Chiraptophobic Cockroaches evading a Torch Light

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Abstract. We propose and study game-theoretic versions of independence in graphs. The games are played by two players - the aggressor and the defender - taking alternate moves on a graph G with tokens located on vertices from an independent set of G. A move of the aggressor is to select a vertex v of G. A move of the defender is to move tokens located on vertices in $N_G(v)$ each along one incident edge. The goal of the defender is to maintain the set of occupied vertices independent while the goal of the aggressor is to make this impossible. We consider the maximum number of tokens for which the aggressor can not win in a strategic and an adaptive version of the game.

Keywords. independence; domination; Roman domination; secure domination

1 Introduction

We consider finite, simple and undirected graphs $G = (V, E)$ and use standard terminology and notation. A set of vertices $I \subseteq V$ is independent in G, if no two vertices in I are adjacent. The independence number $\alpha(G)$ which is the maximum cardinality of an independent set in G is one of the most fundamental and well-studied graph parameters.

In the present paper we propose a game-theoretic version of independence and study two-player games played on a graph $G = (V, E)$. One of the two players – the *defender* – maintains a set of tokens each placed on a different vertex from an independent set of G. A move of the other player – the *aggressor* – consists in choosing a vertex $a \in V$ – *attacking at* a. The defender has to move the tokens located at neighbours of a along the edges of the graph to evacuate $N_G(a)$. Note that it is allowed that one of these tokens moves to a.

Formally, if $I \subseteq V$ is an independent set, then a *legal reply defending* I *against an attack* at a vertex $a \in V$ is an injective mapping

$$f : I \to V$$

such that

- $f(I)$ is independent,
- $f(I) \cap N_G(a) = \emptyset$,
- $f(u) = u$ for every vertex $u \in I \setminus N_G(u)$ and
- $f(u) \in N_G(u)$ for every vertex $u \in I \cap N_G(u)$.

The aggressor wins if and only if at some point during the game the defender has no legal reply to his attack.

We consider a strategic version in which the players take their decisions in turns and an adaptive version in which the infinite sequence of attacks is known to the defender in advance.

If $\mathcal{I}$ denotes the set of all independent sets of G, then a *strategy* is a mapping

$$s : V \times \mathcal{I} \to \mathcal{I} \cup \{\text{lost}\}$$

such that for every vertex $a \in V$ and independent set $I \subseteq V$ either $s(a, I) = \text{lost}$ or there is a legal reply f defending I against an attack at a with $f(I) = s(a, I)$. The *strategic independence number* $\alpha_\infty(G)$ is the maximum cardinality of an independent set $I_0 \subseteq V$ for which there exists a strategy s such that for every sequence $(a_i)_{i\in\mathbb{N}}$ of attacks there exists a sequence $(I_i)_{i\in\mathbb{N}}$ of independent sets such that $I_i = s(a_i, I_{i-1})$ for $i \in \mathbb{N}$.

The *adaptive independence number* $\overline{\alpha}_\infty(G)$ is the maximum integer k such that for every sequence $(a_i)_{i\in\mathbb{N}}$ of attacks there is a sequence $(I_i)_{i\in\{0\}\cup\mathbb{N}}$ of independent sets of cardinality k such that for every $i \in \mathbb{N}$ there exists a legal reply f defending I_{i-1} against an attack at a_i with $f(I_{i-1}) = I_i$.

While the games we consider are reminiscent of cops-and-robber games on graphs and graph searching as considered in [12, 16, 20] our motivation were game-theoretic versions of domination in graphs which have recently attracted much attention. Classically [9], a set of vertices $D \subseteq V$ is a dominating set of a graph $G = (V, E)$, if every vertex $u \in V$ is either in D or adjacent to a vertex in D and the domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. In the corresponding games played by two players on a graph there are tokens located on some vertices, one player attacks at a vertex and the other player has to move a token along an edge to the attacked vertex. The natural question in this context concerns the minimum number of tokens which are needed to defend against the attacks. After two popular articles [19, 21], numerous authors have intensely studied many theoretical aspects of these games [1–7,10,11,13–15] and considered applications [17, 18]. Since in our games we allow the defender only to move the tokens located on neighbours of the attacked vertex, the concepts we propose seem more related to the game-theoretic versions of domination than to the mentioned cops-and-robber or searching games.

