus \mathcal{K}_1 \oplus \operatorname{mul} A$ *of $\mathcal{H} \oplus \mathcal{K}$.*

In the operator case Proposition 2.1 reduces to the following well-known statement, see, e.g., [2, 8, 10].

Corollary 2.2. *Let $A \in \mathcal{C}(\mathcal{H}, \mathcal{K})$ be a closed densely defined linear operator from $\mathcal{H}$ in $\mathcal{K}$. Then the orthogonal projection P_A in $\mathcal{H} \oplus \mathcal{K}$ onto A is given by*

$$P_A = \begin{pmatrix} (I + A^*A)^{-1} & A^*(I + AA^*)^{-1} \\ A(I + A^*A)^{-1} & AA^*(I + AA^*)^{-1} \end{pmatrix}.$$

Proof of Proposition 2.1. Recall that $A^*_{\rm op}A_{\rm op}$ and $A_{\rm op}A^*_{\rm op}$ are nonnegative selfadjoint operators in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{K}_1$, respectively, cf. [6, § V. Theorem 3.24]. In particular, the entries in the matrix representation of P_A are everywhere defined bounded operators.

For $x \in \operatorname{dom} A_{\rm op}$ we have

$$A_{\rm op}x = A_{\rm op}(I + A^*_{\rm op}A_{\rm op})(I + A^*_{\rm op}A_{\rm op})^{-1}x = (I + A_{\rm op}A^*_{\rm op})A_{\rm op}(I + A^*_{\rm op}A_{\rm op})^{-1}x$$

and hence $(I + A_{\rm op}A^*_{\rm op})^{-1}A_{\rm op}x = A_{\rm op}(I + A^*_{\rm op}A_{\rm op})^{-1}x$. As $\overline{\operatorname{dom} A_{\rm op}} = \mathcal{H}_1$ and $(I + A^*_{\rm op}A_{\rm op})^{-1} \in \mathcal{L}(\mathcal{H}_1)$ we conclude

$$\overline{(I + A_{\rm op}A^*_{\rm op})^{-1}A_{\rm op}} = A_{\rm op}(I + A^*_{\rm op}A_{\rm op})^{-1}. \tag{2.5}$$

Making use of (2.5) it follows without difficulties that $P_A^2 = P_A$ holds. Furthermore, from (2.5) we conclude

$$\big(A^*_{\rm op}(I + A_{\rm op}A^*_{\rm op})^{-1}\big)^* = \overline{(I + A_{\rm op}A^*_{\rm op})^{-1}A_{\rm op}} = A_{\rm op}(I + A^*_{\rm op}A_{\rm op})^{-1}. \tag{2.6}$$

Now relation (2.6) together with the selfadjointness of $A^*_{\rm op}A_{\rm op}$ and $A_{\rm op}A^*_{\rm op}$ imply $P_A = P_A^*$. Therefore P_A is an orthogonal projection in $\mathcal{H} \oplus \mathcal{K}$.

It remains to show ran $P_A = A$. According to [6, §V. Theorem 3.24] the (graph of the) restriction $A_{\rm op} \upharpoonright \mathrm{dom}\, A_{\rm op}^* A_{\rm op}$ is dense in $A_{\rm op}$ and hence it follows that the range of the orthogonal projection

$$P_{A_{\rm op}} := \begin{pmatrix} (I + A_{\rm op}^* A_{\rm op})^{-1} & A_{\rm op}^*(I + A_{\rm op}A_{\rm op}^*)^{-1} \\ A_{\rm op}(I + A_{\rm op}^* A_{\rm op})^{-1} & A_{\rm op}A_{\rm op}^*(I + A_{\rm op}A_{\rm op}^*)^{-1} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{K}_1 \end{pmatrix} \to \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{K}_1 \end{pmatrix}$$

is a dense subspace of $A_{\rm op}$. On the other hand ran $P_{A_{\rm op}}$ is closed, therefore ran $P_{A_{\rm op}} = A_{\rm op}$ and with (2.3) we have ran $P_A = A$. Proposition 2.1 is proved. □

For the special case $\mathcal{H} = \mathcal{K}$ the following matrix representation of P_A can be found in [4], where it is called the Stone-de Snoo formula.

Proposition 2.3. *Let $A \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ be a closed linear relation. Then $A^*A \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $AA^* \in \widetilde{\mathcal{C}}(\mathcal{K})$ are nonnegative selfadjoint relations and the orthogonal projection P_A in Proposition 2.1 has the form*

$$P_A = \begin{pmatrix} (I + A^*A)^{-1} & \iota_{\mathcal{H}_1}[A^*(I + AA^*)^{-1}]_{\rm op} \\ \iota_{\mathcal{K}_1}[A(I + A^*A)^{-1}]_{\rm op} & I - (I + AA^*)^{-1} \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \to \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix}, \tag{2.7}$$

where $\iota_{\mathcal{H}_1}$ and $\iota_{\mathcal{K}_1}$ denote the canonical embeddings of $\mathcal{H}_1$ in $\mathcal{H}$ and $\mathcal{K}_1$ in $\mathcal{K}$, respectively.

