Small Perturbation of Selfadjoint and Unitary Operators in Krein Spaces

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Abstract. We investigate the behaviour of the spectrum of selfadjoint operators in Krein spaces under perturbations with uniformly dissipative operators. Moreover we consider the closely related problem of the perturbation of unitary operators with uniformly bi-expansive. The obtained perturbation results give a new characterization of spectral points of positive type and of type π_+ of selfadjoint (resp. unitary) operators in Krein spaces.

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Introduction

A real point λ of the spectrum of a closed operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called a spectral point of positive (negative) type, if for every normed approximative eigensequence (x_n) corresponding to λ all accumulation points of the sequence $([x_n, x_n])$ are positive (resp. negative), see Definition 1.1 below. These spectral points were introduced by P. Lancaster, A. Markus and V. Matsaev in [18] for a bounded operator A which is selfadjoint in the Krein space $(\mathcal{H}, [\cdot,\cdot])$, i.e. the selfadjointness is understood with respect to $[\cdot,\cdot]$. In [20] the existence of a local spectral function was proved for intervals containing only spectral points of positive (negative) type or points of the resolvent set $\rho(A)$. Moreover it was shown that, if A is perturbed by a compact selfadjoint operator, a spectral point of positive type of A becomes either an inner point of the spectrum of the perturbed operator or it becomes an eigenvalue of type π_+. A point from the approximative

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Sadly, our colleague and friend Peter Jonas passed away on July, 18th 2007.

point spectrum of A is of type π_+ if the abovementioned property of approximative eigensequences (x_n) holds only for sequences (x_n) belonging to some linear manifold of finite codimension (see Definition 1.2 below). Every spectral point of a selfadjoint operator in a Pontryagin space with finite rank of negativity is of type π_+. For a detailed study of the properties of the spectrum of type π_+ we refer to [3] and [5].

It is the main aim of this paper to consider perturbations of selfadjoint operators (unitary operators) in some Krein spaces with uniformly dissipative operators (resp. uniformly bi-expansive operators). Let A be a selfadjoint operator in the Krein space $\mathcal{H}$. Let λ_0 be no accumulation point of the non-real spectrum of A and let $(a,b)\setminus\{\lambda_0\}$ consists of spectral points of positive type or of points from the resolvent set of A only. In Section 2 below we show that λ_0 belongs to the spectrum of positive type of A if and only if there exists a fixed open neighbourhood $\mathcal{U}$ of λ_0 such that for all sufficiently small uniformly dissipative operators B the operator $A+B$ has no spectrum inside the intersection of $\mathcal{U}$ and the open lower half-plane. Moreover, the point λ_0 belongs to the spectrum of type π_+ if and only if for all sufficiently small uniformly dissipative operators B the operator $A+B$ has at most finitely many normal eigenvalues inside the intersection of $\mathcal{U}$ and the open lower half-plane. In particular, we are able to show that the sum of all spectral multiplicities within $\mathcal{U}$ intersected with the open lower half-plane equals the rank of negativity of $\kappa_-(E((a',b'))\mathcal{H})$, where $E(\cdot)$ denotes the local spectral function of A. On the other hand, if for every sufficiently small uniformly dissipative operator B the range of the Riesz-Dunford projector corresponding to $A+B$ and the intersection of $\mathcal{U}$ and the open lower half-plane is of infinite dimension, then λ_0 does not belong to $\sigma_{\pi_+}(A)\cup\rho(A)$.

In Section 3 we show that the above arguments hold true in a similar way for uniformly bi-expansive perturbations of unitary operators.

We view these perturbation results also as a new characterization of the spectral points of positive (resp. negative) type and of type π_+ (resp. π_-) of selfadjoint/unitary operators in Krein spaces. We mention that in the early work of L.S. Pontryagin such arguments were used in a similar manner, cf. [22].

Sign type spectrum is used in the theory of indefinite Sturm-Liouville operators, e.g. [4, 6, 8, 17]. Moreover, it is used in the theory of mathematical system theory, see e.g. [12, 13, 19] and in the study of $\mathcal{PT}$-symmetric problems [9, 10, 21].

1. Preliminaries

Let $(\mathcal{H},[\cdot,\cdot])$ be a Krein space. Let A be a closed operator in $\mathcal{H}$. By $\mathcal{L}_\lambda(A)$ we denote the root subspace of A corresponding to λ, i.e. $\mathcal{L}_\lambda(A)=\cup_{n=1}^{\infty}\ker(A-\lambda)^n$. A point $\lambda_0\in\mathbb{C}$ is said to belong to the *approximative point spectrum* $\sigma_{ap}(A)$ of A if there exists a sequence $(x_n)\subset\mathcal{D}(A)$ with $\|x_n\|=1$, $n=1,2,\dots$, and $\|(A-\lambda_0)x_n\|\to 0$ as $n\to\infty$. The boundary points of the spectrum of a closed operator belong to the approximative point spectrum. For a selfadjoint operator A

in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ all real points of the spectrum of A belong to $\sigma_{ap}(A)$ (see e.g. [7, Corollary VI.6.2]).