In the next section we prove some general bounds on the strategic and adaptive independence numbers, show that these numbers coincide with the domination number for trees and consider their values for cycles. There are natural bounds for $\alpha_\infty(G)$ and $\overline{\alpha}_\infty(G)$ in terms of the independence numbers of graphs related to G and our results for trees and cycles lead us to consider the relation of $\alpha(G^{2k})$ and $\gamma(G^k)$. Here G^k denotes the k-th power of G in which two vertices are adjacent exactly if their distance in G is at most k. We prove the curious and apparently previously unknown fact that $\alpha(T^{2k}) = \gamma(T^k)$ for all trees T and $k \in \mathbb{N}$. Since we feel this observation to be of proper interest unrelated to the games, we prove a best-possible result for cacti, i.e. graphs in which no two cycles share an edge. The paper closes with some open problems.

2 Results

We begin with natural bounds in terms of the classical independence number.

Theorem 1

$$\alpha(G^2) \leq \alpha_\infty(G) \leq \overline{\alpha}_\infty(G) \leq \min\left\{\alpha\left(G\left[V \setminus \bigcup_{v\in I} N_G(v)\right]\right) \mid I \text{ is independent in } G^3\right\}$$

for every graph $G=(V,E)$.

Proof: The second inequality is immediate from the definitions.

For the first inequality we consider a maximum independent set $I^2 \in V$ in the square G^2 of G. Since the closed neighbourhoods of all vertices in I^2 are disjoint, the following strategy allows to defend a set of tokens initially located at the vertices in I^2 against arbitrary sequences of attacks. Throughout the game every token remains in the closed neighbourhood of the same vertex in I^2. Assume that there is an attack at a neighbour of the location of a token. If the attacked vertex is adjacent to the initial location of the token in I^2, then the token moves to the attacked vertex. Otherwise it moves to its initial location in I^2. Clearly, the set of vertices occupied by tokens remains independent throughout the execution of the game.

For the third inequality consider an independent set $I^3 \subseteq V$ of G^3. After consecutive attacks at the vertices of I^3, no token is located on a neighbour of a vertex in I^3. Therefore, the tokens occupy vertices which form an independent set of $G\left[V \setminus \bigcup_{v\in I^3} N_G(v)\right]$. This completes the proof. $\square$

Next we consider trees.

Theorem 2 $\overline{\alpha}_\infty(T) = \alpha_\infty(T) = \gamma(T)$ *for every tree T.*

The proof of the Theorem 2 follows immediately from Theorem 1 and from the following two results.

Proposition 3 $\alpha(T^{2k}) = \gamma(T^k)$ *for every tree T and $k \in \mathbb{N}$.*

Proof: Since $\alpha(G^{2k}) \leq \gamma(G^k)$ for every graph G, we only prove the converse inequality for trees T by induction over the order. Clearly, the statement is true for all trees of diameter at most $2k$, because $\alpha(T^{2k}) = \gamma(T^k) = 1$ in this case.

Therefore, let T be a tree with diameter larger than $2k$ and let $u_0u_1u_2...u_{2k}u_{2k+1}...u_l$ with $l \geq 2k+1$ be a longest path in T. Root T at u_{2k+1} and let U denote the set of all vertices within distance at most k from u_k. Let $C \subseteq U$ be a minimal set of vertices such that

$$T' = T\left[V \setminus (U \setminus C)\right]$$

is connected. Note that $U \setminus C$ contains all descendants of u_k.

Since for every dominating set D' of $(T')^k$ the set $D' \cup \{u_k\}$ is a dominating set of T^k, we have $\gamma(T^k) \leq \gamma((T')^k) + 1$.