Proof. Let us first verify that ran $(I + A^*A) = \mathcal{H}$ holds. Since the orthogonal complement $A^\perp$ of A in $\mathcal{H} \oplus \mathcal{K}$ coincides with $(-A^*)^{-1}$ every element in $\mathcal{H} \oplus \mathcal{K}$ can be written as the sum of an element in A and an element in $(-A^*)^{-1}$. In particular, for arbitrary $s \in \mathcal{H}$ and $\binom{s}{0} \in \mathcal{H} \oplus \mathcal{K}$ we find

$$\begin{pmatrix} s \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} z \\ -y \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} x \\ y \end{pmatrix} \in A \quad \text{and} \quad \begin{pmatrix} z \\ -y \end{pmatrix} \in (-A^*)^{-1}.$$

As $\binom{y}{z} \in A^*$ we have $\binom{x}{z} \in A^*A$ and $\binom{x}{s} = \binom{x}{x+z} \in I + A^*A$, therefore $s \in$ ran $(I + A^*A)$.

Next we show that A^*A (and hence also $I + A^*A$) is a symmetric relation in $\mathcal{H}$, that is, $A^*A \subset (A^*A)^*$. For this fix $\binom{u}{w} \in A^*A$. We have to verify

$$(z, u) = (x, w) \quad \text{for all} \quad \begin{pmatrix} x \\ z \end{pmatrix} \in A^*A. \tag{2.8}$$

In fact, for $\binom{u}{w}, \binom{x}{z} \in A^*A$ there exist $v, y \in \mathcal{K}$ such that $\binom{u}{v}, \binom{x}{y} \in A$ and $\binom{v}{w}, \binom{y}{z} \in A^*$. Hence $(z, u) = (y, v)$ and $(w, x) = (v, y)$, and this implies (2.8).

As $I + A^*A$ is symmetric and surjective it follows that $I + A^*A$ (and hence also A^*A) is selfadjoint. Indeed, for $\binom{s}{t} \in (I + A^*A)^*$ there exists $s' \in \mathrm{dom}\,(I + A^*A)$ such that $\binom{s'}{t} \in I + A^*A \subset (I + A^*A)^*$. This implies $\binom{s-s'}{0} \in (I + A^*A)^*$ and hence

$$s - s' \in \ker\big((I + A^*A)^*\big) = \big(\mathrm{ran}\,(I + A^*A)\big)^\perp = \{0\},$$

that is $s = s'$ and $\binom{s}{t} \in (I + A^*A)$. Therefore $I + A^*A$ and A^*A are selfadjoint relations in $\mathcal{H}$. Furthermore A^*A is nonnegative, as for any $\binom{x}{z} \in A^*A$ we have $(z, x) = \|y\|^2$ for some $y \in \mathcal{K}$ with $\binom{x}{y} \in A$ and $\binom{y}{z} \in A^*$.

It is easy to see that mul $(I + A^*A) =$ mul $A^*A =$ mul A^* holds. Let $\mathcal{H}_1$ and $\mathcal{K}_1$ be as in (2.1) and (2.2), respectively, and let $A_{\rm op} \in \mathcal{C}(\mathcal{H}_1, \mathcal{K}_1)$ and $A_{\rm op}^* \in \mathcal{C}(\mathcal{K}_1, \mathcal{H}_1)$ be the operator parts of A and A^*, respectively. It follows that the operator part

$(A^*A)_{\rm op} = A^*A \cap (\mathcal{H}_1 \times \mathcal{H}_1)$ coincides with the selfadjoint operator $A^*_{\rm op}A_{\rm op} \in \mathcal{C}(\mathcal{H}_1)$. Therefore

$$I + A^*A = \left\{ \begin{pmatrix} x \\ x + A^*_{\rm op}A_{\rm op}x + y' \end{pmatrix} : x \in \mathrm{dom}\, A,\ y' \in \mathrm{mul}\, A^* \right\}.$$

Its inverse $(I + A^*A)^{-1}$ is a bounded everywhere defined operator in $\mathcal{H}$ with

$$\ker(I + A^*A)^{-1} = \mathrm{mul}\,(I + A^*A) = \mathrm{mul}\, A^*$$

and it follows that $(I + A^*A)^{-1}$ has the matrix representation

$$(I + A^*A)^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & (I + A^*_{\rm op}A_{\rm op})^{-1} \end{pmatrix} : \begin{pmatrix} \mathrm{mul}\, A^* \\ \mathcal{H}_1 \end{pmatrix} \to \begin{pmatrix} \mathrm{mul}\, A^* \\ \mathcal{H}_1 \end{pmatrix}. \tag{2.9}$$

Similar arguments show that $I + AA^*$ and AA^* are nonnegative selfadjoint relations in $\mathcal{K}$ with $\mathrm{mul}\, AA^* = \mathrm{mul}\, A$, $\ker(I+AA^*)^{-1} = \mathrm{mul}\, A$ and $(AA^*)_{\rm op} = A_{\rm op}A^*_{\rm op}$, so that

$$(I + AA^*)^{-1} = \begin{pmatrix} (I + A_{\rm op}A^*_{\rm op})^{-1} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathrm{mul}\, A \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \mathrm{mul}\, A \end{pmatrix}. \tag{2.10}$$

By comparing (2.9) and (2.10) with the representation of P_A in Proposition 2.1 it follows that the diagonal entries of P_A with respect to the decomposition $\mathcal{H} \oplus \mathcal{K}$ are as in (2.7).