The following definition is from [1]. In [18], [20] it was given for the case of a bounded selfadjoint operator.

Definition 1.1. *For a closed operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of* positive (negative) type *of A if $\lambda_0 \in \sigma_{ap}(A)$ and for every sequence $(x_n) \subset \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0).$$

We denote the set of all points of positive (negative) type of A by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$).

If the operator A is selfadjoint then the sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\mathbb{R}$ (cf. [20])

In a similar way as in Definition 1.1 we introduce now some subsets of $\sigma(A)$ containing $\sigma_{++}(A)$ and $\sigma_{--}(A)$, respectively, which will play an important role in the following (cf. [1] and for special case of a selfadjoint operator see [3]).

Definition 1.2. *For a closed operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of* type π_+ (type π_-) *of A if $\lambda_0 \in \sigma_{ap}(A)$ and if there exists a subspace $\mathcal{H}_0 \subset \mathcal{H}$ with* $\operatorname{codim} \mathcal{H}_0 < \infty$ *such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, we have*

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0).$$

We denote the set of all points of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$). We call $\mathcal{H}_0$ of minimal codimension *if for each subspace $\mathcal{H}_1 \subset \mathcal{H}$ with* $\operatorname{codim} \mathcal{H}_1 < \operatorname{codim} \mathcal{H}_0$ *there exists a sequence $(x_n) \subset \mathcal{H}_1 \cap \mathcal{D}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0)x_n\| \to 0$ as $n \to \infty$, such that*

$$\liminf_{n\to\infty} [x_n, x_n] \leq 0 \quad (\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] \geq 0).$$

Observe, that for a point $\lambda_0 \in \sigma_{\pi_+}(A)$ we have that $\lambda_0 \in \sigma_{++}(A)$ if and only if the subspace $\mathcal{H}_0$ from Definition 1.2 can be choosen as $\mathcal{H}_0 = \mathcal{H}$.

Recall that an operator C in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called *uniformly dissipative* if there exists some $\alpha > 0$ such that for $x \in \mathcal{D}(C)$ we have $\operatorname{Im}[Cx, x] \geq \alpha\|x\|^2$.

The second part of the following lemma is well-known, nevertheless we give a proof for the sake of completeness.

We set $\mathbb{C}^{\pm} := \{z \in \mathbb{C} | \pm \operatorname{Im} z > 0\}$.

Lemma 1.3. *Let C be a closed uniformly dissipative operator in a Krein space $(\mathcal{H}, [\cdot,\cdot])$. Then*

$$\sigma_{ap}(C) \cap \mathbb{C}^- \subset \sigma_{--}(C).$$

If $\lambda \in \sigma_p(C) \cap \mathbb{C}^-$ then for each $x \in \mathcal{L}_\lambda(C)$, $x \neq 0$, it follows

$$[x, x] < 0.$$

Proof. Let $\lambda_0 \in \sigma_{ap}(C) \cap \mathbb{C}^-$. Then the first statement of Lemma 1.3 follows from the fact that for every sequence $(x_n) \subset \mathcal{D}(C)$ with $\|x_n\| = 1$ and $\|(C - \lambda_0)x_n\| \to 0$, $n \to \infty$, we have

$$|\mathrm{Im}\,[Cx_n, x_n] - \mathrm{Im}\,\lambda_0[x_n, x_n]| \le \|(C - \lambda_0)x_n\| \to 0, n \to \infty.$$

Let $\lambda \in \sigma_p(C) \cap \mathbb{C}^-$. It follows from [2, Ch. 2, Corollary 2.17] that for each $y \in \mathcal{L}_\lambda(C)$ we have $[y, y] \le 0$. Assume that there exists an $x \in \mathcal{L}_\lambda(C)$, $x \ne 0$, with $[x, x] = 0$. Then we have $[x, y] = 0$ for all $y \in \mathcal{L}_\lambda(C)$. Hence

$$0 = \mathrm{Im}\,[(C - \lambda)x, x],$$

which is a contradiction to the assumption that C is uniformly dissipative. □

2. Uniformly dissipative perturbations of selfadjoint operators in Krein spaces

Let A be a selfadjoint operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$, that is, $A = A^+$. Let B be a bounded uniformly dissipative operator in the Krein space $(\mathcal{H}, [\cdot, \cdot])$. Then the operator $A + B$, which is defined on $\mathcal{D}(A)$, is uniformly dissipative.