Let I be a maximum independent set of $(T')^{2k}$ chosen such that $|I \cap C|$ is as small as possible. If $v \in I \cap C$, then v is a descendant of, or equal to, u_j for some $k+1 \leq j \leq 2k$. By the choice of I and C, this implies that no descendant of v is in I and that there is some descendant v' of v which does not belong to U. Now $I' = (I \setminus \{v\}) \cup \{v'\}$ is a maximum independent set of $(T')^{2k}$ with $|I' \cap C| < |I \cap C|$, which is a contradiction. Hence $|I \cap C| = 0$, $I \cup \{u_0\}$ is an independent set of T^{2k} and hence $\alpha(T^{2k}) \geq \alpha((T')^{2k}) + 1$.

Altogether, we obtain by induction $\alpha(T^{2k}) - \gamma(T^k) \geq (\alpha((T')^{2k}) + 1) - (\gamma((T')^k) + 1) \geq 0$ and the proof is complete. □

Lemma 4 $\overline{\alpha}_\infty(T) \leq \gamma(T)$ *for every tree* T.

Proof: Let $D = \{v_1, v_2, ..., v_d\}$ be a minimum dominating set of T. We root T at v_d and assume that $v_1, v_2, ..., v_d$ are ordered according to non-increasing depth. Since D is dominating, there is a partition $V_1 \cup V_2 \cup ... \cup V_d$ such that V_i induces a star with center v_i for all $1 \leq i \leq d$.

For an arbitrary independent set $I_0 \subseteq V$ we consider the sequence $v_1, v_2, ..., v_d$ of attacks. We may assume that there are independent sets $I_1, I_2, ..., I_d$ such that there are legal replies f_i defending I_{i-1} against an attack at v_i with $f(I_{i-1}) = I_i$ for $1 \leq i \leq d$. Clearly, $I_i \cap V_i \subseteq \{v_i\}$ for $1 \leq i \leq d$. In view of the order of the attack sequence, it follows that $|I_d \cap V_i| \leq 1$ for $1 \leq i \leq d$. Therefore, $|I_d| \leq \gamma(T)$ which completes the proof. □

We proceed to cycles.

Theorem 5 $\alpha_\infty(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil & \text{for } 3 \leq n \leq 7, \\ \lceil \frac{n-1}{3} \rceil & \text{for } n \geq 8. \end{cases}$

Proof: For $3 \leq n \leq 7$ with $n \neq 6$, we have $\alpha_\infty(C_n) = \alpha(C_n)$ because – up to isomorphism – there is a unique maximum independent set in these cases which can be defended against an arbitrary attack. For $n = 6$ Theorem 1 implies $\alpha_\infty(C_6) \geq \alpha(C_6^2) = 2$. Since the unique maximum independent set I of C_6 cannot be defended against an attack outside of I, we have $\alpha_\infty(C_6) < 3$ and hence $\alpha_\infty(C_6) = 2$.

Now let $n \geq 8$.

Claim 1 $\alpha_\infty(C_n) \leq \lceil \frac{n-1}{3} \rceil$.

Proof of Claim 1: For contradiction, we assume that there exists an independent set I of cardinality $\lceil \frac{n-1}{3} \rceil + 1$ which can be defended against an arbitrary sequence of attacks. Let $C_n = v_1 v_2 v_3 ... v_n v_1$. We identify indices modulo n.

The essential observation is that we can shift a distance of cyclically consecutive elements of I which equals 2 over consecutive distances equal to 3. More precisely, if there is a set of cyclically consecutive vertices $U = \{v_i, v_{i+1}, ..., v_{i+2+3l+k}\}$ such that the cyclically consecutive elements of I in U have distances $2, 3, 3, ..., 3, k$ with $k \geq 3$, i.e.

$$I \cap U = \{v_i, v_{i+2}, v_{i+2+3}, v_{i+2+6}, v_{i+2+9}, ..., v_{i+2+3l}, v_{i+2+3l+k}\}$$

for some $l, k \in \mathbb{N}$ with $k \geq 3$, then consecutive attacks at

$$v_{i+3}, v_{i+6}, v_{i+9}, ..., v_{i+3(l+1)}$$

result in an independent set I' such that the cyclically consecutive elements of I' in U have distances $3, 3, ..., 3, k-1$, i.e.

$$I' \cap U = \{v_i, v_{i+3}, v_{i+6}, v_{i+9}, ..., v_{i+3(l+1)}, v_{i+2+3l+k}\}.$$

In view of the cardinality of I and the fact that no two vertices in I have distance less than 2, this implies that we may assume that the distances of the cyclically consecutive elements of I are all either 2 or 3 and that all consecutive pairs of distance 2 are consecutive.

