G-Selfadjoint Operators in Almost Pontryagin Spaces

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Abstract. An Almost Pontryagin space $(\mathcal{H}, [\cdot\,,\cdot])$ admits a decomposition

$$\mathcal{H} = \mathcal{H}_+ [\dot{+}] \mathcal{H}_- [\dot{+}] \mathcal{H}^\circ,$$

where $(\mathcal{H}_+, [\cdot\,,\cdot])$ and $(\mathcal{H}_-, -[\cdot\,,\cdot])$ are Hilbert spaces and $\mathcal{H}_-$ as well as $\mathcal{H}^\circ$ are finite dimensional. Based on the theory of linear relations we introduce the notion of G-selfadjoint operators in Almost Pontryagin spaces and study their spectral properties. In particular, we construct a spectral function for G-selfadjoint operators in Almost Pontryagin spaces. Finally, we apply our results to the Klein-Gordon equation.

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1. Introduction

Pontryagin spaces are inner product spaces which can be written as a direct and orthogonal sum of a Hilbert space and a finite dimensional anti-Hilbert space.

An Almost Pontryagin space is an inner product space $(\mathcal{H}, [\cdot\,,\cdot])$ which can be written as the direct and orthogonal sum of a Hilbert space, a finite dimensional anti-Hilbert space and a finite dimensional neutral space. Such a decomposition defines a Hilbert space topology $\mathcal{O}$ on $\mathcal{H}$ in a natural way such that the inner product $[\cdot\,,\cdot]$ is continuous with respect to $\mathcal{O}$.

Conversely, if $(\mathcal{H}, (\cdot\,,\cdot))$ is a Hilbert space and if G is a bounded selfadjoint operator in $\mathcal{H}$ such that $\sigma(G) \cap (-\infty, \varepsilon)$ consists of finitely many eigenvalues of G with finite multiplicities for some $\varepsilon > 0$, then $(\mathcal{H}, [\cdot\,,\cdot])$,

$$[\cdot\,,\cdot] := (G\,\cdot\,,\cdot),$$

equipped with the Hilbert space topology induced by $(\cdot\,,\cdot)$ is an Almost Pontryagin space. The main subject of this paper are Almost Pontryagin spaces and operators therein.

We call a densely defined linear operator A in an Almost Pontryagin space $\mathcal{H}$ G-symmetric if $[Ax, y] = [x, Ay]$ holds for all $x, y \in \operatorname{dom} A$, or, what is the same, if GA is a symmetric operator in the Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$. This is equivalent to $A \subset A^+$, where A^+ is the adjoint of A, i.e.

$$A^+ = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : [Ax, u] = [x, v] \text{ for all } x \in \operatorname{dom} A \right\}. \tag{1.1}$$

The adjoint A^+ in (1.1) is not an operator, in general, but a closed linear relation in $\mathcal{H} \times \mathcal{H}$, cf. [DS]. Obviously, if the isotropic part $\mathcal{H}^\circ$ of $\mathcal{H}$ is not trivial, we have for all operators A in $\mathcal{H}$

$$A \neq A^+.$$

The question arises if there is a class of symmetric operators in Almost Pontryagin spaces which complies with that of selfadjoint operators in Pontryagin spaces. We call a densely defined operator A in $\mathcal{H}$ G-selfadjoint if $GA = (GA)^*$ and if A is closed. Obviously, if $(\mathcal{H}, [\cdot\,,\cdot])$ is a Pontryagin space (i.e. $\mathcal{H}^\circ = \{0\}$), an operator A in $\mathcal{H}$ is G-selfadjoint if and only if $A = A^+$. Such operators are necessarily closed. However, in an Almost Pontryagin space there are densely defined operators A with $GA = (GA)^*$ which are not closable (see Example 4.9 below).

We show that G-selfadjoint operators in Almost Pontryagin spaces have similar properties as selfadjoint operators in Pontryagin spaces. For instance, their spectrum is real, with possible exception of a finite set of non-real eigenvalues with finite multiplicities. Moreover, they possess a spectral function with finitely many singularities.

Almost Pontryagin spaces and operators therein were considered in various situations, we mention only [KW1, KW2, KW3, LMT, W1, W2]. Here, we consider as an application the Klein-Gordon equation with assumptions leading to a setting with an Almost Pontryagin space.

We proceed as follows. In Section 2 we recall some commonly known notions which are related to inner product spaces. Section 3 starts with the definition of Almost Pontryagin spaces. Some properties concerning the geometry are collected and we recall some results from [KWW]. We focus especially on the differences between Almost Pontryagin spaces and Pontryagin spaces. E.g., given a subspace $\mathcal{L}$ in a Pontryagin space $(\Pi, [\cdot\,,\cdot])$ with $\mathcal{L} = \mathcal{L}_+ [\dotplus] \mathcal{L}_- [\dotplus] \mathcal{L}^\circ$ we have the well-known decomposition

$$\Pi = \mathcal{L}_+ [\dotplus] \mathcal{L}_- [\dotplus] (\mathcal{L}^\circ \dotplus \mathcal{P}) [\dotplus] \mathcal{M},$$

where $\mathcal{L}^\circ \dotplus \mathcal{P}$ is non-degenerate, cf. [B]. We generalize this fact to a subspace $\mathcal{L}$ in an Almost Pontryagin space $(\mathcal{H}, [\cdot\,,\cdot])$ and obtain a decomposition

$$\mathcal{H} = \mathcal{L}_+ [\dotplus] \mathcal{L}_- [\dotplus] \mathcal{L}_{00} [\dotplus] (\mathcal{L}_{01} \dotplus \mathcal{P}) [\dotplus] \mathcal{M},$$

where $\mathcal{L}_{00} = \mathcal{L}^\circ \cap \mathcal{H}^\circ$, $\mathcal{L}^\circ = \mathcal{L}_{00} [\dotplus] \mathcal{L}_{01}$, $\mathcal{L}_{01} \dotplus \mathcal{P}$ is non-degenerate and $\mathcal{L}^{[\perp]} = \mathcal{L}^\circ [\dotplus] \mathcal{M}$.

In Section 4 we define G-symmetric and G-selfadjoint operators in Almost Pontryagin spaces and give several necessary and sufficient conditions for G-selfadjointness. In particular, we show that a G-selfadjoint operator A admits a matrix representation with respect to any fundamental decomposition $\mathcal{H} = \mathcal{H}^\circ[\dot{+}]\Pi$,

$$A = \begin{pmatrix} A_0 & A_{12} \\ 0 & \widetilde{A} \end{pmatrix}, \quad \operatorname{dom} A = \mathcal{H}^\circ \dot{+} \operatorname{dom} \widetilde{A}, \tag{1.2}$$

where $\widetilde{A}$ is a selfadjoint operator in the Pontryagin space $(\Pi, [\cdot\,,\cdot])$. Moreover, we prove that this is equivalent to

$$\mathcal{H}^\circ \subset \operatorname{dom} A \quad \text{and} \quad A^+ = A^{++}.$$

Finally, we describe the spectrum of G-selfadjoint operators and show in Section 5 that G-selfadjoint operators in Almost Pontryagin spaces possess a spectral function with singularities.

In Section 6 we apply the results of Sections 3-5 to the Klein-Gordon equation which describes the motion of a relativistic spinless particle of mass m and charge e in an electrostatic field with potential q,

$$\left(\left(\frac{\partial}{\partial t} - ieq \right)^2 - \Delta + m^2 \right) \psi(t, \vec{x}) = 0, \quad t \in \mathbb{R}, \ \ \vec{x} \in \mathbb{R}^n.$$

We rewrite this equation as in [LNT] and obtain a first order differential equation for $\mathbf{x} = \binom{x}{y}$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = i\hat{A}\mathbf{x}, \quad \hat{A} = \begin{pmatrix} 0 & I \\ H_0 - V^2 & 2V \end{pmatrix}, \tag{1.3}$$

where H_0 is a strictly positive selfadjoint operator and V a symmetric operator in a Hilbert space. As $\hat{A}$ may not even be densely defined nor closed, suitable assumptions have to be imposed (see Section 6 below) on V such that we can associate a closed operator A with the block operator matrix $\hat{A}$. In general, $\hat{A}$ does not exhibit symmetry in any Hilbert space. But it is symmetric with respect to a so-called energy inner product which is in general an indefinite inner product.

