Normal form for linear systems with respect to its vector relative degree

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Abstract

For multi-input multi-output (MIMO) linear systems with existing vector relative degree a normal form is constructed. This normal form is not only structural simple but allows to characterize the system's zero dynamics for the design of feedback controllers. A characterization of the zero dynamics in terms of the normal form is given.

Keywords: linear ystems, MIMO systems, vector relative degree, normal form

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Nomenclature

$\left[l_1^{(n)}, \ldots, l_m^{(n)}\right] = L \in \mathbb{R}^{n\times m}$,	where $l_i^{(n)} \in \mathbb{R}^n$ denotes the i-th column of L and the superscript (n) remarks the dimension of the vector,
$\left[\left(l_{(m)}^1\right)^T, \ldots, \left(l_{(m)}^n\right)^T\right]^T = L \in \mathbb{R}^{n\times m}$,	where $l_{(m)}^j \in \mathbb{R}^{1\times m}$ denotes the j-th row of L and the subscript (m) remarks the dimension of the row-vector,
$e_k^{(n)} := \left[0_{1\times(k-1)}, 1, 0_{1\times(n-k)}\right]^T$,	the k-th row unit vector in $\mathbb{R}^n$,
$e_{(m)}^k := \left[0_{1\times(k-1)}, 1, 0_{1\times(m-k)}\right]$,	the k-th row unit vector in $\mathbb{R}^{1\times m}$,
$0_{n\times m} \in \mathbb{R}^{n\times m}$,	the 0-matrix of dimension $n \times m$,
$\mathcal{X}_{n\times m} \in \mathbb{R}^{n\times m}$,	an arbitrarily matrix of dimension $n\times m$; note that the use of this symbol implicates that the specific entries of the matrix are not important but only the dimension,
$I_n \in \mathbb{R}^{n\times n}$,	the identity matrix of dimension $n \times n$,
$\mathcal{C}^m([0,\infty) \to \mathbb{R}^n)$,	the set of m-times continuously differentiable maps from $[0,\infty)$ to $\mathbb{R}^n$,
$\mathcal{C}_{\mathrm{pw}}([0,\infty) \to \mathbb{R}^m)$,	the set of piecewise continuous maps from $[0,\infty)$ to $\mathbb{R}^m$.

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1 Introduction

In the present work linear systems with m inputs and m outputs of the form

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + \underbrace{\left[b_1^{(n)}, \ldots, b_m^{(n)}\right]}_{=B} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}}_{=u} \\ \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{=y} &=& \underbrace{\begin{bmatrix} c_{(n)}^1 \\ \vdots \\ c_{(n)}^m \end{bmatrix}}_{=C} x \end{array}\right\} \tag{1.1}$$

are considered, where $n, m \in \mathbb{N}$ with $m \leq n$ and $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$.

It is well known that a linear single-input single-output (SISO) system (1.1), i.e. $m = 1$, has relative degree $r \in \mathbb{N}$ if, and only if, r is exactly the number of times one has to differentiate the output to have the input appear explicitly.

In case of MIMO system (1.1), for every $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, m\}$, one can consider the SISO-system relating input u_j to output y_i given by

$$\left.\begin{array}{rcl} \dot{x} &=& Ax + b_j^{(n)} u_j \\ y_i &=& c_{(n)}^i x\,, \qquad\qquad i, j \in \{1, \ldots, m\}\,. \end{array}\right\} \tag{1.2}$$

Let $r_{i,j} \in \mathbb{N}$ be the relative degree of (1.2). Then, for $i \in \{1, \ldots, m\}$, $r_i := \min_{j\in\{1,\ldots,m\}} r_{i,j}$ is exactly the number one has to differentiate the i-th output to have at least one of the m inputs appear explicitly. The vector $(r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ is called the vector relative degree of the MIMO-system (1.1) if, for all $j \in \{1, \ldots, m\}$, the rows $c_{(n)}^j A^{r_j-1} B$ are linearly independent, see Definition 2.1(a).

Isidori [9] presents a local definition of the vector relative degree for nonlinear MIMO-systems.

Liberzon et al. [10] give a generalization of the relative degree for time-invariant nonlinear systems which is extended in [5] for time-varying linear and nonlinear systems. However in these papers only SISO-systems and MIMO-systems with strict relative degree (see Definition 2.1(c)) are considered.

The relative degree of a system leads to a normal form. For linear SISO-systems one can construct an invertible matrix $U \in \mathbb{R}^{n\times n}$ such that the coordinate transformation $\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right) = Ux$ converts a linear system

$$\dot{x} = Ax + bu\,, \quad y = cx\,, \tag{1.3}$$

with $A \in \mathbb{R}^{n\times n}$ and $b, c^T \in \mathbb{R}^n$, which has relative degree $r \in \mathbb{N}$, into

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} &=& \begin{bmatrix} 0 & 1 & & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ 0 & \ldots & 0 & 1 & 0 \\ R_1 & \ldots & & R_r & S \\ P & 0 & \ldots & 0 & Q \end{bmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ cA^{r-1}b \\ 0 \end{bmatrix} u \\ y &=& [1, 0, \ldots, 0] \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \end{array} \tag{1.4}$$

where $R_1, \ldots, R_r \in \mathbb{R}$, $S \in \mathbb{R}^{1\times n-r}$, $P \in \mathbb{R}^{n-r}$ and $Q \in \mathbb{R}^{(n-r)\times(n-r)}$ can be presented explicitly in terms of the system matrices A, b and c, see [7]. This result is implicitly contained in [9, Chapter 4.1].

The Byrnes–Isidori normal form for nonlinear and linear SISO-systems, introduced in [1], is widely used in control theory for the design of local and global feedback stabilization of nonlinear systems [2], [3], [4], for the design of adaptive observers [11], for the design of adaptive controllers for linear systems [8], [6], to name but a few applications. Thus a construction of a normal form for MIMO-systems will assist the design of controllers and observers for MIMO-systems.

Isidori [9, Chapter 5] presents a local normal form for nonlinear MIMO-systems systems, without specifying the diffeomorphism in terms of the system data which converts the system into a normal form. In the present work, for the linear case a transformation in terms of A, B and C is designed which leads to "many zeros and ones" in the normal form and allows to read off the zero dynamics very easily. The reader will find that the normal form (2.1) for linear MIMO-systems is, roughly speaking, structured as a "diagonal form of m copies of SISO normal forms (1.4)". Furthermore the matrices of the normal form will be characterised explicitly by the system matrices.

The present paper is structured as follows. In Section 2 the main results, i.e. the normal form for linear MIMO-systems is presented and the system's zero dynamics is characterized. Furthermore the inverse of the system ([9, Chapter 5.1]) is presented. Section 3 contains all the proofs.

2 Normal form and zero dynamics

Consider, for $n, m \in \mathbb{N}$ with $m \leq n$ and $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, a linear system (1.1), that is a linear system with m-dimensional input u and output y of form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx\,. \end{aligned}$$

For linear MIMO-systems the vector relative degree is defined as follows.

Definition 2.1

(a) A linear system (A, B, C) of form (1.1) has *(vector) relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ if, and only if,

(i) $\forall\, j \in \{1, \ldots, m\}\ \forall\, k \in \{0, \ldots, r_j - 2\}\ :\ c^j_{(n)} A^k B = 0_{1\times m}$,

(ii) $\operatorname{rk} \begin{bmatrix} c^1_{(n)} A^{r_1-1} B \\ c^2_{(n)} A^{r_2-1} B \\ \vdots \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix} = m.$

(b) A linear system (A, B, C) of form (1.1) has *ordered (vector) relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ if, and only if, (1.1) has (vector) relative degree $r = (r_1, \ldots, r_m)$ with $r_1 \geq r_2 \geq \ldots \geq r_m$.

(c) A linear system (A, B, C) of form (1.1) has *strict relative degree* $\varrho \in \mathbb{N}$ if, and only if, (1.1) has (vector) relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ with $\varrho = r_1 = r_2 = \ldots = r_m$.

Remark 2.2

(i) Note that Definition 2.1(a) coincides with the definition of the vector relative degree for nonlinear MIMO-systems, see [9, Chapter 5.1].

(ii) The linear independence of the rows $c^j_{(n)}A^{r_j-1}B$, although a quite restrictive requirement, is significant for the construction of a coordinate transformation and with it the normal form.

(iii) There exist linear systems (A,B,C) of form (1.1) which do not satisfy condition (ii) in Definition 2.1(a). For a system that does not satisfy both conditions in Definition 2.1(a) the vector relative degree does not exist.

(iv) Note that in literature sometimes the relative degree is called uniform instead of strict.

The following lemma shows that the assumption of ordered vector relative degree is not restrictive.

Lemma 2.3 Let (A,B,C) be a linear system of form (1.1) with vector relative degree $r = (r_1,\dots,r_m) \in \mathbb{N}^{1\times m}$. Then there exists a permutation matrix $P \in \mathbb{R}^{m\times m}$ such that the system (A,B,PC) has ordered vector relative degree $rP = (\widetilde{r}_1,\dots,\widetilde{r}_m)$.

The following theorem presents a normal form for linear systems (A,B,C) of form (1.1) with ordered vector relative degree. The normal form has similar structural properties as the normal form for linear SISO-systems and linear MIMO-systems with strict relative degree, respectively, see (1.4).