If there are three consecutive distances equal to 2, i.e. $v_i, v_{i+2}, v_{i+4}, v_{i+6} \in I$ for some i, then an attack at v_{i+3} cannot be defended. Therefore, we may assume that the consecutive distances are $2, 2, 3, 3, ..., 3$, i.e. $n \equiv 1 (\text{mod } 3)$ and

$$I = \{v_1, v_3, v_5, v_{5+3}, v_{5+6}, ..., v_{5+3 \cdot \frac{n-7}{3}}\}.$$

An attack at v_7 either creates three consecutive distances equal to 2 or the consecutive distances $2, 2, 4, 2, 3, 3, ..., 3$. We have already noted that the defender loses in the first case. In the second case, we can apply the above observation and shift the distance 2 over the consecutive distances equal to 3 which also results in three consecutive distances equal to 2 and the defeat of the defender. This contradiction completes the proof of the claim. $\square$

Claim 2 $\alpha_\infty(C_n) \geq \lceil \frac{n-1}{3} \rceil$.

Proof of Claim 2: For $n \not\equiv 2 (\text{mod } 3)$, Theorem 1 implies $\alpha_\infty(C_n) \geq \alpha(C_n^2) = \lceil \frac{n-1}{3} \rceil$. Hence we may assume that $n = 3k + 2$ for some $k \in \mathbb{N}$ with $k \geq 2$. Let $C_n = v_1 v_2 v_3 ... v_{3k+2} v_1$ where we again identify indices modulo $n = 3k + 2$. We will argue that the defender can win by initially placing his $k + 1$ tokens at the vertices in $I_0 = \{v_1, v_3, v_6, v_9, ..., v_{3k-3}, v_{3k}\}$.

If $I \subseteq V$ is an independent set of C_n, then an interval of consecutive vertices $U = \{v_i, v_{i+1}, ..., v_j\}$ is called an *overfull* interval of I, if $|U \cap I| = l + 4$ and $|U| = j - i + 1 \leq 3l + 7$ for some $l \geq 0$. If I has no overfull interval, I is called *sparse*. Note that I_0 is sparse and that there is always a legal reply to a single attack, if the $k + 1$ tokens are located on the elements of a sparse independent set. Hence it suffices to describe a strategy using only sparse independent sets.

Let the $k + 1$ tokens be located on the elements of a sparse independent set $I \subseteq V$ and consider an attack at a vertex $v_i \in V \setminus I$. We consider different cases. In each case we describe a sparse independent set I' such that there is a legal reply f defending I against an attack at v_i with $f(I) = I'$.

Case 1 $v_{i-3}, v_{i-1} \in I$.

If $v_{i+1} \notin I$, then let $I' = (I \setminus \{v_{i-1}\}) \cup \{v_i\}$ and if $v_{i+1} \in I$, then let $I' = (I \setminus \{v_{i-1}, v_{i+1}\}) \cup \{v_i, v_{i+2}\}$. If I' has an overfull interval, then it has an overfull intervall U' which begins at v_i

and the interval $U = U' \cup \{v_{i-3}, v_{i-2}, v_{i-1}\}$ would be overfull in I, which is a contradiction. Hence I' is sparse.

Case 2 $v_{i-1} \in I$ and $v_{i-3}, v_{i+1} \notin I$.