For the Klein-Gordon equation the formal operator-matrix $\hat{A}$ in the energy inner product has been studied in a number of papers. We mention here only [J, Kk, N1, N2, N3, LNT] and the references given in [LNT].

In [LNT] the assumptions on V were chosen such that the operator A is selfadjoint in some Pontryagin space. Then (cf. [LNT]) the differential equation (1.3) is solvable for any given initial value. We weaken the assumptions from [LNT] in such a way that A is selfadjoint in an Almost Pontryagin space and we obtain with the help of the spectral function for selfadjoint operators in Almost Pontryagin spaces (Section 5) similar results as in [LNT].

As a final remark we mention that some of our results can also be obtained by considering the factor space $\mathcal{H}/\mathcal{H}^\circ$, which is a Pontryagin space, see e.g. [IKL]. E.g., the operator A induces a selfadjoint operator in this Pontryagin space which corresponds to $\widetilde{A}$ in 1.2. However, it is the aim of this paper to show that one can obtain a spectral theory for G-selfadjoint operators in Almost Pontryagin spaces $\mathcal{H}$ without changing the underlying space $\mathcal{H}$.

2. Preliminaries

In this section we introduce basic notions related to inner product spaces and collect some properties. In particular we consider spaces with an inner product which is continuous with respect to some Hilbert space scalar product.

An inner product space $(\mathcal{H}, [\cdot\,,\cdot])$ is a complex vector space $\mathcal{H}$ with an inner product $[\cdot\,,\cdot] : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that for $x, y, z \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$ we have

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z] \quad \text{and} \quad \overline{[x, y]} = [y, x].$$

Let $(\mathcal{H}, [\cdot\,,\cdot])$ be an inner product space. If $\mathcal{L} \subset \mathcal{H}$ is a linear manifold, the *orthogonal companion of* $\mathcal{L}$ *(in* $\mathcal{H}$*)* is defined by

$$\mathcal{L}^{[\perp]} := \{x \in \mathcal{H} : [x, \ell] = 0 \ \text{ for all } \ \ell \in \mathcal{L}\}.$$

The *isotropic part of* $\mathcal{L}$ is the set of all vectors in $\mathcal{L}$ which are $[\cdot\,,\cdot]$-orthogonal to $\mathcal{L}$, i.e.

$$\mathcal{L}^{\circ} := \mathcal{L} \cap \mathcal{L}^{[\perp]}.$$

We call the linear manifold $\mathcal{L}$ *non-degenerate* if $\mathcal{L}^{\circ} = \{0\}$. If $\mathcal{N} \subset \mathcal{H}$ is a linear manifold with $\mathcal{N} \cap \mathcal{L} = \{0\}$ and $\mathcal{N} \subset \mathcal{L}^{[\perp]}$, then by $\mathcal{N}[\dot{+}]\mathcal{L}$ we denote the *direct orthogonal sum of* $\mathcal{N}$ *and* $\mathcal{L}$.

A vector $x \in \mathcal{H}$ is called positive (negative, neutral) if $[x, x] > 0$ (resp. $[x, x] < 0$, $[x, x] = 0$), and nonnegative (nonpositive) if x is not negative (resp. not positive). A linear manifold $\mathcal{L} \subset \mathcal{H}$ is called positive (negative, neutral, nonnegative, nonpositive) if all vectors in $\mathcal{L} \setminus \{0\}$ are positive (resp. negative, neutral, nonnegative, nonpositive). The linear manifold $\mathcal{L}$ is called maximal positive if it is positive and if there is no positive linear manifold $\mathcal{L}' \neq \mathcal{L}$ containing $\mathcal{L}$.

Lemma 2.1. *Let* $(\mathcal{H}, [\cdot\,,\cdot])$ *be a nonnegative inner product space. Then a linear manifold* $\mathcal{L} \subset \mathcal{H}$ *is maximal positive if and only if* $\mathcal{H} = \mathcal{L} \dot{+} \mathcal{H}^{\circ}$*. If there exists a maximal positive linear manifold* $\mathcal{H}_+ \subset \mathcal{H}$ *such that* $(\mathcal{H}_+, [\cdot\,,\cdot])$ *is a Hilbert space, then for all maximal positive linear manifolds* $\mathcal{H}'_+ \subset \mathcal{H}$ *the inner product space* $(\mathcal{H}'_+, [\cdot\,,\cdot])$ *is a Hilbert space.*

Proof. If $\mathcal{H} = \mathcal{L} \dot{+} \mathcal{H}^{\circ}$ then $\mathcal{L}$ is maximal positive by [AI, 1.25, Chapter 1]. Let $\mathcal{L}$ be maximal positive in $\mathcal{H}$. If $x_0 \notin \mathcal{L} \dot{+} \mathcal{H}^{\circ}$, then with [AI, 1.17, Chapter 1] we obtain that for $x \in \operatorname{span}\{x_0\} \setminus \{0\}$ and $\ell \in \mathcal{L}$ the vector $x + l$ is positive. Hence, $\mathcal{L}' := \mathcal{L} \dot{+} \operatorname{span}\{x_0\}$ is positive which contradicts the fact that $\mathcal{L}$ is maximal positive. This implies $\mathcal{H} = \mathcal{L} \dot{+} \mathcal{H}^{\circ}$.

Let $\mathcal{H}'_+$ be maximal positive and let (x'_n) be a Cauchy sequence in the pre-Hilbert space $(\mathcal{H}'_+, [\cdot\,,\cdot])$. With $x_n^+ \in \mathcal{H}_+$ and $x_n^{\circ} \in \mathcal{H}^{\circ}$ such that $x'_n = x_n^+ + x_n^{\circ}$ it follows that (x_n^+) is a Cauchy sequence in $(\mathcal{H}_+, [\cdot\,,\cdot])$. Thus, there exists $x^+ \in \mathcal{H}_+$

such that $[x_n^+ - x^+, x_n^+ - x^+] \to 0$ as $n \to \infty$. As $\mathcal{H}'_+$ is maximal positive there exist vectors $x'_+ \in \mathcal{H}'_+$ and $x'_0 \in \mathcal{H}^\circ$ such that $x^+ = x'_+ + x'_0$. Thus, we have

$$\begin{aligned}[x'_n - x'_+, x'_n - x'_+] &= [x_n^+ + x_n^\circ - x'_+, x_n^+ + x_n^\circ - x'_+] = [x_n^+ - x'_+, x_n^+ - x'_+] \\ &= [x_n^+ - x^+ + x'_0, x_n^+ - x^+ + x'_0] = [x_n^+ - x^+, x_n^+ - x^+],\end{aligned}$$

which converges to zero as $n \to \infty$. □

Now, suppose that $\mathcal{O}$ is a Hilbert space topology on $\mathcal{H}$ and that the inner product $[\cdot\,,\cdot]$ is $\mathcal{O}$-continuous, i.e. for any Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ which induces $\mathcal{O}$ there exists some $c > 0$ such that

$$|[x, y]| \leq c\,\|x\|\|y\| \quad \text{for all} \;\; x, y \in \mathcal{H}.$$

In the following all topological notions are related to the Hilbert space topology $\mathcal{O}$. By a subspace we always understand a closed linear manifold. Note that $\mathcal{L}^{[\perp]}$ is a subspace for every linear manifold $\mathcal{L} \subset \mathcal{H}$, and we have $\mathcal{L}^{[\perp]} = \overline{\mathcal{L}}^{[\perp]}$. Recall (cf. [B, Theorem IV.5.2]) that a subspace $\mathcal{L} \subset \mathcal{H}$ always admits a decomposition

$$\mathcal{L} = \mathcal{L}_+ [\dot{+}] \mathcal{L}_- [\dot{+}] \mathcal{L}^\circ, \tag{2.1}$$

where $\mathcal{L}_+$ is a positive subspace, $\mathcal{L}_-$ is a negative subspace and the projections in $\mathcal{L}$ onto $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}^\circ$, respectively, corresponding to the decomposition (2.1) are continuous. We shall call a decomposition (2.1) with the above properties a *fundamental decomposition of* $\mathcal{L}$. It is easily seen that the numbers

$$\kappa_+(\mathcal{L}) := \dim \mathcal{L}_+, \;\; \kappa_-(\mathcal{L}) := \dim \mathcal{L}_- \;\; \text{and} \;\; \kappa_0(\mathcal{L}) := \dim \mathcal{L}^\circ$$

do not depend on the fundamental decomposition. We call them the *rank of positivity, rank of negativity* and *rank of degeneracy of* $\mathcal{L}$, respectively. Furthermore, we call the sums

$$\kappa_-(\mathcal{L}) + \kappa_0(\mathcal{L}) \;\; \text{and} \;\; \kappa_+(\mathcal{L}) + \kappa_0(\mathcal{L})$$

the *rank of non-positivity* and the *rank of non-negativity of* $\mathcal{L}$, respectively.