Theorem 2.4 Consider a linear system (A,B,C) of form (1.1) with ordered vector relative degree $r = (r_1,\dots,r_m) \in \mathbb{N}^{1\times m}$. Set $r^s := \sum_{j=1}^m r_j$. Then there exists an invertible matrix $U \in \mathbb{R}^{n\times n}$ such that the coordinate transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} := Ux\,, \quad \xi(t) = \Big(y_1(t),\dots,y_1^{(r_1-1)}(t)\Big|\ \dots\ \Big|\, y_m(t),\dots,y_m^{(r_m-1)}(t)\Big)^T \in \mathbb{R}^{r^s}\,,\ \eta(t) \in \mathbb{R}^{n-r^s}\,,$$

converts (A,B,C) into

$$\left.\begin{array}{rcl}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &=& \widetilde{A}\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \widetilde{B}u\\ y &=& \widetilde{C}\begin{pmatrix}\xi\\ \eta\end{pmatrix}\end{array}\right\} \tag{2.1}$$

where

$$\left[\begin{array}{c|c}\widetilde{A} & \widetilde{B}\\ \hline \widetilde{C} & 0\end{array}\right] = \left[\begin{array}{cccc|cccc|c|cccc|c||c}
0 & 1 & & 0 & 0 & \cdots & & 0 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m}\\
\vdots & \ddots & \ddots & & \vdots & & & \vdots & \cdots & \vdots & & & \vdots & \vdots & \vdots\\
0 & \cdots & 0 & 1 & 0 & \cdots & & 0 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m}\\
R^1_{1,1} & \cdots & & R^1_{1,r_1} & R^1_{2,1} & \cdots & & R^1_{2,r_2} & & R^1_{m,1} & \cdots & & R^1_{m,r_m} & S^1 & c^1_{(n)}A^{r_1-1}B\\
\hline
0 & \cdots & & 0 & 0 & 1 & & 0 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m}\\
\vdots & & & \vdots & \vdots & \ddots & \ddots & & & \vdots & & & \vdots & \vdots & \vdots\\
0 & \cdots & & 0 & 0 & \cdots & 0 & 1 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m}\\
R^2_{1,1} & \cdots & & R^2_{1,r_1} & R^2_{2,1} & \cdots & & R^2_{2,r_2} & & R^2_{m,1} & \cdots & & R^2_{m,r_m} & S^2 & c^2_{(n)}A^{r_2-1}B\\
\hline
 & \vdots & & & & & & & \ddots & & & & & \vdots & \vdots\\
\hline
0 & \cdots & & 0 & 0 & \cdots & & 0 & & 0 & 1 & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m}\\
\vdots & & & \vdots & \vdots & & & \vdots & & \vdots & \ddots & \ddots & & \vdots & \vdots\\
0 & \cdots & & 0 & 0 & \cdots & & 0 & & 0 & \cdots & 0 & 1 & 0_{1\times(n-r^s)} & 0_{1\times m}\\
R^m_{1,1} & \cdots & & R^m_{1,r_1} & R^m_{2,1} & \cdots & & R^m_{2,r_2} & & R^m_{m,1} & \cdots & & R^m_{m,r_m} & S^m & c^m_{(n)}A^{r_m-1}B\\
\hline
P_1 & 0 & \cdots & 0 & P_2 & 0 & \cdots & 0 & \cdots & P_m & 0 & \cdots & 0 & Q & 0_{(n-r^s)\times m}\\
\hline
1 & 0 & \cdots & 0 & 0 & \cdots & & 0 & & 0 & \cdots & & 0 & & \\
0 & \cdots & & 0 & 1 & 0 & \cdots & 0 & & \vdots & & & \vdots & & \\
\vdots & & & \vdots & 0 & \cdots & & 0 & \cdots & & & & & 0_{m\times(n-r^s)} & 0\\
 & & & & \vdots & & & \vdots & & 0 & \cdots & & 0 & & \\
0 & \cdots & & 0 & 0 & \cdots & & 0 & & 1 & 0 & \cdots & 0 & &
\end{array}\right] \tag{2.2}$$

and $R^j_{i,k} \in \mathbb{R}$, for $i,j \in \{1,\dots,m\}$ and $k \in \{1,\dots,r_i\}$, $S^1,\dots,S^m \in \mathbb{R}^{1\times(n-r^s)}$, $P_1,\dots,P_m \in \mathbb{R}^{n-r^s}$ and $Q \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$.

More precisely, set, for $i \in \{1,\dots,r_1\}$,

$$m_i := \#\{r_j \,|\, r_j \geq i\,,\ j \in \{1,\dots,m\}\} \tag{2.3}$$

the number of r_j's, $j \in \{1,\dots,m\}$, such that $r_j \geq i$, and define

$$\Gamma := \begin{bmatrix} c^1_{(n)} A^{r_1-1} B \\ \vdots \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{2.4}$$

$$\mathcal{C} := \left[\begin{array}{c} c^1_{(n)} \\ \vdots \\ c^1_{(n)} A^{r_1-1} \\ \hline c^2_{(n)} \\ \vdots \\ c^2_{(n)} A^{r_2-1} \\ \hline \vdots \\ \hline c^m_{(n)} \\ \vdots \\ c^m_{(n)} A^{r_m-1} \end{array}\right] \in \mathbb{R}^{r^s\times n} \tag{2.5}$$

$$\mathcal{B} := \left[B\Gamma^{-1}\left[e^{(m)}_1,\dots,e^{(m)}_{m_1}\right], AB\Gamma^{-1}\left[e^{(m)}_1,\dots,e^{(m)}_{m_2}\right],\dots,A^{r_1-1}B\Gamma^{-1}\left[e^{(m)}_1,\dots,e^{(m)}_{m_{r_1}}\right]\right] \tag{2.6}$$
$$\in \mathbb{R}^{n\times r^s}$$

$$\mathcal{V} \in \mathbb{R}^{n\times(n-r^s)} \;:\; \operatorname{im}\mathcal{V} = \ker\mathcal{C}\,, \quad \text{and} \quad \operatorname{rk}\mathcal{V}^T\mathcal{V} = n - r^s \tag{2.7}$$

$$\widehat{U} := \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \in \mathbb{R}^{n\times n} \quad \text{and} \quad \mathcal{N} := (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right] \tag{2.8}$$

$$T_i := \left[\begin{array}{c|c|c|c} 0_{(r_i+n-r^s)\times(\sum_{j=1}^{i-1} r_j)} & \begin{matrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{matrix} & 0_{(r_i+n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & \begin{matrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{matrix} \end{array}\right] \tag{2.9}$$
$$\in \mathbb{R}^{(r_i+n-r^s)\times n}$$

$$\widehat{\mathcal{C}}_i := \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \in \mathbb{R}^{r_i\times(r_i+n-r^s)} \tag{2.10}$$

$$\widehat{\mathcal{B}}_i := \left[e^{(r_i+n-r^s)}_{r_i}, \left(T_i\widehat{U}A\widehat{U}^{-1}T_i^T\right)e^{(r_i+n-r^s)}_{r_i},\dots,\left(T_i\widehat{U}A\widehat{U}^{-1}T_i^T\right)^{r_i-1}e^{(r_i+n-r^s)}_{r_i}\right] \tag{2.11}$$
$$\in \mathbb{R}^{(r_i+n-r^s)\times r_i}$$

$$\widehat{\mathcal{N}}_i := \left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right]\left[I_{r_i+n-r^s} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i\right] \in \mathbb{R}^{(n-r^s)\times(r_i+n-r^s)} \tag{2.12}$$

$$\widehat{U}_i := \left[\begin{array}{c|c} I_{r^s} & 0_{r^s\times(n-r^s)} \\ \hline 0_{(n-r^s)\times(\sum_{j=1}^{i-1} r_j)}\,,\ \widehat{\mathcal{N}}_i\begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix}\,,\ 0_{(n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & I_{n-r^s} \end{array}\right] \in \mathbb{R}^{n\times n}\,, \tag{2.13}$$

for $i \in \{1,\dots,m\}$, and finally

$$U := \widehat{U}_m \cdot \widehat{U}_{m-1} \cdot \ldots \cdot \widehat{U}_1 \cdot \widehat{U}\,. \tag{2.14}$$

Then, for $i, j \in \{1, \dots m\}$, the entries in (2.1) are given by

$$\left[R^i_{j,1}, \dots, R^i_{j,r_j}\right] = \left[c^i_{(n)} A^{r_i} \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} 0_{(\sum_{\mu=1}^{j-1} r_\mu)\times r_j} \\ I_{r_j} \\ 0_{(\sum_{\mu=j+1}^{m} r_\mu)\times r_j} \end{bmatrix} \right.$$
$$\left. + c^i_{(n)} A^{r_i} \mathcal{V} \left[0_{(n-r^s)\times r_j}, I_{n-r^s}\right] \widehat{\mathcal{B}}_j (\widehat{\mathcal{C}}_j \widehat{\mathcal{B}}_j)^{-1} \right] \tag{2.15}$$

$$S^i = c^i_{(n)} A^{r_i} \mathcal{V} \tag{2.16}$$

$$[P_i, 0, \dots, 0] = \widehat{\mathcal{N}}_i \left(T_i \widehat{U} A \widehat{U}^{-1} T_i^T \right) \widehat{\mathcal{B}}_i (\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i)^{-1} \tag{2.17}$$

$$Q = \mathcal{N} A \mathcal{V} \overset{(2.8)}{=} (\mathcal{V}^T \mathcal{V})^{-1} \mathcal{V}^T \left[I - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\right] A \mathcal{V}\,. \tag{2.18}$$

Note that the coordinate transformation does not affect the input u and output y of the original system (A, B, C).

Next the definition of the zero dynamics of a linear system (A, B, C) of form (1.1) is given. Furthermore, asymptotical stability of linear systems and asymptotical stability of the zero dynamics of a linear system is defined.

Definition 2.5

(i) The *zero dynamics* of a linear system (A, B, C) of form (1.1) are defined as the real vector space of trajectories

$$\mathcal{ZD}(A, B, C) := \Big\{ (x, u) \in \mathcal{C}^1([0, \infty) \to \mathbb{R}^n) \times \mathcal{C}_{\text{pw}}([0, \infty) \to \mathbb{R}^m) \;\Big|\; (x, u) \text{ solves (1.1) with } y \equiv 0 \text{ on } [0, \infty) \Big\}\,.$$

(ii) A linear system $\dot{x} = Ax$, for $A \in \mathbb{R}^{n\times n}$, is called *asymptotically stable* on $[0, \infty)$ if, and only if,

$$\exists\, M, \lambda > 0\ \forall\, t \geq 0\ :\ \|x(t)\| \leq M e^{-\lambda t} \|x(0)\|\,,$$

for all solutions x of $\dot{x} = Ax$.