Lemma 2.1. *Let A be a selfadjoint operator and let B be a bounded uniformly dissipative operator in $\mathcal{H}$. Then*

$$\mathbb{R} \subset \rho(A + B).$$

Proof. Set $C := A + B$. We choose $\alpha > 0$ such that $\mathrm{Im}\,[Bx, x] \ge \alpha\|x\|^2$, $x \in \mathcal{H}$. We have $\mathcal{D}(C) = \mathcal{D}(A) = \mathcal{D}(C^+)$ and, therefore, for $\lambda \in \mathbb{R}$ and $x \in \mathcal{D}(C)$, $x \ne 0$, it follows

$$\|x\|\|(C - \lambda)x\| \ge |[(C - \lambda)x, x]| \ge |\mathrm{Im}\,[Bx, x]| \ge \alpha\|x\|^2$$

and

$$\|(C^+ - \lambda)x\| \ge \alpha\|x\|.$$

As C is a closed operator the point λ belongs to $\rho(C)$. □

Lemma 2.2. *Let $\mu \in \sigma_{++}(A)$. Then there exists a $\delta > 0$ and an $\varepsilon > 0$ such that for all bounded uniformly dissipative operators in $\mathcal{H}$ with $\|B\| \le \varepsilon$ it follows that the intersection of $\mathbb{C}^-$ and the disc around μ with radius δ belongs to $\rho(A + B)$.*

Proof. Assume that the assertion of Lemma 2.2 is not true. Then there exist a sequence of bounded uniformly dissipative operators (B_n) in $\mathcal{H}$ with $\|B_n\| \to 0$, $n \to \infty$, and a sequence (λ_n) in $\sigma(A+B_n) \cap \mathbb{C}^-$ which converges to μ, $\mu \in \sigma_{++}(A)$. We assume $\lambda_n \in \sigma_{ap}(A+B_n)$, $n \in \mathbb{N}$. In view of Lemma 2.1 this is no restriction. By Lemma 1.3 there exists a sequence (x_n), $x_n \in \mathcal{D}(A + B_n) = \mathcal{D}(A)$ with $\|x_n\| = 1$, $[x_n, x_n] < 0$ and $\|(A + B_n - \lambda_n)x_n\| \le \frac{1}{n}$, $n \in \mathbb{N}$. Then $\liminf_{n\to\infty}[x_n, x_n] \le 0$ and

$$(A - \mu)x_n = (A + B_n - \lambda_n)x_n + (\lambda_n - \mu)x_n - B_n x_n \to 0, \quad n \to \infty,$$

which contradicts $\mu \in \sigma_{++}(A)$. □

Proposition 2.3. *Let A be a selfadjoint operator. Assume that λ_0, $\lambda_0 \in (a,b)$, is not an accumulation point of the non-real spectrum of A and that*

$$(a,b) \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A) \tag{2.1}$$

holds. Let $a < a' < \lambda_0 < b' < b$. Then there exists a $\delta' > 0$ such that the strip

$$\{\lambda \in \mathbb{C}^- : a' \leq Re\,\lambda \leq b', -\delta' \leq Im\,\lambda < 0\}$$

belongs to the resolvent set of A. Moreover, if $\gamma_{\delta'}$ denotes the closed oriented curve in the complex plane which consists of the line segments connecting the points $b', b' - i\delta', a' - i\delta', a'$ and b' then there exists an $\varepsilon_0 > 0$ such that for all bounded uniformly dissipative operators B in $\mathcal{H}$ with $\|B\| \leq \varepsilon_0$ we have

$$\gamma_{\delta'} \subset \rho(A + B). \tag{2.2}$$

Proof. The first statement of Proposition 2.3 follows from [20] (or [3]). In order to show (2.2) we choose $\varepsilon_0 > 0$ so small that, cf. Lemma 2.2, for all bounded uniformly dissipative operators B in $\mathcal{H}$ with $\|B\| \leq \varepsilon_0$ the line segments connecting the points b' and $b' - i\delta'$ and the points a' and $a' - i\delta'$ belong to $\rho(A + B)$. Moreover, we choose ε_0 so small that

$$\varepsilon_0 < \frac{1}{\max_{\lambda \in \Gamma} \|(A - \lambda)^{-1}\|}$$

holds, where Γ is the line segment connecting the points $b' - i\delta'$ and $a' - i\delta'$. As $A + B - \lambda = (I + B(A - \lambda)^{-1})(A - \lambda)$, Γ is a subset of $\rho(A + B)$. Moreover, by Lemma 2.1, $\mathbb{R} \subset \rho(A + B)$, hence Proposition 2.3 is proved. □

The following theorem can be considered as the main result of this paper. Recall that for a selfadjoint operator satisfying (2.1) there exists a local spectral function E defined on subintervals of (a,b) with endpoints not equal to a, b or λ_0, cf. [3], [15]. In particular there exists the spectral projection $E((a',b'))$ corresponding to the interval (a',b') with $a < a' < \lambda_0 < b' < b$.