Observe that the operator parts of $A^*(I+AA^*)^{-1}$ and $A(I+A^*A)^{-1}$ are everywhere defined bounded operators given by

$$\left[A^*(I + AA^*)^{-1}\right]_{\rm op} = A^*(I + AA^*)^{-1} \cap \left(\mathcal{K} \times \mathcal{H}_1\right) \in \mathcal{L}(\mathcal{K}, \mathcal{H}_1)$$

and

$$\left[A(I + A^*A)^{-1}\right]_{\rm op} = A(I + A^*A)^{-1} \cap \left(\mathcal{H} \times \mathcal{K}_1\right) \in \mathcal{L}(\mathcal{H}, \mathcal{K}_1),$$

respectively. Together with the canonical embedding of $\mathcal{H}_1$ in $\mathcal{H}$ and $\mathcal{K}_1$ in $\mathcal{K}$ it follows that the offdiagonal entries

$$\begin{pmatrix} 0 & 0 \\ A^*_{\rm op}(I + A_{\rm op}A^*_{\rm op})^{-1} & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathrm{mul}\, A \end{pmatrix} \to \begin{pmatrix} \mathrm{mul}\, A^* \\ \mathcal{H}_1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & A_{\rm op}(I + A^*_{\rm op}A_{\rm op})^{-1} \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathrm{mul}\, A^* \\ \mathcal{H}_1 \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \mathrm{mul}\, A \end{pmatrix}$$

of P_A in Proposition 2.1 coincide with the ones in (2.7). This completes the proof of Proposition 2.3. □

Let again $A \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$. Following [9] we define the operators $\cos A$ and $\sin A$ by

$$\cos A := (I + A^*A)^{-1/2} \in \mathcal{L}(\mathcal{H})$$

and

$$\sin A := \iota_{\mathcal{K}_1}\left[A(I + A^*A)^{-1/2}\right]_{\rm op} \in \mathcal{L}(\mathcal{H}, \mathcal{K}),$$

where $\iota_{\mathcal{K}_1}$ is the canonical embedding of $\mathcal{K}_1$ into $\mathcal{K}$. Now Propositions 2.1 and 2.3 yield the following corollary, which is a slight generalization of the main result in [9], see also [7].

Corollary 2.4. *Let $A \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ be a closed linear relation. Then the orthogonal projection P_A in $\mathcal{H}\oplus\mathcal{K}$ onto A has the form*

$$P_A = \begin{pmatrix} \cos^2 A & \cos A \sin A^* \\ \cos A^* \sin A & I - \cos^2 A^* \end{pmatrix} = \begin{pmatrix} \cos^2 A & \sin A^* \cos A^* \\ \sin A \cos A & I - \cos^2 A^* \end{pmatrix}.$$

Proof. It is clear from Proposition 2.3 that the diagonal entries of P_A are given by $\cos^2 A$ and $I - \cos^2 A^*$. In order to see the form of the offdiagonal entries denote by $E_{A_{\rm op}^* A_{\rm op}}(\cdot)$ and $E_{A_{\rm op} A_{\rm op}^*}(\cdot)$ the spectral functions of the selfadjoint operators $A_{\rm op}^* A_{\rm op} \in \mathcal{C}(\mathcal{H}_1)$ and $A_{\rm op} A_{\rm op}^* \in \mathcal{C}(\mathcal{K}_1)$, respectively. Then

$$\begin{aligned} A_{\rm op} E_{A_{\rm op}^* A_{\rm op}}(\cdot)x &= E_{A_{\rm op} A_{\rm op}^*}(\cdot) A_{\rm op} x, \qquad x \in \operatorname{dom} A_{\rm op}, \\ A_{\rm op}^* E_{A_{\rm op} A_{\rm op}^*}(\cdot)y &= E_{A_{\rm op}^* A_{\rm op}}(\cdot) A_{\rm op}^* y, \qquad y \in \operatorname{dom} A_{\rm op}^*, \end{aligned}$$

imply that the identities

$$\begin{aligned} A_{\rm op}(I + A_{\rm op}^* A_{\rm op})^{-1} &= (I + A_{\rm op} A_{\rm op}^*)^{-1/2} A_{\rm op} (I + A_{\rm op}^* A_{\rm op})^{-1/2}, \\ A_{\rm op}^*(I + A_{\rm op} A_{\rm op}^*)^{-1} &= (I + A_{\rm op}^* A_{\rm op})^{-1/2} A_{\rm op}^* (I + A_{\rm op} A_{\rm op}^*)^{-1/2} \end{aligned}$$