(iii) The *zero dynamics* of a linear system (A, B, C) of form (1.1) are called *asymptotically stable* if, and only if,

$$\exists\, M, \lambda > 0\ \forall\, (x, u) \in \mathcal{ZD}(A, B, C)\ \forall\, t \geq 0\ :\ \|(x(t), u(t))\| \leq M e^{-\lambda t} \|x(0)\|\,.$$

For linear systems (A, B, C) of form (1.1) with ordered vector relative degree $r \in \mathbb{N}^{1\times m}$ the zero dynamics of (A, B, C) can be read off from normal form (2.1) given by Theorem 2.4. Proposition 2.6 provides a characterization of the systems zero dynamics in terms of the normal form. Furthermore asymptotical stability of the zero dynamics of (A, B, C) will be characterized.

Proposition 2.6 For any linear system (A, B, C) of form (1.1) with ordered relative degree $r = (r_1, \dots, r_m) \in \mathbb{N}^{1\times m}$ and normal form (2.1), (2.2) the following holds:

(i) For $S := \left[S^{1T}, \ldots, S^{mT}\right]^T$, with $S^1, \ldots, S^m$ defined in (2.16), $\mathcal{V}$ defined by (2.7) and Q defined in (2.18), the zero dynamics of (A, B, C) are given by

$$\mathcal{ZD}(A,B,C) = \left\{ (\mathcal{V}\eta, -\Gamma^{-1}S\eta) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \times \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \,\middle|\, \dot\eta = Q\eta \right\},$$

(ii) The zero dynamics of (A, B, C) are asymptotically stable if, and only if, $\dot\eta = Q\eta$ is an asymptotically stable linear system.

Given a sufficiently smooth reference signal $y_R\colon \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ one can determine an input $u = u_R$ such that the output y of (1.1) matches this signal straightforward by using the normal form (2.1), (2.2). A system (A, B, C) is called *right-invertible* if this tracking problem can be solved [12]. The following proposition presents the solution to this problem.

Proposition 2.7 Consider a linear system (A, B, C) of form (1.1) with ordered relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ and normal form (2.1), (2.2). Let $y_R = (y_{R1}, \ldots, y_{Rm})^T : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ with $y_{Rj} \in \mathcal{C}^{r_j}([0,\infty) \to \mathbb{R})$, $j \in \{1, \ldots, m\}$. Let y be the output of (1.1). Then the following are equivalent

(i) $y = y_R$,

(ii) the input u of (1.1) is given by

$$u = u_R = \Gamma^{-1}\left(\begin{bmatrix} {y_{R1}}^{(r_1)} \\ \vdots \\ {y_{Rm}}^{(r_m)} \end{bmatrix} - \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi - \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta \right) \tag{2.19}$$

where, for arbitrary $\eta^0 \in \mathbb{R}^{n-r^s}$, η is a solution of the initial value problem

$$\dot\eta = Q\eta + [P_1, \ldots P_m]\, y_R\,, \qquad \eta(0) = \eta^0\,, \tag{2.20}$$

$\xi = \left(y_{R1}, \ldots, y_{R1}^{(r_1-1)} \middle| y_{R2}, \ldots, y_{R2}^{(r_2-1)} \middle| \ldots \middle| y_{Rm}, \ldots, y_{Rm}^{(r_m-1)} \right)^T$, Q is defined in (2.18), $P_1, \ldots, P_m$ are defined in (2.17), Γ is defined in (2.4), $S^1, \ldots, S^m$ are defined in (2.16), $R^j := \left[R^j_{1,1}, \ldots, R^i_{1,r_1} \middle| \ldots \middle| R^j_{m,1}, \ldots, R^i_{m,r_m} \right]$ and $R^j_{i,k}$ is defined in (2.15).

System (2.19), (2.20) is called the *inverse system* of system (1.1) [9].

3 Proofs

This section contains all proofs for the results given in Section 2. It is structured as follows: First it is shown that for every system with vector relative degree $r \in \mathbb{N}^{1\times m}$ one can find an permutation of the output such that the system with permuted output has an ordered vector relative degree. Next linearly independence of the matrices $\mathcal{C}$ and $\mathcal{B}$, defined by (2.5) and (2.6), respectively, is shown. Then the proof for the normal form including the construction of the coordinate transformation is given. A proof for characterization and stability of the system's zero dynamics is presented and finally right-invertibility of (A, B, C) is shown.

3.1 Ordered vector relative degree

Proof of Lemma 2.3. Let $\sigma\colon \{1,\dots,m\} \to \{1,\dots,m\}$ be a permutation such that $r_{\sigma(1)} \geq r_{\sigma(2)} \geq \dots \geq r_{\sigma(m)}$. Furthermore set

$$P := \begin{bmatrix} e^{\sigma(1)}_{(m)} \\ \vdots \\ e^{\sigma(m)}_{(m)} \end{bmatrix}.$$

Then

$$PC = \begin{bmatrix} e^{\sigma(1)}_{(m)} \\ \vdots \\ e^{\sigma(m)}_{(m)} \end{bmatrix} \begin{bmatrix} c^{1}_{(m)} \\ \vdots \\ c^{m}_{(m)} \end{bmatrix} = \begin{bmatrix} c^{\sigma(1)}_{(m)} \\ \vdots \\ c^{\sigma(1)}_{(m)} \end{bmatrix},$$

and by the assumption on the relative degree it follows that

$$\forall\, j \in \{1,\dots,m\}\ \forall\, k \in \{0,\dots,r_{\sigma(j)}-2\}\ :\ (PC)^{j}_{(n)} A^k B = c^{\sigma(j)}_{(n)} A^k B = 0_{1\times m}\,,$$

and

$$\operatorname{rk} \begin{bmatrix} (PC)^{1}_{(n)} A^{r_1-1} B \\ \vdots \\ (PC)^{m}_{(n)} A^{r_m-1} B \end{bmatrix} = m\,,$$

whence the linear system (A,B,PC) has relative degree $Pr = (r_{\sigma(1)},\dots,r_{\sigma(m)})$ with $r_{\sigma(1)} \geq \dots \geq r_{\sigma(m)}$. □

3.2 Linearly independence of $\mathcal{C}$ and $\mathcal{B}$

Recall the matrices $\mathcal{C} \in \mathbb{R}^{r^s\times n}$, defined by (2.5), and $\mathcal{B} \in \mathbb{R}^{n\times r^s}$ defined by (2.6). Note that, for m_i, $i \in \{1,\dots,r_1\}$, defined by (2.3), it holds true that $m = m_1 \geq m_2 \geq \dots \geq m_{r_1} \geq 1$ and

$$r^s = \sum_{j=1}^{m} r_j = \sum_{j=1}^{m}\sum_{i=1}^{r_1} \frac{\max\{r_j - i + 1, 0\}}{\max\{r_j - i + 1, 1\}} = \sum_{i=1}^{r_1} \underbrace{\sum_{j=1}^{m} \frac{\max\{r_j - i + 1, 0\}}{\max\{r_j - i + 1, 1\}}}_{\#\{r_j \,|\, r_j \geq i,\ j\in\{1,\dots,m\}\}} = \sum_{i=1}^{r_1} m_i\,.$$

The following lemma shows that $\mathcal{C}$ and $\mathcal{B}$ have full rank.

Lemma 3.1 If a linear system (A,B,C) of form (1.1) has ordered vector relative degree $r = (r_1,\dots,r_m) \in \mathbb{N}^{1\times m}$, then $\mathcal{C}$ and $\mathcal{B}$, defined by (2.5) and (2.6), respectively, have full rank.

Proof. Note that $\sum_{j=1}^{m} r_j \leq n$. It suffices to show that $\mathcal{C}\mathcal{B} \in \mathbb{R}^{r^s\times r^s}$ is invertible.
First consider the first $m_1 = m$ rows of $\mathcal{C}\mathcal{B}$. Since (A,B,C) has relative degree $r = (r_1,\dots,r_m) \in N^{1\times m}$ it follows that

$$\mathcal{C}B = \left[\begin{array}{c} c^{1}_{(n)} \\ \vdots \\ c^{1}_{(n)} A^{r_1-1} \\ \hline \vdots \\ \hline c^{m}_{(n)} \\ \vdots \\ c^{m}_{(n)} A^{r_m-1} \end{array}\right] B = \left[\begin{array}{c} c^{1}_{(n)} A^0 B \\ \vdots \\ c^{1}_{(n)} A^{r_1-1} B \\ \hline \vdots \\ \hline c^{m}_{(n)} A^0 B \\ \vdots \\ c^{m}_{(n)} A^{r_m-1} B \end{array}\right] = \left[\begin{array}{c} 0_{(r_1-1)\times m_1} \\ \Gamma^{1}_{(m_1)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m_1} \\ \Gamma^{m}_{(m_1)} \end{array}\right] \tag{3.1}$$

where $\Gamma^i_{(m_1)}$, $i \in \{1, \ldots, m\}$, is the i-th row of Γ. Thus

$$\mathcal{C}B\Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_1}^{(m)}\right] = \left[\begin{array}{c} 0_{(r_1-1)\times m_1} \\ \Gamma^1_{(m_1)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m_1} \\ \Gamma^m_{(m_1)} \end{array}\right] \Gamma^{-1} I_{m_1} = \left[\begin{array}{c} 0_{(r_1-1)\times m_1} \\ e^1_{(m_1)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m_1} \\ e^{m_1}_{(m_1)} \end{array}\right],$$

which shows that $\mathcal{C}B\Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_1}^{(m)}\right]$ has rank $m_1 = m$.