Theorem 2.4. *Let A be a selfadjoint operator in the Krein space $\mathcal{H}$. Assume that λ_0, $\lambda_0 \in (a,b)$, is not an accumulation point of the non-real spectrum of A and that*

$$(a,b) \setminus \{\lambda_0\} \subset \sigma_{++}(A) \cup \rho(A). \tag{2.3}$$

Let a', b', δ', ε_0 and $\gamma_{\delta'}$ be as in Proposition 2.3. Then the following assertions are valid.

(i) *The point λ_0 belongs to $\sigma_{++}(A) \cup \rho(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the operator $A + B$ has no spectrum inside the curve $\gamma_{\delta'}$.*

(ii) *The point λ_0 belongs to $\sigma_{\pi_+}(A)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ the spectrum of $A + B$ inside the curve $\gamma_{\delta'}$ consists of at most finitely many normal eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_{\lambda_j}(A + B) : 1 \leq j \leq k\}$$

is of finite dimension. Moreover, in this case, the dimension of $\mathcal{M}_-$ *is equal to the rank of negativity* $\kappa_-(E((a',b'))\mathcal{H})$ *of the Pontryagin space* $E((a',b'))\mathcal{H}$*, that is*

$$\dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}).$$

(iii) *The point* λ_0 *does not belong to* $\sigma_{\pi_+}(A) \cup \rho(A)$ *if and only if there exists an* $\varepsilon_1 > 0$ *such that for every uniformly dissipative operator* B *acting in* $\mathcal{H}$ *with* $\|B\| < \varepsilon_1$ *the range of the Riesz-Dunford projector corresponding to* $A + B$ *and* $\gamma_{\delta'}$ *is of infinite dimension.*

Proof. Let a', b', δ', ε_0 and $\gamma_{\delta'}$ be as in Proposition 2.3. Set $\mathcal{K} := (I - E((a',b')))\mathcal{H}$. Then the space $\mathcal{H}$ decomposes

$$\mathcal{H} = E((a',b'))\mathcal{H}\,[\dot{+}]\,\mathcal{K}, \tag{2.4}$$

where $[\dot{+}]$ denote the direct sum of spaces which are orthogonal with respect to $[\cdot,\cdot]$. Moreover,

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_0 & B_{01} \\ B_{10} & B_1 \end{bmatrix}$$

with respect to the decomposition (2.4). The operators B_0 and B_1 are uniformly dissipative operators in $E((a',b'))\mathcal{H}$ and $\mathcal{K}$, respectively. As E is the spectral function of A, we have

$$\sigma(A_0) \subset [a',b'] \quad \text{and} \quad \sigma(A_1) \subset \mathbb{R} \setminus (a',b').$$

By assumption, a' and b' belong to $\sigma_{++}(A) \cup \rho(A)$. Lemma 2.2 implies the existence of $\delta > 0$ and $\varepsilon > 0$ such that for all bounded uniformly dissipative operators in $\mathcal{H}$ with $\|B\| \le \varepsilon$ it follows that the intersection of $\mathbb{C}^-$ and the discs around a' and b' with radius δ belong to the resovent set of the operator

$$\begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} + \begin{bmatrix} B_0 & 0 \\ 0 & B_1 \end{bmatrix}.$$

Denote by $\Gamma_{\delta'}$ the open set in $\mathbb{C}$ which has as its boundary the curve $\gamma_{\delta'}$, that is $\Gamma_{\delta'} = \{\lambda \in \mathbb{C} : a' < \operatorname{Re}\lambda < b', -\delta' < \operatorname{Im}\lambda < 0\}$. It follows from [16, IV.§3.1] that there is an $\varepsilon_1 > 0$, $\varepsilon_1 < \min\{\varepsilon, \varepsilon_0\}$, such that for all uniformly dissipative operators B acting in $\mathcal{H}$ with $\|B\| < \varepsilon_1$ we have

$$\sigma(A_0 + B_0) \subset \{\lambda \in \mathbb{C} : \operatorname{dist}(\lambda, [a',b']) < \min\{\delta, \delta'\}\} \tag{2.5}$$