Let $I'_1 = (I \setminus \{v_{i-1}\}) \cup \{v_i\}$ and $I'_2 = (I \setminus \{v_{i-1}\}) \cup \{v_{i-2}\}$. If $\{I'_1, I'_2\}$ contains a sparse interval I', we are done. Hence we may assume that I'_1 has an overfull interval U'_1 which begins at v_i and that I'_2 has an overfull interval U'_2 which ends at v_{i-2}. Now $U'_2 \cup \{v_{i-1}\} \cup U'_1$ is an overfull interval of I, which is a contradiction.

Case 3 $v_{i-1}, v_{i+1} \in I$ and $v_{i-3}, v_{i+3} \notin I$.

Let $I' = (I \setminus \{v_{i-1}, v_{i+1}\}) \cup \{v_{i-2}, v_{i+2}\}$. If I' contains an overfull intervall U', then it either ends at v_{i-2} or begins at v_{i+2}. In both cases $U = U' \cup \{v_{i-1}, v_i, v_{i+1}\}$ would be an overfull intervall of I, which is a contradiction.

Clearly, the considered three cases exhaust all possibilities up to isomorphism, which completes the proof. $\square$

Proposition 6 *$\overline{\alpha}_\infty(C_n) \le \lceil \frac{n}{3} \rceil$ for $n \ge 3$.*

Proof: Let $C_n = v_1 v_2 v_3 ... v_n v_1$ and let $k = \lceil \frac{n}{3} \rceil$. After consecutive attacks at the vertices $v_1, v_4, v_7, ..., v_{3k-5}$ no more than $k-1$ tokens can be located on vertices in $M := \bigcup_{i=1}^{k-1} N_{C_n}[v_{3i-2}]$: Immediately after the attack at v_{3i-2}, the set $N_{C_n}[v_{3i-2}]$ contains at most one token. If it does contain a token, no further token can enter $N_{C_n}[v_{3i-2}]$ in response to a subsequent attack. If no token is located on a vertex of $N_{C_n}[v_{3i-2}]$, only one token may enter $N_{C_n}[v_{3i-2}]$, namely in response to the attack at v_{3i+1}.

If at this point $V \setminus M$ contains at most one token, the result follows. Hence we may assume that $V \setminus M$ contains two tokens, which implies that $n \equiv 0 (\text{mod } 3)$ and that two tokens are located at v_{3k-3} and v_{3k-1}. Now after further consecutive attacks at $v_2, v_5, ..., , v_{3k-4}$, the set $N := \bigcup_{i=1}^{k-1} N_{C_n}(v_{3i})$ holds at most $k-1$ tokens by the same argument as above. As the token on v_{3k-1} has not moved, there are no more than k tokens on the whole graph. This completes the proof. $\square$

Note that Proposition 6 gives the exact value for $\overline{\alpha}_\infty(C_n)$ unless $n \equiv 1 (\text{mod } 3)$. Using a computer we checked that $\overline{\alpha}_\infty(C_n) = \lceil \frac{n}{3} \rceil$ for $n \in \{10, 13, 16\}$. For the sake of completeness we give a proof for $n = 10$.

Proposition 7 *$\overline{\alpha}_\infty(C_{10}) = 4$.*

Proof: It remains to prove that the defender can adaptively defend four tokens against arbitrary attack sequences. Let $C_{10} = v_1 v_2 ... v_{10} v_1$. We consider the three independent sets $I_1 = \{v_2, v_4, v_6, v_{10}\}$, $I_2 = \{v_2, v_4, v_7, v_9\}$ and $I_3 = \{v_3, v_6, v_8, v_{10}\}$.

If the four tokens are located at the vertices of I_i for some $1 \leq i \leq 3$ and the next attack is at vertex v_j for some $1 \leq j \leq 10$, then the defender will shift some tokens to locations isomorphic to $I_{k(i,j)}$ with

$$\begin{aligned}
(k(1,1), k(1,2), ..., k(1,10)) &= (2,1,*,1,2,1,3,1,3,1), \\
(k(2,1), k(2,2), ..., k(2,10)) &= (3,2,1,2,3,3,2,1,2,3), \text{ and} \\
(k(3,1), k(3,2), ..., k(3,10)) &= (2,1,3,1,2,3,3,3,3,3).
\end{aligned}$$

Note that only attacking v_3 for tokens located at I_1 creates a problem. Since during every move of the defender which creates a token placement isomorphic to I_1, the defender actually has two choices which differ by an isomorphism not fixing v_3, he can always ensure to have a legal reply for any attack sequence. $\Box$

We have seen that for trees and cycles the values of α_∞, $\overline{\alpha}_\infty$ and γ are closely related. The following two examples show that there is no such relation in general.