Next consider $\mathcal{C}A^{i-1}B\Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_i}^{(m)}\right]$, for $i \in \{2, \ldots, r_1\}$. Since (A, B, C) has relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ it follows with the conventions

(i) $0_{(r_j-i)\times m_i}$ is of dimension zero, if $i \geq r_j$, $j \in \{1, \ldots, m\}$, and

(ii) $\Gamma^j_{(m)}$ and $e^j_{(m_i)}$ do not exist in the following matrices if $j > m_i$, $j \in \{1, \ldots, m\}$, and

(iii) $\mathcal{X}_{\mu\times\nu} \in \mathbb{R}^{\mu\times\nu}$ is an arbitrarily matrix of dimension $\mu \times \nu$,

that

$$\begin{aligned}
&\mathcal{C}A^{i-1}B\Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_i}^{(m)}\right] \\
&= \left[\begin{array}{c} c^1_{(n)}A^{i-1}B \\ \vdots \\ c^1_{(n)}A^{r_1-1}B \\ \vdots \\ c^1_{(n)}A^{r_1+i-2}B \\ \hline \vdots \\ \hline c^m_{(n)}A^{i-1}B \\ \vdots \\ c^m_{(n)}A^{r_m-1}B \\ \vdots \\ c^m_{(n)}A^{r_m+i-2}B \end{array}\right] \Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_i}^{(m)}\right] = \left[\begin{array}{c} 0_{(r_1-i)\times m} \\ \Gamma^1_{(m)} \\ \mathcal{X}_{(\min\{r_1, i-1\})\times m} \\ \hline \vdots \\ \hline 0_{(r_m-i)\times m} \\ \Gamma^m_{(m)} \\ \mathcal{X}_{(\min\{r_m, i-1\})\times m} \end{array}\right] \Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_i}^{(m)}\right] \\
&= \left[\begin{array}{c} 0_{(r_1-i)\times m_i} \\ e^1_{(m_i)} \\ \mathcal{X}_{(\min\{r_1, i-1\})\times m_i} \\ \hline \vdots \\ \hline 0_{(r_m-i)\times m_i} \\ e^m_{(m_i)} \\ \mathcal{X}_{(\min\{r_m, i-1\})\times m_i} \end{array}\right], \qquad i \in \{2, \ldots, r_1\},
\end{aligned}$$

Thus, for all $i \in \{1, \ldots, r_1\}$, the m_i rows of $\mathcal{C}A^{i-1}B\Gamma^{-1}\left[e_1^{(m)}, \ldots, e_{m_i}^{(m)}\right]$ are linearly independent, and since

$$
\mathcal{CB} = \left[\begin{array}{c|c|c|c|c}
\begin{matrix} 0_{(r_1-1)\times m_1} \\ e^1_{(m_1)} \end{matrix} &
\begin{matrix} 0_{(r_1-2)\times m_2} \\ e^1_{(m_2)} \\ \mathcal{X}_{1\times m_2} \end{matrix} &
\iddots &
\begin{matrix} 0_{1\times m_{r_1-1}} \\ e^1_{(m_{r_1-1})} \\ \mathcal{X}_{(r_1-2)\times m_{r_1-1}} \end{matrix} &
\begin{matrix} e^1_{(m_{r_1})} \\ \mathcal{X}_{(r_1-1)\times m_{r_1}} \end{matrix} \\ \hline
\begin{matrix} 0_{(r_2-1)\times m_1} \\ e^2_{(m_1)} \end{matrix} &
\begin{matrix} 0_{(r_2-2)\times m_2} \\ e^2_{(m_2)} \\ \mathcal{X}_{1\times m_2} \end{matrix} &
\iddots &
\begin{matrix} 0_{(r_2-r_1+1)\times m_{r_1-1}} \\ e^2_{(m_{r_1-1})} \\ \mathcal{X}_{(\min\{r_1-2,r_2\})\times m_{r_1-1}} \end{matrix} &
\begin{matrix} e^2_{(m_{r_1})} \\ \mathcal{X}_{(\min\{r_1-1,r_2\})\times m_{r_1}} \end{matrix} \\ \hline
\vdots & \vdots & & \vdots & \vdots \\ \hline
\begin{matrix} 0_{(r_m-1)\times m_1} \\ e^m_{(m_1)} \end{matrix} &
\begin{matrix} 0_{(r_m-2)\times m_2} \\ e^m_{(m_2)} \\ \mathcal{X}_{1\times m_2} \end{matrix} &
\iddots &
\begin{matrix} 0_{(r_m-r_1+1)\times m_{r_1-1}} \\ e^m_{(m_{r_1-1})} \\ \mathcal{X}_{(\min\{r_1-2,r_m\})\times m_{r_1-1}} \end{matrix} &
\begin{matrix} e^m_{(m_{r_1})} \\ \mathcal{X}_{(\min\{r_1-1,r_m\})\times m_{r_1}} \end{matrix}
\end{array}\right]
\begin{matrix} \}\, r_1 \\ \\ \}\, r_2 \\ \vdots \\ \}\, r_m \end{matrix}
\tag{3.2}
$$

$$\underbrace{\qquad}_{m_1} \quad \underbrace{\qquad}_{m_2} \quad \cdots \quad \underbrace{\qquad\qquad}_{m_{r_1-1}} \quad \underbrace{\qquad\qquad}_{m_{r_1}}$$

it follows that $\mathcal{CB}$ is invertible. □

As an immediate consequence of Lemma 3.1 it follows that for linear systems (A, B, C) of form (1.1) with vector relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$, the matrices $C \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ have full rank m.

3.3 Coordinate transformation and normal form

Lemma 3.1 shows that the rows of $\mathcal{C}$ qualify as basis, which, if $r^s = \sum_{j=1}^m r_j < n$, has to be completed, for a coordinate transformation in $\mathbb{R}^n$. Consider a matrix $\mathcal{V} \in \mathbb{R}^{n\times(n-r^s)}$, given by (2.7). For $\widehat{U}$ and $\mathcal{N}$, given by (2.8), it follows from

$$\begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} \left[\mathcal{B}(\mathcal{CB})^{-1}, \mathcal{V}\right] = I_n$$

that $\widehat{U}$ has the inverse

$$\widehat{U}^{-1} = \left[\mathcal{B}(\mathcal{CB})^{-1}, \mathcal{V}\right] . \tag{3.3}$$

Although $\widehat{U}$ already qualifies as coordinate transformation in $\mathbb{R}^n$ we do not obtain a normal form which has the same structure properties as the normal form (1.4) for linear SISO systems (1.3), i.e. the transformation matrix $\widehat{U}$ will not lead in general to a matrix $\widetilde{A}$ as in (2.2). Therefore it is necessary to consider the transformation matrix U, given by (2.14) and $T_i, \widehat{\mathcal{C}}_i, \widehat{\mathcal{B}}_i, \widehat{\mathcal{N}}_i, \widehat{U}_i$, for $i \in \{1, \ldots, m\}$, defined in (2.9)–(2.13), respectively.

Proof of Theorem 2.4. *Step 1*: First it is shown that the coordinate transformation

$$\begin{pmatrix} \chi \\ \zeta \end{pmatrix} := \widehat{U}x\,, \quad \chi(t) = \left(y_1(t), \ldots, y_1^{(r_1-1)}(t)\middle| \ldots \middle| y_m(t), \ldots, y_m^{(r_m-1)}(t)\right)^T \in \mathbb{R}^{r^s}\,,\ \zeta(t) \in \mathbb{R}^{n-r^s}\,,$$

given by (2.7) and (2.8), converts (1.1) into

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\chi\\ \zeta\end{pmatrix} & = & \widehat{A}\begin{pmatrix}\chi\\ \zeta\end{pmatrix} + \widetilde{B}u \\ y & = & \widetilde{C}\begin{pmatrix}\chi\\ \zeta\end{pmatrix} \end{array}\right\} \tag{3.4}$$

where

$$\widehat{A} = \left[\begin{array}{cccc|c} \widehat{A}_{1,1} & \widehat{A}_{1,2} & \dots & \widehat{A}_{1,m} & \widehat{S}_1 \\ \widehat{A}_{2,1} & \widehat{A}_{2,2} & \dots & \widehat{A}_{2,m} & \widehat{S}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \widehat{A}_{m,1} & \widehat{A}_{m,2} & \dots & \widehat{A}_{m,m} & \widehat{S}_m \\ \hline \widehat{P}_1 & \widehat{P}_2 & \dots & \widehat{P}_m & \widehat{Q} \end{array}\right], \tag{3.5}$$

and, for $i, j \in \{1, \dots, m\}$,

$$\widehat{A}_{i,i} := \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \\ \widehat{R}^i_{i,1} & \dots & & \widehat{R}^i_{i,r_i} \end{bmatrix} \in \mathbb{R}^{r_i \times r_i}, \quad \widehat{A}_{i,j} := \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \\ \widehat{R}^i_{j,1} & \dots & \widehat{R}^i_{j,r_j} \end{bmatrix} \in \mathbb{R}^{r_i \times r_j}, \quad j \neq i, \tag{3.6}$$

where $\widehat{R}^j_{i,k} \in \mathbb{R}$, for $k \in \{1, \dots, r_i\}$ and $i, j \in \{1, \dots, m\}$, and, for $i \in \{1, \dots, m\}$,

$$\widehat{S}_i := \begin{bmatrix} 0_{(r_i-1)\times(n-r^s)} \\ S^i \end{bmatrix} \in \mathbb{R}^{r_i \times (n-r^s)}, \quad \widehat{P}_i \in \mathbb{R}^{(n-r^s)\times r_i}, \quad \widehat{Q} \in \mathbb{R}^{(n-r^s)\times(n-r^s)}. \tag{3.7}$$

Step 1a): First the structure of $\widehat{A}$ is proven. By definition of $\widehat{U}$, see (2.8), it follows that

$$\widehat{A} = \widehat{U} A \widehat{U}^{-1} = \begin{bmatrix} \mathcal{C} \\ \mathcal{N} \end{bmatrix} A \left[\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \mathcal{V}\right] = \left[\begin{array}{c|c} \mathcal{C}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} & \mathcal{C}A\mathcal{V} \\ \hline \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} & \mathcal{N}A\mathcal{V} \end{array}\right].$$