and

$$\overline{\Gamma_{\delta'}} \subset \rho(A_1 + B_1). \tag{2.6}$$

Then Lemma 2.1, Proposition 2.3 and (2.5) imply that for all uniformly dissipative operators B with $\|B\| < \varepsilon_1$

$$\sigma(A_0 + B_0) \cap \mathbb{C}^- \subset \Gamma_{\delta'}. \tag{2.7}$$

Now we assume that λ_0 belongs to $\sigma_{\pi_+}(A)$. Then $(E((a',b')), [\cdot,\cdot])$ is a Pontryagin space with a finite rank of negativity and, if $\lambda_0 \in \sigma_{++}(A)$, it is even a Hilbert space (cf. [3, Theorems 23 and 24]). An application of [11, Theorem 11.6] implies that $\sigma(A_0+B_0)\cap\mathbb{C}^-$ consists of at most finitely many eigenvalues and that

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_\lambda(A_0+B_0) : \lambda\in\sigma(A_0+B_0)\cap\mathbb{C}^-\}$$

is a maximal uniformly negative subspace of $E((a',b'))\mathcal{H}$ invariant under A_0+B_0. Therefore

$$\dim\mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H})$$

and relations (2.7) and (2.6) imply that the operator $A+B$ has the properties stated in assertions (i) and (ii) if $B_{01}=B_{10}=0$. If $B_{01}\neq 0$ or $B_{10}\neq 0$ we consider the operators

$$\begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix} + \begin{bmatrix} B_0 & tB_{01} \\ tB_{10} & B_1 \end{bmatrix},$$

where t runs through $[0,1]$. Then by [16, IV.§3.4], Lemma 1.3 and Proposition 2.3 the operator $A+B$ has the properties stated in assertions (i) and (ii).

It remains to consider the case $\lambda_0\notin\sigma_{\pi_+}(A)$. Assume that the range of the Riesz-Dunford projector P_- corresponding to A_0+B_0 and $\gamma_{\delta'}$ is of finite dimension. Then, by Lemma 1.3, it is a uniformly negative subspace of $E((a',b'))\mathcal{H}$. Moreover the range of the Riesz-Dunford projector P_+ corresponding to A_0+B_0 and $\sigma(A_0+B_0)\cap\mathbb{C}^+$ is a nonnegative subspace (cf. [2]) and we have

$$E((a',b'))\mathcal{H} = P_+E((a',b'))\mathcal{H}[\dot{+}]P_-E((a',b'))\mathcal{H}.$$

We claim that $P_-E((a',b'))\mathcal{H}$ is a maximal uniformly negative subspace of the Krein space $E((a',b'))\mathcal{H}$. Indeed, assume that there exists a maximal uniformly negative subspace $\widetilde{\mathcal{M}}_-$ with $P_-E((a',b'))\mathcal{H}\subset\widetilde{\mathcal{M}}_-$ and there exists some x, $x\in\widetilde{\mathcal{M}}_-\setminus P_-E((a',b'))\mathcal{H}$. Then $[x-P_-x,x-P_-x]<0$ holds. But this is a contradiction to $x-P_-x=P_+x\in P_+E((a',b'))\mathcal{H}$.

Therefore the Krein space $E((a',b'))\mathcal{H}$ has a finite dimensional maximal uniformly negative subspace, hence $E((a',b'))\mathcal{H}$ is a Pontryagin space. But this is impossible as $\lambda_0\notin\sigma_{\pi_+}(A)$ (cf. [3, Theorem 24]) and the operator $A+B$ has the properties stated in assertions (iii) if $B_{01}=B_{10}=0$. If $B_{01}\neq 0$ or $B_{10}\neq 0$ then a similar reasoning as above shows that assertion (iii) holds and Theorem 2.4 is proved. □

Corollary 2.5. *Let λ_0, $\lambda_0\in(a,b)$, belongs to $\sigma_{\pi_+}(A)\setminus\sigma_{++}(A)$ and choose $\mathcal{H}_0$ as in Definition 1.2 such that $\mathcal{H}_0$ is of minimal codimension. Assume that λ_0 is not an accumulation point of the non-real spectrum of A and that (2.1) holds. Let a', b', ε_1 and $\mathcal{M}_-$ be as in Theorem 2.4. Then we have for every uniformly dissipative operator B acting in $\mathcal{H}$ with $\|B\|<\varepsilon_1$*

$$\operatorname{codim}\mathcal{H}_0 \le \dim\mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}). \tag{2.8}$$