hold. Now the statement follows from Proposition 2.1,

$$\begin{aligned} \cos A &= \begin{pmatrix} 0 & 0 \\ 0 & (I + A_{\rm op}^* A_{\rm op})^{-1/2} \end{pmatrix} : \begin{pmatrix} \operatorname{mul} A^* \\ \mathcal{H}_1 \end{pmatrix} \to \begin{pmatrix} \operatorname{mul} A^* \\ \mathcal{H}_1 \end{pmatrix}, \\ \cos A^* &= \begin{pmatrix} (I + A_{\rm op} A_{\rm op}^*)^{-1/2} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \operatorname{mul} A \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \operatorname{mul} A \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \sin A &= \begin{pmatrix} 0 & A_{\rm op}(I + A_{\rm op}^* A_{\rm op})^{-1/2} \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \operatorname{mul} A^* \\ \mathcal{H}_1 \end{pmatrix} \to \begin{pmatrix} \mathcal{K}_1 \\ \operatorname{mul} A \end{pmatrix}, \\ \sin A^* &= \begin{pmatrix} 0 & 0 \\ A_{\rm op}^*(I + A_{\rm op} A_{\rm op}^*)^{-1/2} & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \operatorname{mul} A \end{pmatrix} \to \begin{pmatrix} \operatorname{mul} A^* \\ \mathcal{H}_1 \end{pmatrix}. \end{aligned}$$

□

3. Finite rank perturbations of linear relations

In this section we are concerned with finite dimensional perturbations of closed linear relations in Hilbert spaces. The notion of finite rank perturbations introduced below is compatible with the usual notions for unbounded and bounded operators, cf. Corollary 3.4 and Corollary 3.5. Roughly speaking, a linear relation is a finite rank perturbation of another linear relation if both differ by finitely many dimensions. This is made precise in the following theorem, where also an alternative description in terms of orthogonal projections is given.

Theorem 3.1. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ be closed linear relations from $\mathcal{H}$ in $\mathcal{K}$ and let P_A and P_B be the orthogonal projections in $\mathcal{H}\oplus\mathcal{K}$ onto A and B, respectively. Then the following assertions are equivalent:*

(i) *$P_A - P_B$ is a finite rank operator;*
(ii) $\dim A/(A\cap B) < \infty$ *and* $\dim B/(A\cap B) < \infty$.

If (i) *or* (ii) *holds, then A is said to be* a finite rank perturbation of B *and B is said to be* a finite rank perturbation of A.

Proof. The identities

$$\dim \operatorname{ran}\,(P_A - P_{A\cap B}) = \dim A/(A\cap B),$$
$$\dim \operatorname{ran}\,(P_B - P_{A\cap B}) = \dim B/(A\cap B)$$

together with $P_A - P_B = (P_A - P_{A\cap B}) - (P_B - P_{A\cap B})$ show that (ii) implies (i). Assume now that (i) holds. We can assume $B = \mathcal{H}\times\{0\}$ since $\mathcal{H}\oplus\mathcal{K} = B\oplus B^\perp$ and A can also be regarded as a closed linear relation from B to $B^\perp$. Hence in the following we consider the case $B = 0 \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Then

$$P_B = \begin{pmatrix} I & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} : \begin{pmatrix}\text{mul } A^*\\ \mathcal{H}_1\\ \mathcal{K}_1\\ \text{mul } A\end{pmatrix} \to \begin{pmatrix}\text{mul } A^*\\ \mathcal{H}_1\\ \mathcal{K}_1\\ \text{mul } A\end{pmatrix}, \tag{3.1}$$

where $\mathcal{H}_1 = \overline{\text{dom } A}$ and $\mathcal{K}_1 = \overline{\text{dom } A^*}$, and by Proposition 2.1 we have

$$P_A - P_B = \begin{pmatrix} -I & 0 & 0 & 0\\ 0 & -A_{\text{op}}^*A_{\text{op}}(I + A_{\text{op}}^*A_{\text{op}})^{-1} & A_{\text{op}}^*(I + A_{\text{op}}A_{\text{op}}^*)^{-1} & 0\\ 0 & A_{\text{op}}(I + A_{\text{op}}^*A_{\text{op}})^{-1} & A_{\text{op}}A_{\text{op}}^*(I + A_{\text{op}}A_{\text{op}}^*)^{-1} & 0\\ 0 & 0 & 0 & I\end{pmatrix}.$$

Thus, the assumption $\dim \operatorname{ran}\,(P_A - P_B) < \infty$ implies

$$\dim \operatorname{mul} A^* < \infty \qquad \text{and} \qquad \dim \operatorname{mul} A < \infty.$$

Moreover, as $A_{\text{op}} \upharpoonright \text{dom } A_{\text{op}}^*A_{\text{op}}$ is dense in A_{op} it follows that $A_{\text{op}} \in \mathcal{C}(\mathcal{H}_1,\mathcal{K}_1)$ is an operator of finite rank. Therefore $\dim \mathcal{H}_1/\ker A_{\text{op}} < \infty$ and also $\dim \mathcal{H}/\ker A < \infty$. From $A\cap B = \ker A\times\{0\}$ we conclude $\dim A/(A\cap B) < \infty$. Replacing the roles of A and B it follows that also $\dim B/(A\cap B) < \infty$ holds. Hence (i) implies (ii) and Theorem 3.1 is proved. □

Proposition 3.2. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ and $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Then A is a finite rank perturbation of B if and only if $A - T$ is a finite rank perturbation of $B - T$.*