Thus

$$\left[\widehat{P}_1, \dots, \widehat{P}_m\right] = \mathcal{N}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \in \mathbb{R}^{(n-r^s)\times r^s} \tag{3.8}$$

$$\widehat{Q} = \mathcal{N}A\mathcal{V} \in \mathbb{R}^{(n-r^s)\times(n-r^s)}, \tag{3.9}$$

and the definition of $\mathcal{C}$ and $\mathcal{B}$, see (2.5) and (2.6), respectively, yields

$$\mathcal{C}A\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = \left[\begin{array}{c} c^1_{(n)}A \\ \vdots \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline c^m_{(n)}A \\ \vdots \\ c^m_{(n)}A^{r_m} \end{array}\right] \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = \left[\begin{array}{c} \mathcal{C}^2_{(n)} \\ \vdots \\ \mathcal{C}^{r_1}_{(n)} \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline \mathcal{C}^{\sum_{j=1}^{m-1} r_j+2}_{(n)} \\ \vdots \\ \mathcal{C}^{r^s}_{(n)} \\ c^m_{(n)}A^{r_m} \end{array}\right] \mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = \left[\begin{array}{c} (\mathcal{C}\mathcal{B})^2_{(r^s)} \\ \vdots \\ (\mathcal{C}\mathcal{B})^{r_1}_{(r^s)} \\ c^1_{(n)}A^{r_1}\mathcal{B} \\ \hline \vdots \\ \hline (\mathcal{C}\mathcal{B})^{\sum_{j=1}^{m-1} r_j+2}_{(r^s)} \\ \vdots \\ (\mathcal{C}\mathcal{B})^{r^s}_{(r^s)} \\ c^m_{(n)}A^{r_m}\mathcal{B} \end{array}\right] (\mathcal{C}\mathcal{B})^{-1}$$

$$= \left[\begin{array}{c} e^2_{(r^s)} \\ \vdots \\ e^{r_1}_{(r^s)} \\ c^1_{(n)}A^{r_1}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \\ \hline \vdots \\ \hline e^{\sum_{j=1}^{m-1} r_j+2}_{(r^s)} \\ \vdots \\ e^{r^s}_{(r^s)} \\ c^m_{(n)}A^{r_m}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} \end{array}\right]. \tag{3.10}$$

Furthermore, invoking $\operatorname{im}\mathcal{V} = \ker\mathcal{C}$, it follows that

$$\mathcal{C}A\mathcal{V} = \left[\begin{array}{c} c^1_{(n)}A \\ \vdots \\ c^1_{(n)}A^{r_1-1} \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline c^m_{(n)}A \\ \vdots \\ c^m_{(n)}A^{r_m-1} \\ c^m_{(n)}A^{r_m} \end{array}\right]\mathcal{V} = \left[\begin{array}{c} \mathcal{C}^2_{(n)} \\ \vdots \\ \mathcal{C}^{r_1}_{(n)} \\ c^1_{(n)}A^{r_1} \\ \hline \vdots \\ \hline (\mathcal{C})^{\sum_{j=1}^{m-1} r_j+2}_{(n)} \\ \vdots \\ (\mathcal{C})^{r^s}_{(n)} \\ c^m_{(n)}A^{r_m} \end{array}\right]\mathcal{V} = \left[\begin{array}{c} 0_{1\times(n-r^s)} \\ \vdots \\ 0_{1\times(n-r^s)} \\ c^1_{(n)}A^{r_1}\mathcal{V} \\ \hline \vdots \\ \hline 0_{1\times(n-r^s)} \\ \vdots \\ 0_{1\times(n-r^s)} \\ c^m_{(n)}A^{r_m}\mathcal{V} \end{array}\right] \overset{(2.16)}{=} \left[\begin{array}{c} 0_{(r_1-1)\times(n-r^s)} \\ S^1 \\ \hline \vdots \\ \hline 0_{(r_m-1)\times(n-r^s)} \\ S^m \end{array}\right], \tag{3.11}$$

Hence, setting

$$\left[\widehat{R}^i_{1,1}, \ldots, \widehat{R}^i_{1,r_1}\middle| \ldots \middle| \widehat{R}^i_{m,1}, \ldots, \widehat{R}^i_{m,r_m}\right] := c^i_{(n)}A^{r_i}\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, \qquad i \in \{1, \ldots, m\}, \tag{3.12}$$

(3.10) and (3.11) yield the structure of $\widehat{A}$ as given in (3.5)–(3.7).

Step 1b): Next the structure of $\widetilde{B}$ is proven. By the definition of $\widehat{U}$, see (2.8), it follows that

$$\widetilde{B} = \widehat{U}B = \begin{bmatrix} \mathcal{C}B \\ (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[B - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}B\right] \end{bmatrix}.$$

Recall (3.1), i.e.

$$\mathcal{C}B = \begin{bmatrix} 0_{(r_1-1)\times m} \\ \Gamma^1_{(m)} \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m} \\ \Gamma^m_{(m)} \end{bmatrix} = \begin{bmatrix} 0_{(r_1-1)\times m} \\ c^1_{(n)} A^{r_1-1} B \\ \hline \vdots \\ \hline 0_{(r_m-1)\times m} \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix}.$$

Furthermore

$$\begin{aligned}
(\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left[B - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}B\right] &= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\Big[B\Gamma^{-1}\Gamma - \mathcal{B}(\mathcal{C}\mathcal{B})^{-1}\mathcal{C}\,\underbrace{B\Gamma^{-1}\left[e^{(m)}_1,\ldots,e^{(m)}_m\right]}_{=\left[\mathcal{B}^{(n)}_1,\ldots,\mathcal{B}^{(n)}_m\right]}\Gamma\Big] \\
&= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\Big[B\Gamma^{-1}\Gamma - \mathcal{B}\,\underbrace{(\mathcal{C}\mathcal{B})^{-1}\left[(\mathcal{C}\mathcal{B})^{(r^s)}_1,\ldots,(\mathcal{C}\mathcal{B})^{(r^s)}_m\right]}_{=\left[e^{(r^s)}_1,\ldots,e^{(r^s)}_m\right]}\Gamma\Big] \\
&= (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T\left(\left[\mathcal{B}^{(n)}_1,\ldots,\mathcal{B}^{(n)}_m\right] - \left[\mathcal{B}^{(n)}_1,\ldots,\mathcal{B}^{(n)}_m\right]\right)\Gamma \\
&= 0_{(n-r^s)\times m},
\end{aligned}$$

which shows the structure of $\widetilde{B}$ as in (2.2).

Step 1c): Now the structure of $\widetilde{C}$ is shown. Since the rows of C are also rows of $\mathcal{C}$, i.e.

$$C = \begin{bmatrix} c^1_{(n)} \\ c^2_{(n)} \\ \vdots \\ c^m_{(n)} \end{bmatrix} = \begin{bmatrix} \mathcal{C}^1_{(n)} \\ \mathcal{C}^{r_1+1}_{(n)} \\ \vdots \\ \mathcal{C}^{r^s-r_m+1}_{(n)} \end{bmatrix}$$

and since $\operatorname{im}\mathcal{V} = \ker\mathcal{C}$ it follows that $C\mathcal{V} = 0_{m\times(n-r^s)}$. Furthermore

$$C\mathcal{B}(\mathcal{C}\mathcal{B})^{-1} = \begin{bmatrix} (\mathcal{C}\mathcal{B})^1_{(r^s)} \\ (\mathcal{C}\mathcal{B})^{r_1+1}_{(r^s)} \\ \vdots \\ (\mathcal{C}\mathcal{B})^{r^s-r_m+1}_{(r^s)} \end{bmatrix}(\mathcal{C}\mathcal{B})^{-1} = \begin{bmatrix} 1 & 0_{1\times(r_1-1)} & 0 & 0_{1\times(r_2-1)} & 0 & \ldots & 0 & 0_{1\times(r_m-1)} \\ 0 & 0_{1\times(r_1-1)} & 1 & 0_{1\times(r_2-1)} & 0 & \ldots & 0 & 0_{1\times(r_m-1)} \\ 0 & 0_{1\times(r_1-1)} & 0 & 0_{1\times(r_2-1)} & 1 & & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots & 0 & 0_{1\times(r_m-1)} \\ 0 & 0_{1\times(r_1-1)} & 0 & 0_{1\times(r_2-1)} & 0 & & 1 & 0_{1\times(r_m-1)} \end{bmatrix}.$$

Hence

$$\widetilde{C} = C\widehat{U}^{-1} = \left[C\mathcal{B}(\mathcal{C}\mathcal{B})^{-1}, C\mathcal{V}\right]$$

yields the structure of $\widetilde{C}$ as in (2.2).

Step 2: We show that the coordinate transformation

$$\begin{pmatrix}\xi \\ \eta\end{pmatrix} := Ux, \quad \xi(t) = \Big(y_1(t),\ldots,y_1^{(r_1-1)}(t)\Big|\ \ldots\ \Big|\, y_m(t),\ldots,y_m^{(r_m-1)}(t)\Big)^T \in \mathbb{R}^{r^s},\ \eta(t) \in \mathbb{R}^{n-r^s},$$

given by (2.7)–(2.14) converts the linear system (A, B, C) of form (1.1) into (2.1) with $\widetilde{A}, \widetilde{B}, \widetilde{C}$ as in (2.2) with matrix components of $\widetilde{A}$ as in (2.15)–(2.18).