A positive (negative) linear manifold $\mathcal{L} \subset \mathcal{H}$ is called ($\mathcal{O}$-)*uniformly positive* (resp. ($\mathcal{O}$-)*uniformly negative*) if the topology on $\mathcal{L}$ which is induced by the norm $[x, x]^{1/2}$ (resp. $(-[x, x])^{1/2}$), $x \in \mathcal{L}$, coincides with the relative topology induced by $\mathcal{O}$. This is equivalent to the fact that for any norm $\|\cdot\|$ on $\mathcal{H}$ which induces $\mathcal{O}$ there exists a $\delta > 0$ such that

$$[x, x] \geq \delta \|x\|^2 \quad (\text{resp.} \; -[x, x] \geq \delta \|x\|^2) \;\; \text{for all} \;\; x \in \mathcal{L}.$$

A *subspace* $\mathcal{L} \subset \mathcal{H}$ is uniformly positive (uniformly negative) if and only if $(\mathcal{L}, [\cdot\,,\cdot])$ (resp. $(\mathcal{L}, -[\cdot\,,\cdot])$) is a Hilbert space. We will call a linear manifold *uniformly definite* if it is either uniformly positive or uniformly negative.

The linear manifold $\mathcal{L} \subset \mathcal{H}$ is called *ortho-complemented* (*in* $\mathcal{H}$) if

$$\mathcal{H} = \mathcal{L} + \mathcal{L}^{[\perp]}.$$

Lemma 2.2. *Let $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{H}$ be uniformly definite subspaces with $\mathcal{L}_1[\perp]\mathcal{L}_2$. Then their sum is direct, $\mathcal{L} := \mathcal{L}_1[\dot{+}]\mathcal{L}_2$ is a subspace, and we have:*

(i) *$\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}$ are ortho-complemented.*
(ii) *If $\mathcal{L}_1$ and $\mathcal{L}_2$ both are uniformly positive (uniformly negative), then $\mathcal{L}$ is uniformly positive (resp. uniformly negative).*

Proof. We evidently have $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$. It was shown in [LMM, Lemma 2.1] that $\mathcal{L}_1$ and $\mathcal{L}_2$ are ortho-complemented. If by $\mathcal{M}$ we denote the orthogonal companion of $\mathcal{L}_2$ in $\mathcal{L}_1^{[\perp]}$, we have $\mathcal{H} = \mathcal{L}_1[\dot{+}]\mathcal{L}_2[\dot{+}]\mathcal{M}$. From this it follows that $\mathcal{L}$ is a subspace. Hence, $\mathcal{M} = \mathcal{L}^{[\perp]}$, and $\mathcal{L}$ is ortho-complemented. In order to show (ii) let $\mathcal{L}_1$ and $\mathcal{L}_2$ be uniformly positive and let $\|\cdot\|$ be a norm inducing $\mathcal{O}$. Then there exists a $\delta > 0$ such that $[x_i, x_i] \geq 2\delta\|x_i\|^2$ for all $x_i \in \mathcal{L}_i$ $(i = 1, 2)$. Consequently, $[x_1 + x_2, x_1 + x_2] = [x_1, x_1] + [x_2, x_2] \geq \delta(2\|x_1\|^2 + 2\|x_2\|^2) \geq \delta(\|x_1\| + \|x_2\|)^2 \geq \delta\|x_1 + x_2\|^2$ for all $x_1 \in \mathcal{L}_1$ and all $x_2 \in \mathcal{L}_2$. □

3. Geometry of Almost Pontryagin Spaces

We consider now a special case of the spaces from the previous section.

Definition 3.1. Let $(\mathcal{H}, [\cdot\,,\cdot])$ be an inner product space and let $\mathcal{O}$ be a Hilbert space topology on $\mathcal{H}$. The triplet $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ is called an *Almost Pontryagin space with finite rank of non-positivity* (*non-negativity*) if

(i) the inner product $[\cdot\,,\cdot]$ is $\mathcal{O}$-continuous and
(ii) there exists a $\mathcal{O}$-uniformly positive (resp. $\mathcal{O}$-uniformly negative) $\mathcal{O}$-closed linear manifold $\mathcal{L} \subset \mathcal{H}$ with $\operatorname{codim} \mathcal{L} < \infty$.

In the sequel, we will collect some properties of Almost Pontryagin spaces. The following lemma justifies the term "with finite rank of non-positivity (non-negativity)" in Definition 3.1.

Lemma 3.2. *Let $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ be an Almost Pontryagin space with finite rank of non-positivity (non-negativity). Then $\kappa_-(\mathcal{H}) + \kappa_0(\mathcal{H}) < \infty$ (resp. $\kappa_+(\mathcal{H}) + \kappa_0(\mathcal{H}) < \infty$). Moreover, there exists a fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-[\dot{+}]\mathcal{H}^\circ \tag{3.1}$$

of $\mathcal{H}$ in which $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is uniformly positive (resp. uniformly negative). Obviously the fundamental decomposition (3.1) *can be written in the following way*

$$\mathcal{H} = \Pi[\dot{+}]\mathcal{H}^\circ, \tag{3.2}$$

where $(\Pi, [\cdot\,,\cdot])$, $\Pi = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$, is a Pontryagin space.

Proof. Let $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ be an Almost Pontryagin space with finite rank of non-positivity. A similar reasoning applies to Almost Pontryagin spaces with finite rank of non-negativity. Let $\mathcal{L} \subset \mathcal{H}$ be a uniformly positive subspace with $\operatorname{codim} \mathcal{L} < \infty$ and let $\mathcal{N} := \mathcal{L}^{[\perp]}$. If $\mathcal{N} = \mathcal{N}_+[\dot{+}]\mathcal{N}_-[\dot{+}]\mathcal{N}^\circ$ is a fundamental decomposition of $\mathcal{N}$, then, since $\mathcal{L}$ is ortho-complemented (cf. Lemma 2.2), $\mathcal{N}$ is finite dimensional and with $\mathcal{H}_+ := \mathcal{L}[\dot{+}]\mathcal{N}_+$ and $\mathcal{H}_- := \mathcal{N}_-$ we obtain the desired fundamental decomposition of $\mathcal{H}$, cf. Lemma 2.2. □

Proposition 3.3. *Let* $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ *be an Almost Pontryagin space with finite rank of non-positivity* (*non-negativity*) *and let* $\widetilde{\mathcal{O}}$ *be another Hilbert space topology on* $\mathcal{H}$*, such that* $[\cdot\,,\cdot]$ *is* $\widetilde{\mathcal{O}}$*-continuous. Then also* $(\mathcal{H}, \widetilde{\mathcal{O}}, [\cdot\,,\cdot])$ *is an Almost Pontryagin space with the same rank of non-positivity* (*non-negativity*).