The graph $H_l = (V_l, E_l)$ with $V_l = \{u, v\} \cup \{u_i, v_i \mid 1 \leq i \leq l\}$ and $E_l = \{uu_i, u_iv_i, v_iv \mid 1 \leq i \leq l\}$ satisfies $\gamma(H_l) = 2$, $\alpha_\infty(H_l) = \overline{\alpha}_\infty(H_l) = l$.

The graph $H'_l = (V'_l, E'_l)$ with $V'_l = \{v_i \mid 1 \leq i \leq l\} \cup \{u_{i,j} \mid 1 \leq i < j \leq l\}$ and $E'_l = \{v_iu_{i,j}, v_ju_{i,j} \mid 1 \leq i < j \leq l\} \cup \{u_{i,j}u_{i',j'} \mid 1 \leq i < j \leq l, 1 \leq i' < j' \leq l, (i,j) \neq (i',j')\}$ satisfies $\gamma(H_l) = \lceil \frac{l}{2} \rceil$ and $\alpha_\infty(H_l) \leq 2$.

In view of the last example, Proposition 3 and the fact that $\alpha(G^2) \leq \gamma(G)$ for every graph G, it is an interesting problem which restrictions on the cycle structure of graphs G in some class of graphs imply the existence of a non-trivial lower bound for $\alpha(G^2)$ in terms of $\gamma(G)$. Our last result establishes such a result for cacti.

Theorem 8 *$2\alpha(G^2) \geq \gamma(G)$ for every cactus $G = (V, E)$.*

Proof: We assume that G is a counterexample for which $|V| + |E|$ is minimum. Since the result clearly holds for K_2 and cycles, G has more than two blocks. Furthermore, it is easy to see that the result holds for graphs which arise by attaching arbitrarily many endvertices to the vertices of a cycle and hence G does not have this structure.

We consider a longest path P in the block-cutvertex incidence tree of G [8]. Let $P = B_1v_1B_0....$ Note that B_1 is an endblock and that v_1 is the cutvertex common to the blocks B_0 and B_1. If P has length at least 3, then let $P = B_1v_1B_0v_0...$ and $V_0 = \{v_0\}$, otherwise let $V_0 = \emptyset$. This implies that all blocks $B_1, B_2, ..., B_l$ which intersect the block B_0 in cutvertices in $B_0 \setminus V_0$ are endblocks. Let $B_i \cap B_0 = \{v_i\}$ for $1 \leq i \leq l$.

We prove a sequence of claims which will imply the desired result. Several times we will consider subgraphs G' of G and use the following abbreviations $\alpha = \alpha(G^2)$, $\gamma = \gamma(G)$, $\alpha' = \alpha((G')^2)$ and $\gamma' = \gamma(G')$.

Claim 1 All blocks in $\{B_1, B_2, ..., B_l\}$ are either K_2, or C_4 or C_5.

Proof of the claim: For contradiction, we assume that $B_1 \cong C_k$ for some $k \in \{3, 6, 7, 8, ...\}$. If $k = 3$, then let G' arise from G by deleting the edge of B_1 not incident with v_1. Clearly,

$\alpha = \alpha'$, $\gamma = \gamma'$ and we obtain the contradiction $2\alpha - \gamma = 2\alpha' - \gamma' \geq 0$. If $k \geq 6$, then let G' arise from G by contracting three consecutive edges of B_1. It is easy to see that $\alpha = \alpha' + 1$ and $\gamma = \gamma' + 1$, which implies a similar contradiction as above. $\Box$

Claim 2 If $v_i = v_j$ for some $1 \leq i < j \leq l$, then $B_i, B_j \cong K_2$.