Proof. Assume that A is a finite rank perturbation of B. Then it follows from Theorem 3.1 (ii) that there exists a finite dimensional subspace $N \subset A$ such that each element $\binom{u}{v} \in A$ can be written as

$$\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}u_1\\ v_1\end{pmatrix} + \begin{pmatrix}u_2\\ v_2\end{pmatrix}, \text{ where } \begin{pmatrix}u_1\\ v_1\end{pmatrix} \in A\cap B, \quad \begin{pmatrix}u_2\\ v_2\end{pmatrix} \in N.$$

Hence

$$\begin{pmatrix}u\\ v - Tu\end{pmatrix} = \begin{pmatrix}u_1\\ v_1 - Tu_1\end{pmatrix} + \begin{pmatrix}u_2\\ v_2 - Tu_2\end{pmatrix},$$

that is, $A - T = ((A\cap B) - T) \,\dot{+}\, M$, where

$$M = \left\{\begin{pmatrix}u_2\\ v_2 - Tu_2\end{pmatrix} : \begin{pmatrix}u_2\\ v_2\end{pmatrix} \in N\right\}$$

and $\dot{+}$ denotes the sum of two linear manifolds. Since

$$(A - T)\cap(B - T) = (A\cap B) - T$$

and $\dim M < \infty$ it follows that

$$\dim(A - T)/\big((A - T)\cap(B - T)\big) < \infty.$$

Similarly, we get

$$\dim(B-T)/\big((A-T)\cap(B-T)\big)<\infty,$$

so that, by Theorem 3.1 (ii) $A-T$ is a finite rank perturbation of $B-T$. The converse implication follows when A, B and T are replaced by $A-T$, $B-T$ and $-T$, respectively. □

For $A,B\in\widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ we define

$$\rho(A,B):=\big\{T\in\mathcal{L}(\mathcal{H},\mathcal{K}):(A-T)^{-1},\,(B-T)^{-1}\in\mathcal{L}(\mathcal{K},\mathcal{H})\big\} \tag{3.2}$$

Observe that if in the special case $\mathcal{H}=\mathcal{K}$ the intersection of the resolvent sets $\rho(A)$ and $\rho(B)$ is nonempty, then $\{\lambda I:\lambda\in\rho(A)\cap\rho(B)\}$ is a subset of $\rho(A,B)$.

Proposition 3.3. *Let $A,B\in\widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ and $\rho(A,B)\neq\emptyset$. Then A is a finite rank perturbation of B if and only if $(A-T)^{-1}-(B-T)^{-1}$ is a finite rank operator for some (and hence for all) $T\in\rho(A,B)$.*

Proof. Suppose that A is a finite rank perturbation of B and let $T\in\rho(A,B)$. By Proposition 3.2 $A-T$ is a finite rank perturbation of $B-T$ and Theorem 3.1 (ii) implies that the closed linear relations $A-T$ and $B-T$ are both finite dimensional extensions of the linear relation $(A-T)\cap(B-T)$. Hence the same holds for the inverses, that is,

$$\dim(A-T)^{-1}/((A-T)^{-1}\cap(B-T)^{-1})<\infty \tag{3.3}$$

and

$$\dim(B-T)^{-1}/((A-T)^{-1}\cap(B-T)^{-1})<\infty. \tag{3.4}$$

Now the statement follows from $(A-T)^{-1}-(B-T)^{-1}\in\mathcal{L}(\mathcal{K},\mathcal{H})$. Conversely, if for some $T\in\rho(A,B)$ the operator $(A-T)^{-1}-(B-T)^{-1}\in\mathcal{L}(\mathcal{K},\mathcal{H})$ is of finite rank, then (3.3) and (3.4) hold. This implies that $A-T$ and $B-T$ are finite dimensional extensions of $(A-T)\cap(B-T)$ and therefore A is a finite rank perturbation of B by Theorem 3.1 (ii) and Proposition 3.2. □

We complete this section with two corollaries. The first one shows that for closed linear relations (and operators) in the same Hilbert space and a common point in their resolvent sets the notion of finite rank pertubations suggested above is compatible with the usual definition via resolvent differences.

Corollary 3.4. *Let $A,B\in\widetilde{\mathcal{C}}(\mathcal{H})$ and $\rho(A)\cap\rho(B)\neq\emptyset$. Then A is a finite rank perturbation of B if and only if $(A-\lambda)^{-1}-(B-\lambda)^{-1}$ is a finite rank operator for some (and hence for all) $\lambda\in\rho(A)\cap\rho(B)$.*

Corollary 3.5. *Let $A,B\in\mathcal{L}(\mathcal{H},\mathcal{K})$. Then A is a finite rank perturbation of B if and only if $A-B$ is a finite rank operator.*

Proof. Suppose that A is a finite rank perturbation of B, i.e., P_A-P_B is a finite rank operator. From Corollary 2.2 it follows that the entries in the first column of P_A-P_B are given by

$$(I+A^*A)^{-1}-(I+B^*B)^{-1}\ \text{ and }\ A(I+A^*A)^{-1}-B(I+B^*B)^{-1},$$

respectively, and are finite rank operators. Multiplying the first operator from the left with B and subtracting the second one yields that $A-B$ is a finite rank operator.