Recall the structure of $\widehat{A}$ given by (3.5)–(3.7). For $i \in \{1, \dots, m\}$, consider the matrices

$$\widehat{A}_i := \left[\begin{array}{cccc|c} 0 & 1 & \dots & 0 & \\ \vdots & & \ddots & & \widehat{S}_i \\ 0 & \dots & 0 & 1 & \\ \widehat{R}^i_{i,1} & \dots & & \widehat{R}^i_{i,r_i} & \\ \hline & \widehat{P}_i & & & \widehat{Q} \end{array}\right] = \left[\begin{array}{c|c} \widehat{A}_{i,i} & \widehat{S}_i \\ \hline \widehat{P}_i & \widehat{Q} \end{array}\right] = T_i \widehat{A} T_i^T = T_i \widehat{U} A \widehat{U}^{-1} T_i^T \in \mathbb{R}^{(r_i+n-r^s)\times(r_i+n-r^s)},$$

$$\widehat{C}_i := \left[1, 0_{1\times(r_i+n-r^s-1)}\right] = e^1_{(r_i+n-r^s)} = e^i_{(m)} \widetilde{C} T_i^T \in \mathbb{R}^{1\times(r_i+n-r^s)},$$

$$\widehat{B}_i := \left[\begin{array}{c} 0_{(r_i-1)\times 1} \\ 1 \\ \hline 0_{(n-r^s)\times 1} \end{array}\right] = e^{(r_i+n-r^s)}_{r_i} = T_i \widetilde{B} \Gamma^{-1} e^{(m)}_i \in \mathbb{R}^{r_i+n-r^s}.$$

Then

$$\begin{bmatrix} \widehat{C}_i \\ \widehat{C}_i \widehat{A}_i \\ \vdots \\ \widehat{C}_i \widehat{A}_i^{r_i-1} \end{bmatrix} = \begin{bmatrix} e^1_{(r_i+n-r^s)} \\ e^2_{(r_i+n-r^s)} \\ \vdots \\ e^{r_i}_{(r_i+n-r^s)} \end{bmatrix} = \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \overset{(2.10)}{=} \widehat{\mathcal{C}}_i \in \mathbb{R}^{r_i\times(r_i+n-r^s)}$$

$$\left[\widehat{B}_i, \widehat{A}_i \widehat{B}_i, \dots, \widehat{A}_i^{r_i-1} \widehat{B}_i\right] = \left[e^{(r_i+n-r^s)}_{r_i}, \dots, \widehat{A}_i^{r_i-1} e^{(r_i+n-r^s)}_{r_i}\right] \overset{(2.11)}{=} \widehat{\mathcal{B}}_i \in \mathbb{R}^{(r_i+n-r^s)\times r_i}.$$

More precisely $\widehat{\mathcal{B}}_i$ is structured as follows:

$$\widehat{\mathcal{B}}_i = \left[\begin{array}{c|cccc} 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \iddots & 1 & * \\ 0 & 0 & \iddots & \iddots & \vdots \\ 0 & 1 & * & \dots & * \\ 1 & * & \dots & & * \\ \hline 0_{(n-r^s)\times 1} & & \mathcal{X}_{(n-r^s)\times(r_i-1)} & & \end{array}\right] \in \mathbb{R}^{(r_i+n-r^s)\times r_i}. \tag{3.13}$$

Since $\widehat{C}_i \widehat{A}_i^j \widehat{B}_i = 0$, for all $j \in \{0, \dots, r_i - 2\}$, and $\widehat{C}_i \widehat{A}_i^{r_i-1} \widehat{B}_i = 1$, it follows that the linear SISO-system

$$\left.\begin{array}{rcl} \dot{z} &=& \widehat{A}_i z + \widehat{B}_i v \\ w &=& \widehat{C}_i z \end{array}\right\}$$

has relative degree r_i. Furthermore, it follows that

$$\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \iddots & \iddots & * \\ 0 & 1 & \iddots & \vdots \\ 1 & * & \dots & * \end{bmatrix} \quad \text{and} \quad \left(\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i\right)^{-1} = \begin{bmatrix} * & \dots & * & 1 \\ \vdots & \iddots & \iddots & 0 \\ * & 1 & \iddots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix} \tag{3.14}$$

and thus

$$\widehat{B}_i \left(\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i\right)^{-1} = \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 0 \\ * & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ * & \dots & * & 1 & 0 \\ * & \dots & & * & 1 \\ \hline \multicolumn{4}{c|}{\mathcal{X}_{(n-r^s)\times(r_i-1)}} & 0_{(n-r^s)\times 1} \end{array}\right] \in \mathbb{R}^{(r_i+n-r^s)\times r_i} \tag{3.15}$$

$$\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \widehat{B}_i \left(\widehat{\mathcal{C}}_i \widehat{\mathcal{B}}_i\right)^{-1} = \left[\mathcal{X}_{(n-r^s)\times(r_i-1)}, 0_{(n-r^s)\times 1}\right] . \tag{3.16}$$

Set $\widehat{\mathcal{V}}_i := \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix}$. Then $\ker \widehat{\mathcal{C}}_i = \operatorname{im} \widehat{\mathcal{V}}_i$ and thus

$$\begin{bmatrix} \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \\ \underbrace{\left(\widehat{\mathcal{V}}_i \widehat{\mathcal{V}}_i^T\right)^{-1} \widehat{\mathcal{V}}_i^T}_{=\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right]} \left[I_{r_i+n-r^s} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i\right] \end{bmatrix} \overset{(2.10),\,(2.12)}{=} \begin{bmatrix} \widehat{\mathcal{C}}_i \\ \widehat{\mathcal{N}}_i \end{bmatrix} \tag{3.17}$$

is invertible with inverse

$$\begin{bmatrix} \widehat{\mathcal{C}}_i \\ \widehat{\mathcal{N}}_i \end{bmatrix}^{-1} = \left[\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}, \widehat{\mathcal{V}}_i\right] . \tag{3.18}$$

Furthermore

$$\begin{aligned} \widehat{\mathcal{C}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} &= \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} = 0_{r_i\times(n-r^s)} \\ \widehat{\mathcal{C}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix} &= \left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix} = I_{r_i} \\ \widehat{\mathcal{N}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} &= \left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right] \left[\begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix} - \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\widehat{\mathcal{C}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix}\right] = I_{n-r^s} . \end{aligned}$$

and thus

$$\left[\begin{array}{c|c|c|c} I_{\sum_{j=1}^{i-1} r_j} & 0_{(\sum_{j=1}^{i-1} r_j)\times r_i} & 0_{(\sum_{j=1}^{i-1} r_j)\times(\sum_{j=i+1}^{m} r_j)} & 0_{(\sum_{j=1}^{i-1} r_j)\times(n-r^s)} \\ \hline 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\widehat{\mathcal{C}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix}}_{=I_{r_i}} & 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\widehat{\mathcal{C}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix}}_{=0_{r_i\times(n-r^s)}} \\ \hline 0_{(\sum_{j=i+1}^{m} r_j)\times(\sum_{j=1}^{i-1} r_j)} & 0_{(\sum_{j=i+1}^{m} r_j)\times r_i} & I_{\sum_{j=i+1}^{m} r_j} & 0_{(\sum_{j=i+1}^{m} r_j)\times(n-r^s)} \\ \hline 0_{(n-r^s)\times(\sum_{j=1}^{i-1} r_j)} & \widehat{\mathcal{N}}_i \begin{bmatrix} I_{r_i} \\ 0_{(n-r^s)\times r_i} \end{bmatrix} & 0_{(n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & \underbrace{\widehat{\mathcal{N}}_i \begin{bmatrix} 0_{r_i\times(n-r^s)} \\ I_{n-r^s} \end{bmatrix}}_{=I_{n-r^s}} \end{array}\right] \overset{(2.13)}{=} \widehat{U}_i$$

and, since

$$\left[I_{r_i}, 0_{r_i\times(n-r^s)}\right] \widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1} = \widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1} = I_{r_i} ,$$

it follows from (3.17) and (3.18) that

$$\widehat{U}_i^{-1} =$$
$$\left[\begin{array}{c|c|c|c} I_{\sum_{j=1}^{i-1} r_j} & 0_{(\sum_{j=1}^{i-1} r_j)\times r_i} & 0_{(\sum_{j=1}^{i-1} r_j)\times(\sum_{j=i+1}^{m} r_j)} & 0_{(\sum_{j=1}^{i-1} r_j)\times(n-r^s)} \\ \hline 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\begin{matrix}\left[I_{r_i}, 0_{r_i\times(n-r^s)}\right]\\ \cdot\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\end{matrix}}_{=I_{r_i}} & 0_{r_i\times(\sum_{j=1}^{i-1} r_j)} & \underbrace{\left[I_{r_i}, 0_{r_i\times(n-r^s)}\right]\widehat{\mathcal{V}}_i}_{=0_{r_i\times(n-r^s)}} \\ \hline 0_{(\sum_{j=i+1}^{m} r_j)\times(\sum_{j=1}^{i-1} r_j)} & 0_{(\sum_{j=i+1}^{m} r_j)\times r_i} & I_{\sum_{j=i+1}^{m} r_j} & 0_{(\sum_{j=i+1}^{m} r_j)\times(n-r^s)} \\ \hline 0_{(n-r^s)\times(\sum_{j=1}^{i-1} r_j)} & \begin{matrix}\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right]\\ \cdot\widehat{\mathcal{B}}_i(\widehat{\mathcal{C}}_i\widehat{\mathcal{B}}_i)^{-1}\end{matrix} & 0_{(n-r^s)\times(\sum_{j=i+1}^{m} r_j)} & \underbrace{\left[0_{(n-r^s)\times r_i}, I_{n-r^s}\right]\widehat{\mathcal{V}}_i}_{=I_{n-r^s}} \end{array}\right]. \tag{3.19}$$