Proof of the claim: For contradiction, we assume that B_i is either C_4 or C_5. If $G' = G - (B_i \setminus \{v_i\})$, then $\gamma \leq \gamma' + 2$. Since $(G')^2$ has a maximum independent set which does not contain v_i, we obtain $\alpha \geq \alpha' + 1$ and hence $2\alpha - \gamma \geq 2(\alpha' + 1) - (\gamma' + 2) \geq 0$. $\Box$

Claim 3 $B_i \cong K_2$ for $1 \leq i < j \leq l$.

Proof of the claim: For contradiction, we assume that B_i is either C_4 or C_5 for some $1 \leq i \leq l$. By Claim 2, v_i belongs exactly to the two blocks B_0 and B_i. If $G' = G - B_i$, then $\gamma \leq \gamma' + 2$ and $\alpha \geq \alpha' + 1$, which implies a contradiction. $\Box$

Claim 4 If B_0 is a cycle and $1 \leq i \leq l$, then v_i has no neighbour of degree 2 in $B_0 \setminus V_0$.

Proof of the claim: For contradiction, we assume that v_i' is such a neighbour. If $G' = G - v_i'$, then $\gamma = \gamma'$ and $\alpha \geq \alpha'$, which implies a contradiction. $\Box$

Claim 5 $B_0 \cong K_2$.

Proof of the claim: For contradiction, we assume that $B_0 \cong C_k$ for some $k \geq 3$. By Claims 3 and 4, all vertices in $B_0 \setminus V_0$ are cutvertices and the intersecting blocks are all K_2's. If $k = 3$, then let $G' = G - (B_1 \cup B_2 \cup ... \cup B_l)$. Clearly, $\gamma \leq \gamma' + 2$. If I' is a maximum independent set of $(G')^2$, then $I' \setminus \{v_0\}$ together with two endvertices adjacent to the two cutvertices in $B_0 \setminus V_0$ is an independent set in G^2. Hence $\alpha \geq \alpha' + 1$, which implies a contradiction. If $k \geq 4$, then let $G' = G - B_i$ for some $1 \leq i \leq l$ such that v_i has distance at least 2 from V_0. Since $(G')^2$ has a maximum independent set not containing the two neighbours of v_2 in B_0, we obtain $\gamma \leq \gamma' + 1$ and $\alpha \geq \alpha' + 1$, which implies a contradiction. $\Box$

Claim 6 $V_0 \neq \emptyset$.

Proof of the claim: If V_0 is empty, then G is a star and the result clearly holds. $\Box$

Claim 7 v_0 is contained in more than two blocks.

Proof of the claim: If v_0 is contained in exactly two blocks, then $G' = G - (B_0 \cup B_1 \cup ... \cup B_l)$ satisfies $\gamma \leq \gamma' + 1$ and $\alpha \geq \alpha' + 1$, which implies a contradiction. $\Box$

Let $G' = G - ((B_0 \cup B_1 \cup ... \cup B_l) \setminus V_0)$. By the choice of P is follows that $(G')^2$ has a maximum independent set which does not contain v_0. Hence $\gamma \leq \gamma' + 1$ and $\alpha \geq \alpha' + 1$, which implies a final contradiction. $\Box$

We close with some open problems: *What is the exact value of* $\overline{\alpha}_\infty$ *for cycles? What are the values of* α_∞ *and* $\overline{\alpha}_\infty$ *for powers of trees and for cacti? Is it true that* $\frac{2}{3}\gamma(G) \leq \alpha_\infty(G) < 2\gamma(G)$ *if* G *is a cactus (note that the first inequality would be best-possible in view of the graph that arises by identifying one vertex from each of two cycles* C_4 *and that the second inequality would be best-possible in view of the graph that arises by identifying one vertex from each of several cycles* C_6*)? Is* $\overline{\alpha}_\infty(G) \leq \gamma(G)$ *for powers of trees or even for chordal graphs?*

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