Conversely, if $A - B$ is a finite rank operator, then also $A^* - B^*$, $A^*A - B^*B$ and $AA^* - BB^*$ are finite rank operators. Making use of Corollary 2.2 it is not difficult to see that $P_A - P_B$ is a finite rank operator and hence A is a finite rank perturbation of B □

4. Compact perturbations of linear relations

Recall that the *gap* between two closed subspaces M and N of a Hilbert space is defined by

$$\hat{\delta}(M,N) := \max\Big\{ \sup_{u \in M, \|u\|=1} \text{dist}\,(u,N), \sup_{v \in N, \|v\|=1} \text{dist}\,(v,M) \Big\}.$$

If P_M and P_N denote the orthogonal projections onto M and N, respectively, then the gap between M and N is

$$\hat{\delta}(M,N) = \|P_M - P_N\|,$$

cf. [6]. The following lemma is known for the special case that A and B are closed operators, see [6, Theorem IV.2.17]. The proof for the relation case is almost the same, however, for the convenience of the reader we present the details.

Lemma 4.1. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$, $T \in \mathcal{L}(\mathcal{H},\mathcal{K})$ and*

$$\gamma := 2(1 + \|T\|^2).$$

Denote by P_A, P_B, P_{A-T} and P_{B-T} the orthogonal projections in $\mathcal{H} \oplus \mathcal{K}$ onto A, B, $A - T$ and $B - T$, respectively. Then the following estimate holds:

$$\frac{1}{\gamma}\|P_{A-T} - P_{B-T}\| \le \|P_A - P_B\| \le \gamma \|P_{A-T} - P_{B-T}\|. \tag{4.1}$$

Proof. It suffices to verify the first estimate in (4.1), the second estimate follows when $A - T$, $B - T$ and T are replaced by A, B and $-T$, respectively.

Let $\varphi \in A - T$, $\|\varphi\| = 1$, and choose $\binom{u}{v} \in A$ such that

$$\varphi = \begin{pmatrix} u \\ v - Tu \end{pmatrix} \in A - T, \qquad \|\varphi\|^2 = \|u\|^2 + \|v - Tu\|^2 = 1. \tag{4.2}$$

Then $r^2 := \|u\|^2 + \|v\|^2 > 0$, and $r^{-1} \binom{u}{v}$ belongs to the unit sphere of A. Therefore, for any $\delta' > \|P_A - P_B\| = \hat{\delta}(A,B)$ the element $r^{-1}\binom{u}{v}$ has a distance less than δ' from B. Hence there exists an element $\binom{x}{y} \in B$ with $\|r^{-1}u - x\|^2 + \|r^{-1}v - y\|^2 < \delta'^2$, i.e.,

$$\|u - rx\|^2 + \|v - ry\|^2 < r^2 \delta'^2. \tag{4.3}$$

We define an element ψ of $B - T$ by

$$\psi := \begin{pmatrix} rx \\ ry - rTx \end{pmatrix}$$

With the help of (4.3) we find

$$\begin{aligned} \|\varphi - \psi\|^2 &= \|u - rx\|^2 + \|(v - ry) - T(u - rx)\|^2 \\ &\le \|u - rx\|^2 + 2\|v - ry\|^2 + 2\|T\|^2\|u - rx\|^2 \\ &\le 2(1 + \|T\|^2)(\|u - rx\|^2 + \|v - ry\|^2) < 2(1 + \|T\|^2) r^2 \delta'^2 \end{aligned}$$

and on the other hand

$$r^2 = \|u\|^2 + \|v - Tu + Tu\|^2 \le \|u\|^2 + 2\|v - Tu\|^2 + 2\|T\|^2\|u\|^2.$$

Then (4.2) implies $r^2 \leq 2(1+\|T\|^2)$ and hence

$$\|\varphi - \psi\|^2 \leq 4(1+\|T\|^2)^2\delta'^2. \tag{4.4}$$

As φ is an element of the unit sphere of $A-T$, $\psi \in B-T$ and δ' is an arbitrary number greater than $\|P_A - P_B\|$ it follows that

$$\sup_{\varphi\in A-T, \|\varphi\|=1} \text{dist}\,(\varphi, B-T) \leq 2(1+\|T\|^2)\|P_A - P_B\|$$

holds. The estimate

$$\sup_{\varphi\in B-T, \|\varphi\|=1} \text{dist}\,(\varphi, A-T) \leq 2(1+\|T\|^2)\|P_A - P_B\|$$

is obtained by interchanging A and B in the above considerations and therefore $\|P_{A-T} - P_{B-T}\| \leq \gamma\|P_A - P_B\|$, where $\gamma = 2(1+\|T\|^2)$. □

In the next theorem, which is the main result in this section, two equivalent notions for compact perturbations of linear relations are introduced. In analogy to Theorem 3.1 a linear relation is a compact perturbation of another linear relation if the difference of the corresponding orthogonal projections is compact. In connection with (semi-)Fredholm theory of linear relations this notion was already used in [8, Proposition 18].