Recall $\widehat{A} = \widehat{U}A\widehat{U}^{-1}$ given by (3.5)–(3.7). First apply the transformation $\widehat{U}_1$. Then, omitting the dimensions of the zeros and identity matrices in $\widehat{U}_1$, it follows that

$$\begin{aligned}
\widehat{U}_1\widehat{A}\widehat{U}_1^{-1} &= \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline \widehat{\mathcal{N}}_1\begin{bmatrix} I \\ 0\end{bmatrix} & 0 & \underbrace{\widehat{\mathcal{N}}_1\begin{bmatrix} 0 \\ I\end{bmatrix}}_{=I} \end{array}\right] \left[\begin{array}{c|ccc|c} \widehat{A}_{1,1} & \dots & \widehat{A}_{1,m} & & \widehat{S}_1 \\ \hline \vdots & \ddots & \vdots & & \vdots \\ \widehat{A}_{m,1} & \dots & \widehat{A}_{m,m} & & \widehat{S}_m \\ \hline \widehat{P}_1 & \dots & \widehat{P}_m & & \widehat{Q} \end{array}\right] \left[\begin{array}{c|c|c} \underbrace{[I,0]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}}_{=I} & 0 & 0 \\ \hline 0 & I & 0 \\ \hline [0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & 0 & I \end{array}\right] \\
&= \left[\begin{array}{c|c|c} \widehat{A}_{1,1} & \widehat{A}_{1,2} \quad \dots \quad \widehat{A}_{1,m} & \widehat{S}_1 \\ \hline \widehat{A}_{2,1} & \widehat{A}_{2,2} \quad \dots \quad \widehat{A}_{2,m} & \widehat{S}_2 \\ \vdots & \vdots \qquad\qquad \vdots & \vdots \\ \widehat{A}_{m,1} & \widehat{A}_{m,2} \quad \dots \quad \widehat{A}_{m,m} & \widehat{S}_m \\ \hline \widehat{\mathcal{N}}_1\begin{bmatrix} I \\ 0\end{bmatrix}\widehat{A}_{1,1} + \widehat{\mathcal{N}}_1\begin{bmatrix} 0 \\ I\end{bmatrix}\widehat{P}_1 & \begin{matrix}\widehat{\mathcal{N}}_1\begin{bmatrix} I \\ 0\end{bmatrix}\left[\widehat{A}_{1,2},\dots,\widehat{A}_{1,m}\right] \\ +I\left[\widehat{P}_2,\dots,\widehat{P}_m\right]\end{matrix} & \widehat{\mathcal{N}}_1\begin{bmatrix} I \\ 0\end{bmatrix}\widehat{S}_1 + \widehat{\mathcal{N}}_1\begin{bmatrix} 0 \\ I\end{bmatrix}\widehat{Q} \end{array}\right] \\
&\quad \cdot \left[\begin{array}{c|c|c} \underbrace{[I,0]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}}_{=I} & 0 & 0 \\ \hline 0 & I & 0 \\ \hline [0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & 0 & I \end{array}\right] \\
&= \left[\begin{array}{c|c|c} \widehat{A}_{1,1}I + \widehat{S}_1\,[0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & \widehat{A}_{1,2} \quad \dots \quad \widehat{A}_{1,m} & \widehat{S}_1 \\ \hline \begin{bmatrix} \widehat{A}_{2,1} \\ \vdots \\ \widehat{A}_{m,1}\end{bmatrix} + \begin{bmatrix} \widehat{S}_2 \\ \vdots \\ \widehat{S}_m\end{bmatrix}[0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} & \begin{matrix} \widehat{A}_{2,2} & \dots & \widehat{A}_{2,m} \\ \vdots & & \vdots \\ \widehat{A}_{m,2} & \dots & \widehat{A}_{m,m}\end{matrix} & \begin{matrix}\widehat{S}_2 \\ \vdots \\ \widehat{S}_m\end{matrix} \\ \hline \begin{matrix} \widehat{\mathcal{N}}_1\left[\begin{bmatrix} I \\ 0\end{bmatrix}\widehat{A}_{1,1} + \begin{bmatrix} 0 \\ I\end{bmatrix}\widehat{P}_1\right][I,0]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1} \\ +\widehat{\mathcal{N}}_1\left[\begin{bmatrix} I \\ 0\end{bmatrix}\widehat{S}_1 + \begin{bmatrix} 0 \\ I\end{bmatrix}\widehat{Q}\right][0,I]\,\widehat{\mathcal{B}}_1(\widehat{\mathcal{C}}_1\widehat{\mathcal{B}}_1)^{-1}\end{matrix} & \begin{matrix}\widehat{\mathcal{N}}_1\begin{bmatrix} I \\ 0\end{bmatrix}\left[\widehat{A}_{1,2},\dots,\widehat{A}_{1,m}\right] \\ +\left[\widehat{P}_2,\dots,\widehat{P}_m\right]\end{matrix} & \widehat{\mathcal{N}}_1\begin{bmatrix} I \\ 0\end{bmatrix}\widehat{S}_1 + \widehat{Q} \end{array}\right].
\end{aligned} \tag{3.20}$$

Furthermore, for $j \in \{2, \ldots, m\}$,

$$\begin{aligned}
&\widehat{A}_{1,1} I_{r_1} + \widehat{S}_1 \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad = \quad \widehat{A}_{1,1} + \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ S^1 \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(3.11)}{=} \quad \begin{bmatrix} 0_{(r_1-1)\times 1}, I_{r_1-1} \\ \left[\widehat{R}^i_{1,1}, \ldots, \widehat{R}^i_{1,r_1}\right] \end{bmatrix} + \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ c^1_{(n)} A^{r_1} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(3.12)}{=} \quad \begin{bmatrix} 0_{(r_1-1)\times 1}, I_{r_1-1} \\ c^1_{(n)} A^{r_1} \mathcal{B} (\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} I_{r_1} \\ 0_{(r_s-r_1)\times r_1} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ c^1_{(n)} A^{r_1} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
&\widehat{A}_{j,1} + \widehat{S}_j \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(3.11)}{=} \quad \begin{bmatrix} 0_{(r_j-1)\times r_1} \\ \left[\widehat{R}^j_{1,1}, \ldots, \widehat{R}^j_{1,r_1}\right] \end{bmatrix} + \begin{bmatrix} 0_{(r_j-1)\times(n-r^s)} \\ c^j_{(n)} A^{r_j} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
&\quad \overset{(3.12)}{=} \quad \begin{bmatrix} 0_{(r_j-1)\times r_1} \\ c^j_{(n)} A^{r_j} \mathcal{B} (\mathcal{C}\mathcal{B})^{-1} \begin{bmatrix} I_{r_1} \\ 0_{(r_s-r_1)\times r_1} \end{bmatrix} \end{bmatrix} + \begin{bmatrix} 0_{(r_j-1)\times(n-r^s)} \\ c^j_{(n)} A^{r_j} \mathcal{V} \end{bmatrix} \left[0_{(n-r^s)\times r_1}, I_{r_1}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \qquad (3.22)
\end{aligned}$$

$$\begin{aligned}
&\widehat{\mathcal{N}}_1 \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \widehat{A}_{1,j} \\
&\quad \overset{(3.17)}{=} \quad \left[0_{(n-r^s)\times r_1}, I_{n-r^s}\right] \left[I_{r_1+n-r^s} - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \widehat{\mathcal{C}}_1\right] \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \left[\widehat{R}^1_{j,1}, \ldots, \widehat{R}^1_{j,r_j}\right] \end{bmatrix} \\
&\quad \overset{(2.10)}{=} \quad \left[0_{(n-r^s)\times r_1}, I_{n-r^s}\right] \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \left[\widehat{R}^1_{j,1}, \ldots, \widehat{R}^1_{j,r_j}\right] \end{bmatrix} \\
&\qquad\quad - \left[0_{(n-r^s)\times r_1}, I_{n-r^s}\right] \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \left[I_{r_1}, 0_{r_1\times(n-r^s)}\right] \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \left[\widehat{R}^1_{j,1}, \ldots, \widehat{R}^1_{j,r_j}\right] \end{bmatrix} \\
&\quad \overset{(3.16)}{=} \quad 0_{r_1\times r_j} - \left[\mathcal{X}_{(n-r^s)\times(r_1-1)}, 0_{(n-r^s)\times 1}\right] \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ \left[\widehat{R}^1_{j,1}, \ldots, \widehat{R}^1_{j,r_j}\right] \end{bmatrix} \\
&\quad = \quad 0_{r_1\times r_j} \qquad (3.23)
\end{aligned}$$

$$\begin{aligned}
&\widehat{\mathcal{N}}_1 \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \widehat{S}_j \\
&\quad \overset{(3.11)}{=} \quad \left[0_{(n-r^s)\times r_1}, I_{n-r^s}\right] \left[I_{r_1+n-r^s} - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \widehat{\mathcal{C}}_1\right] \begin{bmatrix} I_{r_1} \\ 0_{(n-r^s)\times r_1} \end{bmatrix} \begin{bmatrix} 0_{(r_1-1)\times r_j} \\ S^1 \end{bmatrix} \\
&\quad \overset{(3.23)}{=} \quad 0_{r_1\times r_j} \qquad (3.24)
\end{aligned}$$