Proof. By Lemma 3.2 we retrieve a decomposition $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-[\dot{+}]\mathcal{H}^\circ$ of $\mathcal{H}$ in which $(\mathcal{H}_+, [\cdot\,,\cdot])$ is a Hilbert space and finite-codimensional in $\mathcal{H}$ and $\mathcal{H}_-$ is uniformly negative. Let $\langle\cdot\,,\cdot\rangle$ be a Hilbert space inner product (on $\mathcal{H}$) inducing $\widetilde{\mathcal{O}}$ and set $\widehat{\mathcal{H}} := \mathcal{H}_+[\dot{+}]\mathcal{H}^\circ$. This linear manifold is $\widetilde{\mathcal{O}}$-closed since $\widehat{\mathcal{H}} = \mathcal{H}_-^{[\perp]}$ and $[\cdot\,,\cdot]$ is $\widetilde{\mathcal{O}}$-continuous. We now define the linear manifold

$$\mathcal{L} := (\mathcal{H}^\circ)^{\langle\perp\rangle} \cap \widehat{\mathcal{H}}.$$

Here, "$\langle\perp\rangle$" denotes the orthogonal complement (in $\mathcal{H}$) with respect to the scalar product $\langle\cdot\,,\cdot\rangle$. The linear manifold $\mathcal{L}$ is $\widetilde{\mathcal{O}}$-closed and finite-codimensional in $\mathcal{H}$. Furthermore, as $\widehat{\mathcal{H}} = \mathcal{L} \dot{+} \mathcal{H}^\circ$, $\mathcal{L}$ is maximal positive (in $\widehat{\mathcal{H}}$). Thus, by Lemma 2.1 $(\mathcal{L}, [\cdot\,,\cdot])$ is a Hilbert space. This implies that $\mathcal{L}$ is $\widetilde{\mathcal{O}}$- uniformly positive. □

If the Hilbert space inner product $(\cdot\,,\cdot)$ on $\mathcal{H}$ induces the topology $\mathcal{O}$, the bounded and selfadjoint operator G which is uniquely defined by

$$[x, y] = (Gx, y) \quad \text{for all} \quad x, y \in \mathcal{H}$$

is called the *Gram operator of* $[\cdot\,,\cdot]$ *with respect to* $(\cdot\,,\cdot)$. If $\mathcal{L} \subset \mathcal{H}$ is a linear manifold, then evidently

$$\mathcal{L}^{[\perp]} = (G\mathcal{L})^\perp = G^{-1}\left(\mathcal{L}^\perp\right) := \{x \in \mathcal{H} : Gx \in \mathcal{L}^\perp\}, \tag{3.3}$$

where "$\perp$" denotes the orthogonal complement in $\mathcal{H}$ with respect to the Hilbert space inner product $(\cdot\,,\cdot)$.

The following two results were shown in [KWW] for Almost Pontryagin spaces with finite rank of non-positivity. Similar statements hold for Almost Pontryagin spaces with finite rank of non-negativity.

Proposition 3.4. *Let* $(\mathcal{H}, [\cdot\,,\cdot])$ *be an inner product space which admits a decomposition*

$$\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-[\dot{+}]\mathcal{H}^\circ, \tag{3.4}$$

where $(\mathcal{H}_+, [\cdot\,,\cdot])$ *is a Hilbert space,* $\mathcal{H}_-$ *is negative and* $\mathcal{H}_-$ *as well as* $\mathcal{H}^\circ$ *are finite dimensional. If* P_+*,* P_- *and* P_0 *denote the projections onto* $\mathcal{H}_+$*,* $\mathcal{H}_-$ *and* $\mathcal{H}^\circ$*, respectively, corresponding to the decomposition* (3.4)*, then the inner product* $(\cdot\,,\cdot)$*, defined by*

$$(x, y) := [P_+x, P_+y] - [P_-x, P_-y] + (P_0x, P_0y)_0, \quad x, y \in \mathcal{H},$$

where $(\cdot\,,\cdot)_0$ *is any positive definite inner product on* $\mathcal{H}^\circ$*, is a Hilbert space inner product on* $\mathcal{H}$*. If* $\mathcal{O}$ *denotes the Hilbert space topology induced by* $(\cdot\,,\cdot)$*, then*

$(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ is an Almost Pontryagin space, and the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$ is given by $G = P_+ - P_-$.

For every Hilbert space topology $\widetilde{\mathcal{O}}$ on $\mathcal{H}$ such that $(\mathcal{H}, \widetilde{\mathcal{O}}, [\cdot\,,\cdot])$ is an Almost Pontryagin space and $\mathcal{H}_+$ in (3.4) is $\widetilde{\mathcal{O}}$-closed, we have $\widetilde{\mathcal{O}} = \mathcal{O}$.

Proposition 3.5. *Let $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ be an Almost Pontryagin space with finite rank of non-positivity, let $(\cdot\,,\cdot)$ be a Hilbert space inner product inducing $\mathcal{O}$ and denote by G the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$. Then there exists a number $\varepsilon > 0$ such that the set $\sigma(G) \cap (-\infty, \varepsilon)$ consists of finitely many eigenvalues of G with finite multiplicities.*

Conversely, if $(\mathcal{H}, (\cdot\,,\cdot))$ is a Hilbert space with topology $\mathcal{O}$ and G is a bounded selfadjoint operator in $\mathcal{H}$ with the spectral properties described above, then $(\mathcal{H}, \mathcal{O}, (G\cdot,\cdot))$ is an Almost Pontryagin space with finite rank of non-positivity.

For the rest of this section, let $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ be an Almost Pontryagin space with finite rank of non-positivity $\kappa = \kappa_-(\mathcal{H}) + \kappa_0(\mathcal{H})$. For Almost Pontryagin spaces with finite rank of non-negativity similar statements hold. Moreover, we fix a Hilbert space inner product $(\cdot\,,\cdot)$ on $\mathcal{H}$ which defines the topology $\mathcal{O}$ and the norm $\|\cdot\|$.

Remark 3.6. In [KWW] it was shown that if $\dim \mathcal{H} = \infty$ and $\kappa_0(\mathcal{H}) > 0$, there exists another Hilbert space topology $\widetilde{\mathcal{O}} \neq \mathcal{O}$ on $\mathcal{H}$ such that $(\mathcal{H}, \widetilde{\mathcal{O}}, [\cdot\,,\cdot])$ is an Almost Pontryagin space with finite rank of non-positivity κ. That is the reason why we have to fix the topology $\mathcal{O}$ in Definition 3.1.

The topology $\widetilde{\mathcal{O}}$ can be constructed as follows: Choose a fundamental decomposition (3.1) as in Lemma 3.2 and set $\widehat{\mathcal{H}} := \mathcal{H}_+ \dotplus \mathcal{H}^\circ$. Choose a $\mathcal{O}$-*non*-closed linear manifold $\mathcal{H}'_+ \subset \widehat{\mathcal{H}}$ such that $\widehat{\mathcal{H}} = \mathcal{H}'_+ \dotplus \mathcal{H}^\circ$ (to this end let $x_0 \in \mathcal{H}^\circ$, $\|x_0\| = 1$, set $\mathcal{L} := \big(\{x_0\}^\perp \cap \mathcal{H}^\circ\big)^\perp \cap \widehat{\mathcal{H}}$ and choose a non-continuous linear functional ϕ on $\mathcal{L}$ with $\phi(x_0) \neq 1$. Then $\mathcal{H}'_+ := \ker(\phi - (\cdot, x_0))$ is a linear manifold as desired). As a consequence of Lemma 2.1, $(\mathcal{H}'_+, [\cdot\,,\cdot])$ is a Hilbert space. By applying Proposition 3.4 to the decomposition $\mathcal{H} = \mathcal{H}'_+ \dotplus \mathcal{H}_- \dotplus \mathcal{H}^\circ$ we obtain a Hilbert space topology $\widetilde{\mathcal{O}}$ on $\mathcal{H}$ which does not coincide with $\mathcal{O}$ since $\mathcal{H}'_+$ is closed in $\widetilde{\mathcal{O}}$ but not closed in $\mathcal{O}$.

However, if $\kappa_0(\mathcal{H}) = 0$ it is well known that the topology $\mathcal{O}$ is the unique Banach space topology on $\mathcal{H}$, in which the inner product $[\cdot\,,\cdot]$ is continuous (cf. [L2, Proposition 1.2]).