Theorem 4.2. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ be closed linear relations from $\mathcal{H}$ in $\mathcal{K}$ and let P_A and P_B be the orthogonal projections in $\mathcal{H}\oplus\mathcal{K}$ onto A and B, respectively. Then the following assertions are equivalent:*

(i) *$P_A - P_B$ is a compact operator;*
(ii) *For every $\varepsilon > 0$ there exists a linear relation $F \in \widetilde{\mathcal{C}}(\mathcal{H},\mathcal{K})$ such that $P_B - P_F$ is a finite rank operator and*

$$\hat{\delta}(A,F) = \|P_A - P_F\| < \varepsilon.$$

If (i) *or* (ii) *holds, then A is said to be* a compact perturbation of B *and B is said to be* a compact perturbation of A.

Proof. Since $P_A - P_B = P_A - P_F - (P_B - P_F)$ it is clear that (ii) implies (i). Suppose that (i) holds. As in the proof of Theorem 3.1 we can assume that $B = 0$, $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Let $\mathcal{H}_1 = \overline{\text{dom}\, A}$ and $\mathcal{K}_1 = \overline{\text{dom}\, A^*}$. Then the orthogonal projections P_A and P_B are given by (2.4) and (3.1), respectively. Since

$$P_A - P_B = \begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & -A_{\text{op}}^*A_{\text{op}}(I+A_{\text{op}}^*A_{\text{op}})^{-1} & A_{\text{op}}^*(I+A_{\text{op}}A_{\text{op}}^*)^{-1} & 0 \\ 0 & A_{\text{op}}(I+A_{\text{op}}^*A_{\text{op}})^{-1} & A_{\text{op}}A_{\text{op}}^*(I+A_{\text{op}}A_{\text{op}}^*)^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

is compact it is clear that mul A^* and mul A are finite dimensional and the nonnegative selfadjoint operator $A_{\text{op}}^*A_{\text{op}}(I+A_{\text{op}}^*A_{\text{op}})^{-1} \in \mathcal{L}(\mathcal{H}_1)$ is also compact. Therefore

$$\sigma\big(A_{\text{op}}^*A_{\text{op}}(I+A_{\text{op}}^*A_{\text{op}})^{-1}\big)\backslash\{0\}$$

consists only of isolated eigenvalues with finite multiplicity and zero is the only possible accumulation point. It follows from the spectral mapping theorem that $\sigma(A_{\text{op}}^*A_{\text{op}})$ has the same properties, hence $A_{\text{op}}^*A_{\text{op}}$ is a compact operator. Using the polar decomposition of A_{op} it follows that also $A_{\text{op}} \in \mathcal{L}(\mathcal{H}_1,\mathcal{K}_1)$ is compact. Therefore, for each $\varepsilon > 0$ there exists a decomposition $A_{\text{op}} = F_{\text{op}} + G_{\text{op}}$ such that $F_{\text{op}} \in \mathcal{L}(\mathcal{H}_1,\mathcal{K}_1)$ is a finite rank operator, $G_{\text{op}} \in \mathcal{L}(\mathcal{H}_1,\mathcal{K}_1)$ is sufficiently small, and

$\|P_{A_{\rm op}} - P_{F_{\rm op}}\| < \varepsilon$, where $P_{A_{\rm op}}$ and $P_{F_{\rm op}}$ are the orthogonal projections in $\mathcal{H}_1 \oplus \mathcal{K}_1$ onto $A_{\rm op}$ and $F_{\rm op}$, respectively. The norm estimate $\|P_{A_{\rm op}} - P_{F_{\rm op}}\| < \varepsilon$ can easily be verified with the help of Corollary 2.2.

Define the linear relation $F \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ by

$$F := \left\{ \begin{pmatrix} x \\ F_{\rm op}x + x' \end{pmatrix} : x \in \mathcal{H}_1,\, x' \in \text{mul}\, A \right\}.$$

Then mul F = mul A and mul $F^* = (\text{dom}\, F)^\perp = (\text{dom}\, A)^\perp$ = mul A^* imply that the orthogonal proction P_F in $\mathcal{H} \oplus \mathcal{K}$ onto F is given by

$$P_F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (I + F_{\rm op}^* F_{\rm op})^{-1} & F_{\rm op}^*(I + F_{\rm op}F_{\rm op}^*)^{-1} & 0 \\ 0 & F_{\rm op}(I + F_{\rm op}^*F_{\rm op})^{-1} & F_{\rm op}F_{\rm op}^*(I + F_{\rm op}F_{\rm op}^*)^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \tag{4.5}$$

with respect to the decomposition mul $A^* \oplus \mathcal{H}_1 \oplus \mathcal{K}_1 \oplus$ mul A, cf. Proposition 2.1. Hence, by (2.4) and Corollary 2.2 we have $\|P_A - P_F\| = \|P_{A_{\rm op}} - P_{F_{\rm op}}\| < \varepsilon$. As mul $A^*$and mul A are finite dimensional and $F_{\rm op}$ is a finite rank operator it follows from (3.1) and (4.5) that

$$P_B - P_F = \begin{pmatrix} -I & 0 & 0 & 0 \\ 0 & F_{\rm op}^*F_{\rm op}(I + F_{\rm op}^*F_{\rm op})^{-1} & F_{\rm op}^*(I + F_{\rm op}F_{\rm op}^*)^{-1} & 0 \\ 0 & F_{\rm op}(I + F_{\rm op}^*F_{\rm op})^{-1} & F_{\rm op}F_{\rm op}^*(I + F_{\rm op}F_{\rm op}^*)^{-1} & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

is a finite rank operator. This completes the proof of Theorem 4.2. □

The next proposition is the analogue of Proposition 3.2 for compact perturbations.