Moreover, let

$$\ker A = \mathcal{N}_0[\dot{+}]\mathcal{N}_+[\dot{+}]\mathcal{N}_- \quad \textit{and} \quad \mathcal{L}_{\lambda_0}(A) = \mathcal{L}_0[\dot{+}]\mathcal{L}_+[\dot{+}]\mathcal{L}_-$$

be fundamental decompositions of $\ker A$ *and* $\mathcal{L}_{\lambda_0}(A)$*, respectively, that is,* $\mathcal{N}_0 = \mathit{ker}\, A \cap (\mathit{ker}\, A)^{[\perp]}$*,* $\mathcal{L}_0 = \mathcal{L}_{\lambda_0}(A) \cap (\mathcal{L}_{\lambda_0}(A))^{[\perp]}$*,* $\mathcal{N}_+$*,* $\mathcal{L}_+$ *are positive subspaces of* $E((a',b'))\mathcal{H}$ *and* $\mathcal{N}_-$*,* $\mathcal{L}_-$ *are negative subspace of* $E((a',b'))\mathcal{H}$*. We have equality in* (2.8)*, that is,*

$$\operatorname{codim} \mathcal{H}_0 = \dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H})$$

if and only if

$$\mathit{dim}\, \mathcal{N}_0 + \mathit{dim}\, \mathcal{N}_- = \mathit{dim}\, \mathcal{L}_0 + \mathit{dim}\, \mathcal{L}_-.$$

In this case we have

$$\mathit{dim}\, \mathcal{N}_0 + \mathit{dim}\, \mathcal{N}_- = \mathit{dim}\, \mathcal{L}_0 + \mathit{dim}\, \mathcal{L}_- = \operatorname{codim} \mathcal{H}_0 = \dim \mathcal{M}_- = \kappa_-(E((a',b'))\mathcal{H}).$$

Proof. Choose a fundamental decomposition for the Pontryagin space $E((a',b'))\mathcal{H}$, $E((a',b'))\mathcal{H} = \Pi_+ \,[\dot{+}]\, \Pi_-$. Then $\Pi_+ \,[\dot{+}]\, (I - E((a',b')))\mathcal{H}$ is of finite codimension in $\mathcal{H}$ and an easy calculation shows that (2.8) holds. The remaining statements of Corollary 2.5 follows from [5, Theorem 3.6]. □

We refer to [5] for an example such that the inequality in (2.8) is strict.

3. Uniformly bi-expansive perturbations of unitary operators in Krein spaces

A bounded operator U in a Krein space $(\mathcal{H}, [\cdot,\cdot])$ is called unitary if U is surjective and $[Ux, Ux] = [x,x]$ for all $x \in \mathcal{H}$.

A bounded operator V is said to be bi-expansive if both V and V^+ are noncontractive with respect to $[\cdot,\cdot]$, that is,

$$[Vx, Vx] \geq [x,x] \quad \text{and} \quad [V^+x, V^+x] \geq [x,x] \quad \text{for all} \quad x \in \mathcal{H}.$$

The operator V is called uniformly bi-expansive if the operator V is bi-expansive and there is an $\alpha_V > 0$ such that $[Vx, Vx] \geq [x,x] + \alpha_V \|x\|^2$. If V is uniformly bi-expansive then also V^+ is uniformly bi-expansive and $\alpha_{V^+} = \alpha_V$.

For every uniformly bi-expansive operator V we have

$$\mathbb{T} \subset \rho(V), \tag{3.1}$$

where $\mathbb{T}$ denote the unit circle $\mathbb{T} = \{\lambda \mid |\lambda| = 1\}$ (see, e.g., [2, Theorem 2.4.31]). The operator

$$A := i(V+1)(V-1)^{-1} \tag{3.2}$$

is called the Caley-Neumann transformation of V. If V is a uniformly bi-expansive operator then we have for $x \in \mathcal{H}$ with $y := (V-1)x$,

$$\begin{aligned} \operatorname{Im}[Ay, y] &= \operatorname{Re}\,([(V+I)x, (V-I)x]) = \operatorname{Re}\,([Vx,Vx] + [x,Vx] - [Vx,x] - [x,x]) \\ &= [Vx,Vx] - [x,x] \end{aligned}$$

and A is uniformly dissipative.

It is well-known that the classes of selfadjoint and unitary operators (as well as the classes of bounded uniformly dissipative operators and uniformly bi-expansive operators) are closely connected via Caley-Neumann transformation. It is a natural idea to prove similar results as in the previous sections using Caley-Neumann transformation for bi-expansive perturbations of unitary operators. But this does not work in general since the image of an unbounded uniformly dissipative operator $A + B$ need not to be a uniformly bi-expansive operator. Because of this in this section we follow the same ideas as in the previous sections replacing dissipative perturbations of selfadjoint operators by bi-expansive perturbations of unitary operators.