Proposition 3.7. *If $\mathcal{L} \subset \mathcal{H}$ is a linear manifold, then we have*

$$\mathcal{L}^{[\perp][\perp]} = \overline{\mathcal{L}} + \mathcal{H}^\circ. \tag{3.5}$$

Proof. Since $\overline{\mathcal{L}}^{[\perp]} = \mathcal{L}^{[\perp]}$ we may assume that $\mathcal{L}$ is a subspace. Let $(\cdot\,,\cdot)$ be a Hilbert space inner product inducing $\mathcal{O}$ and let G denote the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot,\cdot)$. By Proposition 3.5 G has a closed range and an (at most) finite-dimensional kernel. Hence, $G\mathcal{L}$ is closed (see [Ka, Lemma IV.5.29]), and by (3.3) we have

$$\mathcal{L}^{[\perp][\perp]} = \left((G\mathcal{L})^{\perp}\right)^{[\perp]} = G^{-1}\left((G\mathcal{L})^{\perp}\right)^{\perp} = G^{-1}\left(G\mathcal{L}\right) = \mathcal{L} + \ker(G) = \mathcal{L} + \mathcal{H}^{\circ},$$

and the proposition is proved. □

Lemma 3.8. *For a linear manifold $\mathcal{L} \subset \mathcal{H}$ the following statements hold:*

(i) *If $\mathcal{L}$ is closed and positive, then $\mathcal{L}$ is uniformly positive;*

(ii) *If $\mathcal{L}$ is non-positive, then $\dim \mathcal{L} \leq \kappa$;*

(iii) *If $\mathcal{L}$ is negative, then $\dim \mathcal{L} \leq \kappa_-(\mathcal{H})$;*

(iv) *If $\mathcal{L}$ is maximal negative, then $\dim \mathcal{L} = \kappa_-(\mathcal{H})$.*

Proof. Fix a fundamental decomposition (3.1) for $\mathcal{H}$ and denote by P_+, P_- and P_0 the fundamental projections onto $\mathcal{H}_+$, $\mathcal{H}_-$ and $\mathcal{H}^{\circ}$, respectively. Let $\|\cdot\|$ be a norm on $\mathcal{H}$ which induces $\mathcal{O}$. In order to prove (i), we assume the contrary. Then there exists a sequence $(x_n) \subset \mathcal{L}$ with $\|x_n\| = 1$ and $\lim_{n\to\infty}[x_n, x_n] = 0$ which converges weakly to some $x_0 \in \mathcal{L}$. Since

$$|[x_0, x_0]| \leq |[x_0 - x_n, x_0]| + [x_n, x_n]^{1/2}[x_0, x_0]^{1/2}$$

$x_0 = 0$ follows. We set $u_n := P_+x_n$ and $v_n := (I - P_+)x_n$. Since $\mathcal{H}_- \dotplus \mathcal{H}^{\circ}$ is finite dimensional, $v_n \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} \|u_n\| = 1$. But $\lim_{n\to\infty}[u_n, u_n] = \lim_{n\to\infty}[x_n, x_n] = 0$ which implies $u_n \to 0$ as $n \to \infty$. A contradiction.

Let us now consider a non-positive subspace $\mathcal{L}$ with $(\kappa+1)$ linearly independent vectors $e_1, \ldots, e_{\kappa+1} \in \mathcal{L}$. For $j = 1, \ldots, \kappa+1$ set $x_j^{\pm} := P_{\pm}e_j$ and $x_j^{\circ} := P_0e_j$. Then there exist $\lambda_1, \ldots, \lambda_{\kappa+1} \in \mathbb{C}$ with at least one λ_i different from zero such that $\sum_{j=1}^{\kappa+1} \lambda_j(x_j^- + x_j^{\circ}) = 0$. But then $\sum_{j=1}^{\kappa+1} \lambda_j e_j \in \mathcal{H}_+$ and $\lambda_1 = \cdots = \lambda_{\kappa+1} = 0$ follows which is a contradiction.

If $\mathcal{L}$ is negative, then from (ii) we conclude that $\mathcal{L}$ is finite dimensional and thus uniformly negative and therefore ortho-complemented (cf. Lemma 2.2). Set $\mathcal{N} := \mathcal{L}^{[\perp]}$ and let $\mathcal{N} = \mathcal{N}_+[\dotplus]\mathcal{N}_-[\dotplus]\mathcal{N}^{\circ}$ be a fundamental decomposition of $\mathcal{N}$. Then we see that $\dim \mathcal{L} \leq \dim \mathcal{L} + \dim \mathcal{N}_- = \kappa_-(\mathcal{H})$. If $\mathcal{L}$ is even maximal negative, then assertion (iv) follows from $\mathcal{N}_- = \{0\}$. □

For a set $\Lambda \subset \mathcal{H}$ we denote by $\mathcal{O}_{\Lambda}$ the relative topology on Λ which is induced by $\mathcal{O}$. The following result is an immediate consequence of Lemma 3.8 (see also [KWW, Proposition 3.1]).

Corollary 3.9. *If $\mathcal{L} \subset \mathcal{H}$ is a subspace, then $(\mathcal{L}, \mathcal{O}_{\mathcal{L}}, [\cdot\,,\cdot])$ is an Almost Pontryagin space with finite rank of non-positivity $\kappa' \leq \kappa$.*

Recall, that if $(\Pi, [\cdot\,,\cdot])$ is a Pontryagin space and $\mathcal{L} \subset \Pi$ is a subspace with a fundamental decomposition $\mathcal{L} = \mathcal{L}_+[\dotplus]\mathcal{L}_-[\dotplus]\mathcal{L}^{\circ}$, we have

$$\Pi = \mathcal{L}_+[\dotplus]\mathcal{L}_-[\dotplus](\mathcal{L}^{\circ} \dotplus \mathcal{P})[\dotplus]\mathcal{M},$$

where $\mathcal{L}^{\circ} \dotplus \mathcal{P}$ is non-degenerate (cf. [B, Theorem IX.2.5]). The following lemma can be seen as a generalization of this fact.

Lemma 3.10. *Let $\mathcal{D} \subset \mathcal{H}$ be a dense linear manifold in $\mathcal{H}$. Furthermore, let $\mathcal{L}$ be a subspace, $\mathcal{L} \subset \mathcal{D}$. If $\mathcal{L} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-[\dot{+}]\mathcal{L}^\circ$ is a fundamental decomposition for $\mathcal{L}$, then there exist subspaces $\mathcal{L}_{00}, \mathcal{L}_{01}, \mathcal{P} \subset \mathcal{D}$ and $\mathcal{M} \subset \mathcal{H}$ such that $\mathcal{H}$ admits a decomposition*

$$\mathcal{H} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-[\dot{+}]\mathcal{L}_{00}[\dot{+}](\mathcal{L}_{01} \dotplus \mathcal{P})[\dot{+}]\mathcal{M} \tag{3.6}$$

with the following properties:

(i) $\mathcal{L}_{00} = \mathcal{L}^\circ \cap \mathcal{H}^\circ$ *and* $\mathcal{L}^\circ = \mathcal{L}_{00}[\dot{+}]\mathcal{L}_{01}$*;*

(ii) $\mathcal{P}$ *is neutral and* $\mathcal{G} := \mathcal{L}_{01} \dotplus \mathcal{P}$ *is non-degenerate;*

(iii) $\mathcal{P} \cap \mathcal{L}_{01} = \mathcal{P} \cap \mathcal{L}_{01}^{[\perp]} = \mathcal{P}^{[\perp]} \cap \mathcal{L}_{01} = \{0\}$*;*

(iv) $\kappa_+(\mathcal{G}) = \kappa_-(\mathcal{G}) = \dim \mathcal{P} = \dim \mathcal{L}_{01} < \infty$*;*