Proposition 4.3. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ and $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then A is a compact perturbation of B if and only if $A - T$ is a compact perturbation of $B - T$.*

Proof. Assume that A is a compact perturbation of B and let $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. According to Theorem 4.2 (ii) for given $\varepsilon > 0$ there exists $F \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ such that $P_B - P_F$ is a finite rank operator and $\|P_A - P_F\| < \varepsilon$. According to Theorem 3.1 and Proposition 3.2 also $P_{B-T} - P_{F-T}$ is a finite rank operator and by Lemma 4.1

$$\|P_{A-T} - P_{F-T}\| \leq 2(1 + \|T\|^2)\|P_A - P_F\| < 2(1 + \|T\|^2)\varepsilon$$

holds. This implies that $P_{A-T} - P_{B-T}$ is compact and hence $A - T$ is a compact perturbation of $B - T$ by Theorem 4.2 (i). By replacing A, B and T with $A - T$, $B - T$ and $-T$, respectively, it follows that A is a compact perturbation B when $A - T$ is a compact perturbation of $B - T$. □

Proposition 4.4. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H}, \mathcal{K})$ and $\rho(A, B) \neq \emptyset$. Then A is a compact perturbation of B if and only if $(A - T)^{-1} - (B - T)^{-1}$ is a compact operator for some (and hence for all) $T \in \rho(A, B)$.*

Proof. Assume that A is a compact perturbation of B and let $T \in \rho(A, B)$. By Proposition 4.3, $A - T$ is a compact perturbation of $B - T$ and hence the operator $P_{A-T} - P_{B-T}$ is compact. Observe that P_{A-T} is connected with the orthogonal projection $P_{(A-T)^{-1}} \in \mathcal{L}(\mathcal{K}\oplus\mathcal{H})$ in $\mathcal{K}\oplus\mathcal{H}$ onto $(A-T)^{-1}$ in the following manner: Let $h \in \mathcal{H}$ and $k \in \mathcal{K}$. Then

$$P_{A-T}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} z \\ z' \end{pmatrix} \in A-T$$

if and only if

$$P_{(A-T)^{-1}}\begin{pmatrix} k \\ h \end{pmatrix} = \begin{pmatrix} z' \\ z \end{pmatrix} \in (A-T)^{-1}.$$

The projections P_{B-T} and $P_{(B-T)^{-1}}$ are connected in the same way. Therefore, since the compact operator $P_{A-T} - P_{B-T}$ maps bounded sequences onto sequences with a convergent subsequence, the same is true for $P_{(A-T)^{-1}} - P_{(B-T)^{-1}}$, and hence this operator is compact. From Corollary 2.2 it follows that the entries in the first column of $P_{(A-T)^{-1}} - P_{(B-T)^{-1}}$ are given by

$$\bigl(I + (A^*-T^*)^{-1}(A-T)^{-1}\bigr)^{-1} - \bigl(I + (B^*-T^*)^{-1}(B-T)^{-1}\bigr)^{-1} \tag{4.6}$$

and

$$\begin{aligned}(A-T)^{-1}&\bigl(I + (A^*-T^*)^{-1}(A-T)^{-1}\bigr)^{-1} \\ &- (B-T)^{-1}\bigl(I + (B^*-T^*)^{-1}(B-T)^{-1}\bigr)^{-1}\end{aligned} \tag{4.7}$$

and both are compact operators. Multiplying (4.6) from the left with $(B-T)^{-1}$ and subtracting (4.7) then implies that $(A-T)^{-1} - (B-T)^{-1}$ is a compact operator.

Conversely, suppose that $(A-T)^{-1} - (B-T)^{-1}$ is compact for some $T \in \rho(A,B)$. Then also the operators in (4.6) and (4.7) are compact and with the help of Corollary 2.2 it follows that $P_{(A-T)^{-1}} - P_{(B-T)^{-1}}$ is compact. Therefore the above considerations imply that also $P_{A-T} - P_{B-T}$ is compact. Hence $A-T$ is a compact perturbation of $B-T$ and Proposition 4.3 yields that A is a compact perturbation of B. □

The following two corollaries show that the notion of compact perturbations introduced in Theorem 4.2 reduces to the usual notions if, e.g., both relations (or operators) act in the same Hilbert space and have a common point in their resolvent sets.

Corollary 4.5. *Let $A, B \in \widetilde{\mathcal{C}}(\mathcal{H})$ and $\rho(A)\cap\rho(B) \neq \emptyset$. Then A is a compact perturbation of B if and only if $(A-\lambda)^{-1} - (B-\lambda)^{-1}$ is a compact operator for some (and hence for all) $\lambda \in \rho(A)\cap\rho(B)$.*

Corollary 4.6. *Let $A, B \in \mathcal{L}(\mathcal{H},\mathcal{K})$. Then A is a compact perturbation of B if and only if $A-B$ is a compact operator.*

Corollary 4.6 can be proved in the same way as Corollary 3.5; simply replace the expression "finite rank" in the proof of Corollary 3.5 by "compact".

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