and

$$
\begin{aligned}
& \widehat{\mathcal{N}}_1 \left[\begin{bmatrix} I \\ 0 \end{bmatrix} \widehat{A}_{1,1} + \begin{bmatrix} 0 \\ I \end{bmatrix} \widehat{P}_1 \right] [I,0]\, \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} + \widehat{\mathcal{N}}_1 \left[\begin{bmatrix} I \\ 0 \end{bmatrix} \widehat{S}_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} \widehat{Q} \right] [0,I]\, \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
= \;& \widehat{\mathcal{N}}_1 \left(\begin{bmatrix} \widehat{A}_{1,1} \\ \widehat{P}_1 \end{bmatrix} [I,0] + \begin{bmatrix} \widehat{S}_1 \\ \widehat{Q} \end{bmatrix} [0,I] \right) \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
= \;& \widehat{\mathcal{N}}_1 \widehat{A}_1 \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
= \;& \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T \right)^{-1} \widehat{\mathcal{V}}_1^T \left[I_{r_1+n-r^s} - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \widehat{\mathcal{C}}_1 \right] \widehat{A}_1 \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
= \;& \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T \right)^{-1} \widehat{\mathcal{V}}_1^T \left[\widehat{A}_1 \widehat{\mathcal{B}}_1 - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \widehat{\mathcal{C}}_1 \widehat{A}_1 \widehat{\mathcal{B}}_1 \right] (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
= \;& \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T \right)^{-1} \widehat{\mathcal{V}}_1^T \Big[\left[(\widehat{\mathcal{B}}_1)_2^{(r_1+n-r^s)}, \ldots, (\widehat{\mathcal{B}}_1)_{r_1}^{(r_1+n-r^s)}, * \right] \\
& \qquad\qquad - \widehat{\mathcal{B}}_1 (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \left[(\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)_2^{(r_1)}, \ldots, (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)_2^{(r_1)}, * \right] \Big] (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
= \;& \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T \right)^{-1} \widehat{\mathcal{V}}_1^T \left[\left[(\widehat{\mathcal{B}}_1)_2^{(r_1+n-r^s)}, \ldots, (\widehat{\mathcal{B}}_1)_{r_1}^{(r_1+n-r^s)}, * \right] - \widehat{\mathcal{B}}_1 \begin{bmatrix} 0 & * \\ I_{r_1-1} & * \end{bmatrix} \right] (\widehat{\mathcal{C}}_1 \widehat{\mathcal{B}}_1)^{-1} \\
\overset{(3.14)}{=} \;& \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T \right)^{-1} \widehat{\mathcal{V}}_1^T [0, \ldots, 0, *] \begin{bmatrix} * & \cdots & * & 1 \\ \vdots & \iddots & \iddots & 0 \\ * & 1 & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \\
= \;& \left(\widehat{\mathcal{V}}_1 \widehat{\mathcal{V}}_1^T \right)^{-1} \widehat{\mathcal{V}}_1^T [*, 0, \ldots, 0] \,. \qquad (3.25)
\end{aligned}
$$

Hence the equations (3.21)–(3.25) show that only the first r_1 columns of $\widehat{A}$ change when applying the transformation $\widehat{U}_1$. Furthermore the first r_1 columns of $\widehat{U}_1 \widehat{A} \widehat{U}_1^{-1}$ are equal to the first r_1 columns of $\widetilde{A}$ and by (3.21)–(3.25) equations (2.15) and (2.17) hold for $i = 1$.

Moreover an application of the transformation $\widehat{U}_i$, $i \in \{2, \ldots m\}$, has the similar effect as in (3.20)–(3.25) on the r_i columns from column number $\sum_{j=1}^{i-1} r_j + 1$ to column number $\sum_{j=1}^{i} r_j$ of matrix $\left(\widehat{U}_{i-1} \ldots \widehat{U}_1 \widehat{A} \widehat{U}_1^{-1} \ldots \widehat{U}_{i-1}^{-1} \right)$, which, when finally all transformation matrices $\widehat{U}_i$ are applied, yields (2.2) and (2.15)–(2.18). This completes the proof. □

3.4 Proof of the zero dynamics

Now a proof for the characterization and stability of the zero dynamics of linear MIMO-systems with ordered vector relative degree is given.

Proof of Proposition 2.6

(i) Set

$$
\mathcal{Z} = \left\{ (\mathcal{V}\eta, -\Gamma^{-1} S \eta) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \times \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \,\middle|\, \dot{\eta} = Q\eta \right\} .
$$

"$\subseteq$": If $(x, u) \in \mathcal{ZD}(A, B, C)$ then $y \equiv 0$ on $[0, \infty)$ and so

$$
\xi = \left(y_1, y_1^{(1)}, \ldots, y_1^{(r_1-1)} \middle| y_2, \ldots, y_2^{(r_2-1)} \middle| \ldots \middle| y_m, \ldots, y_m^{(r_m-1)} \right)^T \equiv 0 \,,
$$

which yields, in view of (2.1)–(2.2),

$$0_{r^s\times 1} = \begin{bmatrix} 0_{(r_1-1)\times(n-r^s)} \\ S^1 \\ \vdots \\ 0_{(r_m-1)\times(n-r^s)} \\ S^m \end{bmatrix} \eta + \begin{bmatrix} 0_{(r_1-1)\times m} \\ c^1_{(n)} A^{r_1-1} B \\ \vdots \\ 0_{(r_m-1)\times m} \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix} u$$

$$\dot\eta = Q\eta\,,$$

and

$$0_{m\times 1} = \underbrace{\begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix}}_{=:S} \eta + \begin{bmatrix} c^1_{(n)} A^{r_1-1} B \\ \vdots \\ c^m_{(n)} A^{r_m-1} B \end{bmatrix} u$$

thus (2.4) yields $u = -\Gamma^{-1} S\eta$. Since $x = U^{-1}\left(\xi^T, \eta^T\right)^T$ it follows from (3.3) and (3.19) that $(x, u) = (\mathcal{V}\eta, -\Gamma^{-1}S\eta)$ for η being a solution of $\dot\eta = Q\eta$ and therefore $(x, u) \in \mathcal{Z}$.

"$\supseteq$": If $(\tilde x, \tilde u) = (\mathcal{V}\eta, -\Gamma^{-1}S\eta) \in \mathcal{Z}$, then, by (2.7), we have

$$\tilde y := C\tilde x = C\mathcal{V}\eta \equiv 0\,,$$

and so

$$\tilde\xi = \left(\tilde y_1, \ldots, \tilde y_1^{(r_1-1)} \,\middle|\, \tilde y_2, \ldots, \tilde y_2^{(r_2-1)} \,\middle|\, \ldots \,\middle|\, \tilde y_m, \ldots, \tilde y_m^{(r_m-1)}\right)^T \equiv 0\,,$$

and therefore $\left(\begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tilde u\right)$ solves the first equation in (2.1) with $y \equiv 0$ on $[0, \infty)$. Thus it follows that

$$(\tilde x, \tilde u) = \left(U^{-1}\begin{pmatrix} 0 \\ \eta \end{pmatrix}, \tilde u\right) = (\mathcal{V}\eta, \tilde u) \in \mathcal{ZD}(A, B, C)\,.$$

(ii) From (i) it follows that

$$x = \mathcal{V}\eta \qquad \text{and} \qquad \eta = (\mathcal{V}^T\mathcal{V})^{-1}\mathcal{V}^T x\,,$$

where η is a solution of $\dot\eta = Q\eta$. Thus

$$\exists\, M, \lambda > 0\ \forall\, (x, u) \in \mathcal{ZD}(A, B, C)\ \forall\, t \ge 0\ :\ \|(x(t), u(t))\| \le M e^{-\lambda t}\|x(0)\|\,.$$

if, and only if, $\dot\eta = Q\eta$ is an asymptotically stable system. □

3.5 Proof of the right-invertibility

Proof of Proposition 2.7

"$\Rightarrow$": If $y = y_R$ then by (2.1) $\xi = \left(y_{R1}, \ldots, y_{R1}^{(r_1-1)} \,\middle|\, \ldots \,\middle|\, y_{Rm}, \ldots, y_{Rm}^{(r_m-1)}\right)^T$. Thus, by (2.1) it follows that

$$y_{R_j}^{(r_j)} = y_j^{r_j-1} = \dot\xi_{\sum_{i=1}^j r_i} = R^j\xi + S^j\eta + c^j_{(m)} A^{r_j-1} B u\,, \qquad j \in \{1, \ldots, m\}\,,$$

hence

$$\begin{bmatrix} y_{R_1}^{(r_1)} \\ \vdots \\ y_{Rm}^{(r_m)} \end{bmatrix} = \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi + \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta + \underbrace{\begin{bmatrix} c_{(m)}^1 A^{r_1-1}B \\ \vdots \\ c_{(m)}^m A^{r_m-1}B \end{bmatrix}}_{=\Gamma} u$$

and so

$$u = \Gamma^{-1} \left(\begin{bmatrix} y_{R_1}^{(r_1)} \\ \vdots \\ y_{Rm}^{(r_m)} \end{bmatrix} - \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi - \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta \right).$$

Furthermore it follows that η has to be a solution of

$$\dot{\eta} = \left[P_1, 0, \ldots, 0 \,\middle|\, P_2, 0, \ldots, 0 \,\middle|\, \ldots \,\middle|\, P_m, 0, \ldots, 0\right] \xi + Q\eta = Q\eta + [P_1, \ldots, P_m]\, y_R$$

for any initial value $\eta(0) \in \mathbb{R}^{n-r^s}$.

"$\Leftarrow$": Assume the (ii) holds. By (2.1) it follows that

$$\dot{\xi}_{\sum_{i=1}^{j} r_i} = R^j\xi + S^j\eta + \underbrace{c_{(m)}^j A^{r_j-1}B\Gamma^{-1}}_{=e_{(m)}^j} \left(\begin{bmatrix} y_{R_1}^{(r_1)} \\ \vdots \\ y_{Rm}^{(r_m)} \end{bmatrix} - \begin{bmatrix} R^1 \\ \vdots \\ R^m \end{bmatrix} \xi - \begin{bmatrix} S^1 \\ \vdots \\ S^m \end{bmatrix} \eta \right) = y_{R_j}^{(r_j)}$$

where, for any $\eta^0 \in \mathbb{R}^{n-r^s}$, η is a solution of the initial value problem

$$\dot{\eta} = [P_1, \ldots, P_m]\, y_R + Q\eta\,, \qquad \eta(0) = \eta^0\,.$$

This yields, in view of (2.1), $\xi = \left(y_{R1}, \ldots, y_{R_1}^{(r_1-1)} \,\middle|\, \ldots \,\middle|\, y_{Rm}, \ldots, y_{Rm}^{(r_m-1)}\right)^T$ and thus $y = \left(\xi_1, \xi_{r_1+1}, \ldots, \xi_{\sum_{i=1}^{m-1} r_i+1}\right)^T = y_R$. Hence that last row of the normal form (2.1) reads

$$\dot{\eta} = [P_1, \ldots, P_m]\, y_R + Q\eta = \left[P_1, 0, \ldots, 0 \,\middle|\, P_2, 0, \ldots, 0 \,\middle|\, \ldots \,\middle|\, P_m, 0, \ldots, 0\right] \xi + Q\eta$$

which completes the proof. □

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