(v) $\mathcal{L}^{[\perp]} = \mathcal{L}^\circ[\dot{+}]\mathcal{M}$*.*

Proof. By Lemma 2.2 the subspace $\mathcal{L}_+[\dot{+}]\mathcal{L}_-$ is ortho-complemented. With $\mathcal{K} := (\mathcal{L}_+[\dot{+}]\mathcal{L}_-)^{[\perp]}$ we have $\mathcal{H} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-[\dot{+}]\mathcal{K}$, and $\mathcal{D} \cap \mathcal{K}$ is dense in $\mathcal{K}$. Let now $\mathcal{K}_1$ and $\mathcal{L}_{01}$ be subspaces such that $\mathcal{K} = \mathcal{L}^\circ \dotplus \mathcal{K}_1$ and $\mathcal{L}^\circ = \mathcal{L}_{00} \dotplus \mathcal{L}_{01}$, where $\mathcal{L}_{00} := \mathcal{L}^\circ \cap \mathcal{H}^\circ$. Then we have $\mathcal{K} = \mathcal{L}_{00} \dotplus \mathcal{L}_{01} \dotplus \mathcal{K}_1$. Since $\dim \mathcal{L}_{01} < \infty$, the space $\mathcal{K}_2 := \mathcal{L}_{01} \dotplus \mathcal{K}_1$ is a subspace, and $\mathcal{D}_2 := \mathcal{D} \cap \mathcal{K}_2$ is dense in $\mathcal{K}_2$. We observe that by construction we have

$$\mathcal{H} = \mathcal{L}_+[\dot{+}]\mathcal{L}_-[\dot{+}]\mathcal{L}_{00}[\dot{+}]\mathcal{K}_2, \tag{3.7}$$

which in particular implies $\mathcal{K}_2^\circ \subset \mathcal{H}^\circ$, and from $\mathcal{L}_{01} = \mathcal{L}_{01} \cap \mathcal{K}_2$ we conclude

$$\mathcal{L}_{01} \cap \mathcal{D}_2^{[\perp]} = \mathcal{L}_{01} \cap \mathcal{K}_2^{[\perp]} = \mathcal{L}_{01} \cap \mathcal{K}_2^\circ \subset \mathcal{L}_{01} \cap \mathcal{H}^\circ = \{0\}.$$

According to [B, Lemma I.10.4] there exist a basis $\{e_1, \ldots, e_n\}$ of $\mathcal{L}_{01}$ and vectors $g_1, \ldots, g_n$ in $\mathcal{D}_2$ such that

$$[e_i, g_j] = \delta_{ij} \text{ holds for all } i, j = 1, \ldots, n.$$

It is easy to see that the subspace

$$\mathcal{G} := \operatorname{span}\{e_1, \ldots, e_n, g_1, \ldots, g_n\}$$

is non-degenerate. With a fundamental symmetry J in $\mathcal{G}$ (see e.g. [L2]) define the neutral subspace $\mathcal{P} := J\mathcal{L}_{01}$. Finally, with $\mathcal{M} := \mathcal{G}^{[\perp]} \cap \mathcal{K}_2$, we have $\mathcal{K}_2 = (\mathcal{L}_{01} \dotplus \mathcal{P})[\dot{+}]\mathcal{M}$, and with (3.7) the decomposition (3.6) follows. Now, it is not difficult to see that the statements (i)-(v) hold. □

The following propositions are consequences of Lemma 3.10.

Proposition 3.11. *A non-positive subspace $\mathcal{L}$ is maximal non-positive if and only if it has dimension κ.*

Proof. If $\dim \mathcal{L} = \kappa$, the statement follows from Lemma 3.8. Let $\mathcal{L}$ be maximal non-positive. We use the same notations as in Lemma 3.10. Since $\mathcal{H} = \mathcal{L}_-[\dot{+}]\mathcal{L}_{00}[\dot{+}](\mathcal{L}_{01} \dotplus \mathcal{P})[\dot{+}]\mathcal{M}$, the subspace $\mathcal{M}$ must be positive and thus $\kappa = \dim \mathcal{L}_- + \dim \mathcal{L}_{00} + \kappa_-(\mathcal{G}) = \dim \mathcal{L}_- + \dim \mathcal{L}_{00} + \dim \mathcal{L}_{01} = \dim \mathcal{L}$. □

In a Pontryagin space with finite rank of negativity any dense linear manifold contains a maximal negative subspace (cf. [B, Theorem IX.1.4]). This also holds in an Almost Pontryagin space with finite rank of non-positivity. Indeed, if $\mathcal{H} = \Pi[\dot{+}]\mathcal{H}^\circ$ with a Pontryagin space Π and $\mathcal{D}$ is dense in $\mathcal{H}$, then $\mathcal{D}\cap\Pi$ is dense in Π. This implies the assertion. The analogue statement for maximal non-positive subspaces in Almost Pontryagin spaces is not true, in general.

Proposition 3.12. *Let $\mathcal{D} \subset \mathcal{H}$ be a dense linear manifold in $\mathcal{H}$. Then for every non-positive subspace $\mathcal{L} \subset \mathcal{D}$ we have*

$$\dim \mathcal{L} \leq \kappa_-(\mathcal{H}) + \dim \mathcal{D}^\circ \leq \kappa,$$

and there exists a non-positive subspace $\mathcal{L}' \subset \mathcal{D}$ with $\dim \mathcal{L}' = \kappa_-(\mathcal{H}) + \dim \mathcal{D}^\circ$. In particular, $\mathcal{D}$ contains a maximal non-positive subspace if and only if $\mathcal{H}^\circ \subset \mathcal{D}$.

Proof. Let $\mathcal{L} \subset \mathcal{D}$ be a non-positive subspace. If $\mathcal{L} = \mathcal{L}_-[\dot{+}]\mathcal{L}^\circ$ is a fundamental decomposition of $\mathcal{L}$, we obtain a decomposition

$$\mathcal{H} = \mathcal{L}_-[\dot{+}]\mathcal{L}_{00}[\dot{+}](\mathcal{L}_{01} \dot{+} \mathcal{P})[\dot{+}]\mathcal{M}$$

with the properties stated in Lemma 3.10. Then we have

(i) $\kappa_-(\mathcal{H}) = \dim \mathcal{L}_- + \dim \mathcal{L}_{01} + \kappa_-(\mathcal{M})$ and
(ii) $\mathcal{D} = \mathcal{L}_-[\dot{+}]\mathcal{L}_{00}[\dot{+}](\mathcal{L}_{01} \dot{+} \mathcal{P})[\dot{+}](\mathcal{D}\cap\mathcal{M})$.

From (ii) it easily follows that $\mathcal{D}^\circ = \mathcal{L}_{00} \dot{+} (\mathcal{D}\cap\mathcal{M}^\circ)$. This and (i) imply

$$\begin{aligned}\dim \mathcal{L} &= \dim \mathcal{L}_- + \dim \mathcal{L}_{00} + \dim \mathcal{L}_{01}\\ &= \kappa_-(\mathcal{H}) - \kappa_-(\mathcal{M}) + \dim \mathcal{L}_{00}\\ &\leq \kappa_-(\mathcal{H}) + \dim \mathcal{L}_{00}\\ &\leq \kappa_-(\mathcal{H}) + \dim \mathcal{D}^\circ.\end{aligned}$$

By $\mathcal{D}^\circ \subset \mathcal{H}^\circ$ this number does not exceed κ. In order to show the existence of a subspace $\mathcal{L}' \subset \mathcal{D}$ as above, choose a fundamental decomposition $\mathcal{H} = \Pi[\dot{+}]\mathcal{H}^\circ$ of $\mathcal{H}$ with a Pontryagin space $(\Pi, [\cdot\,,\cdot])$ with finite rank of negativity $\kappa_-(\mathcal{H})$ (see Lemma 3.2). The linear manifold $\mathcal{D}' := \mathcal{D}\cap\Pi$ is dense in Π. Thus, there exists a negative subspace $\mathcal{L}_- \subset \mathcal{D}'$ with $\dim \mathcal{L}_- = \kappa_-(\mathcal{H})$ and the subspace $\mathcal{L}' := \mathcal{L}_-[\dot{+}]\mathcal{D}^\circ$ is non-positive and has dimension $\kappa_-(\mathcal{H}) + \dim \mathcal{D}^\circ$. Hence, $\mathcal{D}$ contains a maximal non-positive (and thus κ-dimensional) subspace if and only if $\dim \mathcal{D}^\circ = \dim \mathcal{H}^\circ$, which, as $\mathcal{D}^\circ \subset \mathcal{H}^\circ$, is equivalent to $\mathcal{D}^\circ = \mathcal{H}^\circ$. But this is again equivalent to $\mathcal{H}^\circ \subset \mathcal{D}$. □