The following lemma is an analog of Lemma 1.3. Let $\mathbb{D}$ denotes the open unit disc,

$$\mathbb{D} := \{\lambda \in \mathbb{C} \mid \lambda| < 1\}.$$

Lemma 3.1. *Let V be a uniformly bi-expansive operator in a Krein space $(\mathcal{H}, [\cdot, \cdot])$. Then*

$$\sigma_{ap}(V) \cap \mathbb{D} \subset \sigma_{--}(V);$$

If $\lambda \in \sigma_p(V) \cap \mathbb{D}$ then for each $x \in \mathcal{L}_\lambda(V)$, $x \neq 0$, it follows $[x, x] < 0$.

Proof. Let $\lambda_0 \in \sigma_{ap}(V) \cap \mathbb{D}$. Let (x_n) be a sequence with $\|x_n\| = 1$, $n \in \mathbb{N}$, and $(V - \lambda_0)x_n \to 0$ as $n \to \infty$. Since

$$[Vx_n, Vx_n] - |\lambda_0|^2[x_n, x_n] = [(V - \lambda_0)x_n, Vx_n] + \lambda_0[x_n, (V - \lambda_0)x_n],$$

we have

$$0 = \liminf_{n\to\infty}[Vx_n, Vx_n] - |\lambda_0|^2 \liminf_{n\to\infty}[x_n, x_n] \geq (1 - |\lambda_0|^2) \liminf_{n\to\infty}[x_n, x_n] + \alpha_V.$$

Hence

$$\liminf_{n\to\infty}[x_n, x_n] \leq -\frac{\alpha_v}{1 - |\lambda_0|^2} < 0.$$

Now we show that $\mathcal{L}_\lambda(V)$ is a negative subspace, i.e. $[x, x] < 0$ for all non-zero $x \in \mathcal{L}_\lambda(V)$. By (3.1), $1 \in \rho(V)$ and we consider the Caley-Neumann transformation A of V, cf. (3.2). The operator A is uniformly dissipative. Since $\mathcal{L}_\lambda(V) = \mathcal{L}_\mu(A)$ for $\mu = i\frac{\lambda+1}{\lambda-1}$, the statement follows from Lemma 1.3. □

Lemma 3.2. *Let U be a unitary operator and let $\mu \in \sigma_{++}(U)$. Then $|\mu| = 1$ and there exist a $\delta > 0$ and an $\varepsilon > 0$ such that for all uniformly bi-expansive operators V with $\|I - V\| \leq \varepsilon$ it follows that the intersection of $\mathbb{D}$ and the disc around μ with radius δ belongs to $\rho(UV)$.*

Proof. First we show that $|\mu| = 1$. Assume the contrary: $|\mu| \neq 1$. Let $\|z_n\| = 1$, $n \in \mathbb{N}$, and $(U - \mu)z_n \to 0$ as $n \to \infty$. Since

$$(1 - |\mu|^2)[z_n, z_n] = [Uz_n, Uz_n] - |\mu|^2[z_n, z_n] = [(U - \mu)z_n, Uz_n] + \mu[z_n, (U - \mu)z_n]$$

we have $\lim_{n\to\infty}[z_n, z_n] = 0$ which contradicts to $\mu \in \sigma_{++}(U)$.

Assume now that the second assertion of the lemma is not true. Then there exists a sequence of uniformly bi-expansive operators V_n in $\mathcal{H}$ with $V_n \to I$ as

$n \to \infty$ and a sequence $(\lambda_n) \subset \sigma(UV_n) \cap \mathbb{D}$ which converges to $\mu \in \sigma_{++}(U)$. In view of (3.1) it is no restriction if we assume that $\lambda_n \in \sigma_{ap}(UV_n)$, $n \in \mathbb{N}$. By Lemma 3.1 there exists a sequence (x_n) with $\|x_n\| = 1$, $[x_n, x_n] < 0$ and $\|(UV_n - \lambda_n)x_n\| \leq \frac{1}{n}$. Then $\liminf_{n\to\infty}[x_n, x_n] \leq 0$ and as $n \to \infty$ we have

$$(U - \mu)x_n = (UV_n - \lambda_n)x_n + (\lambda_n - \mu)x_n - U(V_n - I)x_n \to 0$$

which contradicts $\mu \in \sigma_{++}(U)$. □

Assume

$$\varphi, \psi \in [0, 2\pi), \ \varphi < \psi \quad \text{and} \quad \delta \in (0, 1).$$

Denote by $\omega_{\varphi,\psi}$ the open arc of the unit circle given by

$$\omega_{\varphi,\psi} := \{\lambda = e^{i\eta} \mid \varphi < \eta < \psi\},$$

by $\Omega_{\varphi,\psi,\delta}$ the part of the sector generated by $\omega_{\varphi,\psi}$,

$$\Omega_{\varphi,\psi,\delta} := \{\lambda = re^{i\eta} \mid \varphi \leq \eta \leq \psi, \ 1 - \delta \leq r < 1\},$$

and by $\gamma_{\varphi,\psi,\delta}$ the boundary of $\Omega_{\varphi,\psi,\delta}$.