In a Pontryagin space a subspace $\mathcal{L}$ is ortho-complemented if and only if it is non-degenerate. In an Almost Pontryagin space we have the following

Proposition 3.13. *A subspace $\mathcal{L} \subset \mathcal{H}$ is ortho-complemented in $\mathcal{H}$ if and only if $\mathcal{L}^\circ \subset \mathcal{H}^\circ$.*

Proof. By applying Lemma 3.10 to $\mathcal{L}$ and using the same notations we obtain $\mathcal{L} + \mathcal{L}^{[\perp]} = \mathcal{L} \dotplus \mathcal{M}$. Thus, $\mathcal{L}$ is ortho-complemented if and only if $\mathcal{P} = \{0\}$. But this is equivalent to $\mathcal{L}^\circ \subset \mathcal{H}^\circ$. □

4. G-Symmetric and G-Selfadjoint Operators in Almost Pontryagin Spaces

In this section let $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ be an Almost Pontryagin space with finite rank of non-positivity $\kappa = \kappa_-(\mathcal{H}) + \kappa_0(\mathcal{H})$ and let $(\cdot\,,\cdot)$ be a Hilbert space inner product inducing $\mathcal{O}$. By G we denote the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$. The results of this section also hold for Almost Pontryagin spaces with finite rank of non-negativity.

We recall some notions related to linear relations in $\mathcal{H}$. A *linear relation in $\mathcal{H}$* is a linear manifold $S \subset \mathcal{H} \times \mathcal{H}$. For the basic properties of linear relations we refer to [C, DS, H]. We only mention that for linear relations $S, T \subset \mathcal{H} \times \mathcal{H}$ one defines

$$\operatorname{dom} S := \left\{ x : \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}, \qquad \textit{the domain of } S,$$

$$\operatorname{ran} S := \left\{ y : \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}, \qquad \textit{the range of } S,$$

$$\operatorname{mul} S := \left\{ y : \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}, \qquad \textit{the multivalued part of } S,$$

$$S^{-1} := \left\{ \begin{pmatrix} y \\ x \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}, \qquad \textit{the inverse of } S,$$

$$S + T := \left\{ \begin{pmatrix} x \\ y+z \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in S,\ \begin{pmatrix} x \\ z \end{pmatrix} \in T \right\}, \quad \textit{the sum of } S \textit{ and } T$$

and the *product of S and T*,

$$ST := \left\{ \begin{pmatrix} x \\ z \end{pmatrix} : \text{there exists } y \in \mathcal{H} \text{ with } \begin{pmatrix} y \\ z \end{pmatrix} \in S, \begin{pmatrix} x \\ y \end{pmatrix} \in T \right\}.$$

The elements in a linear relation S will usually be written as column vectors $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, where $x \in \operatorname{dom} S$ and $y \in \operatorname{ran} S$. Linear operators are identified as linear relations via their graphs.

Further on, we define

$$S^+ := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H} \times \mathcal{H} : [y, u] = [x, v] \ \text{ for all } \begin{pmatrix} x \\ y \end{pmatrix} \in S \right\}$$

and call this linear relation the $[\cdot,\cdot]$-*adjoint of* S. If S is (the graph of) an operator, we have

$$S^+ := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H} \times \mathcal{H} : [Sx, u] = [x, v] \ \text{ for all } x \in \operatorname{dom} S \right\}.$$

Lemma 4.1. *If $S \subset \mathcal{H} \times \mathcal{H}$ is a linear relation, then S^+ is a closed linear relation with $\mathcal{H}^\circ \times \mathcal{H}^\circ \subset S^+$ and*

$$S^{++} = \overline{S} + (\mathcal{H}^\circ \times \mathcal{H}^\circ). \tag{4.1}$$

If $\operatorname{dom} S$ is dense in $\mathcal{H}$ we have

$$\operatorname{mul} S^+ = \mathcal{H}^\circ.$$

The relation S^+ is a linear operator if and only if S is densely defined and $\mathcal{H}^\circ = \{0\}$.

Proof. It follows from the definition of S^+ that S^+ is closed and $\mathcal{H}^\circ \times \mathcal{H}^\circ \subset S^+$. As

$$(\operatorname{dom} S)^{[\perp]} = \operatorname{mul} S^+,$$

it remains to show (4.1). By $\mathcal{O} \times \mathcal{O}$ we denote the product topology on $\mathcal{H} \times \mathcal{H}$. We introduce an inner product $[\cdot,\cdot]$ on $\mathcal{H} \times \mathcal{H}$,

$$\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right] := [x, u] + [y, v], \quad x, y, u, v \in \mathcal{H}. \tag{4.2}$$

By [KWW, Proposition 3.1], $(\mathcal{H} \times \mathcal{H}, \mathcal{O} \times \mathcal{O}, [\cdot,\cdot])$ is an Almost Pontryagin space with finite rank of non-positivity and isotropic part $\mathcal{H}^\circ \times \mathcal{H}^\circ$. It is easily seen that $(S^{-1})^{[\perp]} = (S^{[\perp]})^{-1}$, where the orthogonal companion is now with respect to the inner product defined in (4.2). We have

$$S^+ = \left((-S)^{[\perp]} \right)^{-1}.$$

Applying this twice we obtain

$$S^{++} = \left((-S^+)^{[\perp]} \right)^{-1} = \left(((S^{[\perp]})^{-1})^{[\perp]} \right)^{-1} = S^{[\perp][\perp]} = \overline{S} + (\mathcal{H}^\circ \times \mathcal{H}^\circ)$$

by Proposition 3.7. □

We now introduce the notion of G-symmetric operators where the symmetry is understood with respect to the inner product $[\cdot,\cdot]$.

Definition 4.2. A densely defined linear operator A in $\mathcal{H}$ is called *G-symmetric* if the operator GA is symmetric in the Hilbert space $(\mathcal{H}, (\cdot,\cdot))$.

Evidently, for a densely defined linear operator A in $\mathcal{H}$ the following statements are equivalent.

(i) A is G-symmetric;
(ii) $A \subset A^+$;
(iii) $[Ax, y] = [x, Ay]$ for all $x, y \in \operatorname{dom} A$.

We say that a closed and densely defined operator T in $\mathcal{H}$ is a Φ_+-operator if $\dim\ker(T) < \infty$ and if $\operatorname{ran}(T)$ is closed.

Lemma 4.3. *Let A be a closed and densely defined, G-symmetric operator in $\mathcal{H}$. Then for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ the operator $A-\lambda$ is a Φ_+-operator. Moreover,* $\ker(A-\lambda) \neq \{0\}$ *only holds for at most finitely many $\lambda \in \mathbb{C}\setminus\mathbb{R}$.*

Proof. Let $\mathcal{H} = \mathcal{H}_+[\dot{+}]\mathcal{H}_-[\dot{+}]\mathcal{H}^\circ$ be a fundamental decomposition of $\mathcal{H}$. Then, since $\mathcal{H}_+$ is uniformly positive, there exists a $\delta > 0$ such that $[x,x] \geq \delta\|x\|^2$ for $x \in \mathcal{H}_+$. Thus, for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$ and all $x \in \mathcal{H}_+ \cap \operatorname{dom} A$ we have

$$\begin{aligned}\delta|\mathrm{Im}\lambda|\|x\|^2 &\leq |\mathrm{Im}\lambda|[x,x] = |\mathrm{Im}[\lambda x, x]| = |\mathrm{Im}[(A-\lambda)x, x]|\\ &\leq |[(A-\lambda)x,x]| \leq \|G\|\|(A-\lambda)x\|\|x\|,\end{aligned}$$

which implies

$$\|(A-\lambda)x\| \geq \delta\|G\|^{-1}|\mathrm{Im}\lambda|\|x\|.$$

This shows that $A-\lambda$ is a Φ_+-operator for all $\lambda \in \mathbb{C}\setminus\mathbb{R}$. And as the subspace $\overline{\operatorname{span}}\{\ker(A-\lambda) : \lambda \in \mathbb{C}^+\}$ is neutral, the additional assertion follows from Lemma 3.8(ii). □

In [LMM] an everywhere defined and bounded linear operator A was called G-selfadjoint if $[Ax,y] = [x,Ay]$ holds for all $x,y \in \mathcal{H}$. This is obviously equivalent to the condition $(GA)^* = GA$. We extend this notion to unbounded operators.