Proposition 3.3. *Let U be a unitary operator. Assume $\lambda_0 = e^{i\eta_0}$, $\lambda_0 \in \omega_{\varphi,\psi}$, is not an accumulation point of $\sigma(U) \setminus \mathbb{T}$ and that*

$$\omega_{\varphi,\psi} \setminus \{\lambda_0\} \subset \sigma_{++}(U) \cup \rho(U). \tag{3.3}$$

Let

$$\varphi < \varphi' < \eta_0 < \psi' < \psi.$$

Then there exists a $\delta' > 0$ such that $\Omega_{\varphi',\psi',\delta'} \subset \rho(U)$.

Moreover, there exists an $\varepsilon_0 > 0$ such that for all uniformly bi-expansive operators V with $\|I - V\| < \varepsilon_0$ we have $\gamma_{\varphi',\psi',\delta'} \subset \rho(UV)$.

Proof. We omit the proof since it repeats similar arguments as we used in the proof of Proposition 2.3 . □

A unitary operator in a Krein space satisfying (3.3) has a local spectral function E defined on subarcs of $\omega_{\varphi,\psi}$ with endpoints not equal to $e^{i\varphi}, e^{i\psi}$ or λ_0, cf. [15]. In particular there exists the spectral projection $E(\omega_{\varphi',\psi'})$ corresponding to the subarc $\omega_{\varphi',\psi'}$ with $\varphi < \varphi' < \eta_0 < \psi' < \psi$.

Theorem 3.4. *Let U be a unitary operator in the Krein space $\mathcal{H}$. Assume that $\lambda_0 = e^{i\eta_0}$, $\lambda_0 \in \omega_{\varphi,\psi}$, is not an accumulation point of $\sigma(U) \setminus \mathbb{T}$ and that*

$$\omega_{\varphi,\psi} \setminus \{\lambda_0\} \subset \sigma_{++}(U) \cup \rho(U).$$

Let a', b', δ', ε_0 and $\gamma_{\varphi',\psi',\delta'}$ be as in Proposition 3.3. Then the following assertions are valid.

(i) *The point λ_0 belongs to $\sigma_{++}(U) \cup \rho(U)$ if and only if there exists an $\varepsilon_1 > 0$ such that for every uniformly bi-expansive operator V acting in $\mathcal{H}$ with $\|I - V\| < \varepsilon_1$ the operator UV has no spectrum inside the curve $\gamma_{\varphi',\psi',\delta'}$.*

(ii) *The point* λ_0 *belongs to* $\sigma_{\pi_+}(U)$ *if and only if there exists an* $\varepsilon_1 > 0$ *such that for every uniformly bi-expansive operator* V *acting in* $\mathcal{H}$ *with* $\|I - V\| < \varepsilon_1$ *the spectrum of* UV *inside the curve* $\gamma_{\varphi',\psi',\delta'}$ *consists of at most finitely many normal eigenvalues* $\lambda_1, \lambda_2, \ldots, \lambda_k$. *Then*

$$\mathcal{M}_- := \operatorname{span}\{\mathcal{L}_{\lambda_j}(UV) : 1 \leq j \leq k\}$$

is of finite dimension and moreover, in this case, the dimension of $\mathcal{M}_-$ *is equal to the rank of negativity* $\kappa_-(E(\omega_{\varphi',\psi'})\mathcal{H})$ *of the Pontryagin space* $E(\omega_{\varphi',\psi'})\mathcal{H}$, *that is*

$$\dim \mathcal{M}_- = \kappa_-(E(\omega_{\varphi',\psi'})\mathcal{H}).$$

(iii) *The point* λ_0 *does not belong to* $\sigma_{\pi_+}(U) \cup \rho(U)$ *if and only if there exists an* $\varepsilon_1 > 0$ *such that for every uniformly bi-expansive operator* V *acting in* $\mathcal{H}$ *with* $\|I - V\| < \varepsilon_1$ *the range of the Riesz-Dunford projector corresponding to* UV *and* $\gamma_{\varphi',\psi',\delta'}$ *is of infinite dimension.*

Proof. is similar to the proof of Theorem 2.4. □

We left it to the reader to formulate and to prove statements like Corollary 2.5 for operators UV, where U is a unitary and V is a bi-expansive operator.

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