Definition 4.4. A linear densely defined operator A in $\mathcal{H}$ is called G-selfadjoint if $GA = (GA)^*$ and if A is closed.

Obviously, if $(\mathcal{H}, [\cdot\,,\cdot])$ is a Pontryagin space (i.e. $\mathcal{H}^\circ = \{0\}$), an operator A in $\mathcal{H}$ is G-selfadjoint if and only if $A = A^+$. Such operators are necessarily closed. However, in an Almost Pontryagin space there are densely defined operators A with $GA = (GA)^*$ which are not closable (see Example 4.9 below).

If $\langle\cdot\,,\cdot\rangle$ is a second Hilbert space inner product inducing $\mathcal{O}$ and $\widetilde{G}$ is the Gram operator of $[\cdot\,,\cdot]$ with respect to $\langle\cdot\,,\cdot\rangle$, it can be easily seen that an operator A is G-symmetric (G-selfadjoint) if and only if it is $\widetilde{G}$-symmetric (resp. $\widetilde{G}$-selfadjoint). That is, Definition 4.4 does not depend on the choice of the particular Hilbert space inner product (resp. on G).

Theorem 4.5. *For a densely defined operator A in $\mathcal{H}$ the following statements are equivalent:*

(i) $GA = (GA)^*$;
(ii) $A \subset A^+$ *and* $\operatorname{dom} A = \operatorname{dom} A^+$;
(iii) $A^+ = A \dot{+} (\{0\} \times \mathcal{H}^\circ)$;
(iv) *with respect to any fundamental decomposition $\mathcal{H} = \mathcal{H}^\circ[\dot{+}]\Pi$ (cf. (3.2)) the operator A admits a matrix representation*

$$A = \begin{pmatrix} A_0 & A_{12} \\ 0 & \widetilde{A} \end{pmatrix}, \quad \operatorname{dom} A = \mathcal{H}^\circ \dot{+} \operatorname{dom}\widetilde{A}, \tag{4.3}$$

where $\widetilde{A}$ is a selfadjoint operator in the Pontryagin space $(\Pi, [\cdot\,,\cdot])$.

If A is additionally assumed to be closed, then (ii)-(iv) *are equivalent to the G-selfadjointness of the operator A. Moreover, in this case,* (i)-(iv) *are equivalent to one of the following statements.*

(v) $\mathcal{H}^\circ \subset \operatorname{dom} A$ *and* $A^+ = A^{++}$*;*

(vi) *The operator A is G-symmetric and the sets $\rho(A) \cap \mathbb{C}^+$ and $\rho(A) \cap \mathbb{C}^-$ are not empty.*

Proof. The implication (i)⇒(ii) is evident. Suppose that (ii) holds. Then, as $A \subset A^+$ and $\{0\} \times \mathcal{H}^\circ \subset A^+$ (cf. Lemma 4.1), we have $A + (\{0\} \times \mathcal{H}^\circ) \subset A^+$. Let now $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in A^+$. Then $u \in \operatorname{dom} A^+ = \operatorname{dom} A$ and

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ Au \end{pmatrix} + \begin{pmatrix} 0 \\ v - Au \end{pmatrix}.$$

Since for all $x \in \operatorname{dom} A$ we have $[x, v - Au] = [x, v] - [Ax, u] = 0$, it follows that $v - Au \in \mathcal{H}^\circ$ and (iii) follows.

Assume now that (iii) holds and let $\mathcal{H} = \mathcal{H}^\circ [\dot{+}] \Pi$ be a fundamental decomposition of $\mathcal{H}$ as in (3.2). By Lemma 4.1 $\mathcal{H}^\circ \subset \operatorname{dom} A^+ = \operatorname{dom} A$, hence

$$\operatorname{dom} A = \mathcal{H}^\circ \dot{+} (\operatorname{dom} A \cap \Pi) \quad \text{and} \quad A\mathcal{H}^\circ \subset \mathcal{H}^\circ.$$

It is now clear that A admits a representation (4.3). By P we denote the projection in $\mathcal{H}$ onto Π with $\ker P = \mathcal{H}^\circ$. Then we have $\widetilde{A} = P(A \upharpoonright \operatorname{dom} \widetilde{A})$, where $\operatorname{dom} \widetilde{A} = \operatorname{dom} A \cap \Pi$. It is easy to see that $\widetilde{A}$ is symmetric in $(\Pi, [\cdot\,,\cdot])$. If $u, z \in \Pi$ such that $[\widetilde{A}x, u] = [x, z]$ for all $x \in \operatorname{dom} \widetilde{A}$, then one easily verifies $[Ax, u] = [x, z]$ for all $x \in \operatorname{dom} A$ which implies $\left(\begin{smallmatrix} u \\ z \end{smallmatrix}\right) \in A^+ = A \dot{+} (\{0\} \times \mathcal{H}^\circ)$ and thus $u \in \operatorname{dom} A \cap \Pi = \operatorname{dom} \widetilde{A}$. This shows that $\widetilde{A}$ is selfadjoint in the Pontryagin space $(\Pi, [\cdot\,,\cdot])$.

Let us show the implication (iv)⇒(i). To this end we set $\Pi := \operatorname{ran} G$. Then $\mathcal{H} = \mathcal{H}^\circ [\oplus] \Pi$ is a fundamental decomposition of $\mathcal{H}$, cf. Proposition 3.5, and A admits a representation (4.3) with respect to this decomposition. With the projection P in $\mathcal{H}$ onto Π with $\ker P = \mathcal{H}^\circ$ and $\widetilde{G} := P(G \upharpoonright \Pi)$ we have

$$G = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{G} \end{pmatrix} \quad \text{and} \quad GA = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{G}\widetilde{A} \end{pmatrix}.$$

Thus, it remains to show that $\widetilde{G}\widetilde{A}$ is selfadjoint in the Hilbert space $(\Pi, (\cdot\,,\cdot))$. But this is an easy consequence of the fact that $\widetilde{A}$ is selfadjoint in the Pontryagin space $(\Pi, [\cdot\,,\cdot])$ and $\widetilde{G}$ is the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$ in Π.

Let now A be a closed operator. Then (4.1) implies

$$A^{++} = A + (\mathcal{H}^\circ \times \mathcal{H}^\circ)$$

and the equivalence of (iii) and (v) follows with Lemma 4.1.

Let A be a G-selfadjoint operator. The matrix representation (4.3) of A implies

$$\sigma(A) = \sigma(A_0) \cup \sigma(\widetilde{A}), \tag{4.4}$$

hence (iv) implies (vi). Now, suppose that both the upper and the lower half plane contain points from $\rho(A)$. Then by Lemma 4.3 $\mathbb{C} \setminus \mathbb{R}$ belongs with at most finitely many exceptional points to $\rho(A)$. Thus, there exist numbers $\lambda_0, \overline{\lambda_0} \in (\mathbb{C}\setminus\mathbb{R})\cap\rho(A)$. Let $\{h_1, \ldots, h_n\}$ be a basis of $\mathcal{H}^\circ$. Then for all $j = 1, \ldots, n$ and all $y \in \mathcal{H}$ we have

$$[(A-\lambda_0)^{-1}h_j, y] = [h_j, (A-\overline{\lambda_0})^{-1}y] = 0.$$

Hence $(A-\lambda_0)^{-1}h_j \in \mathcal{H}^\circ$. But these vectors are linearly independent, and therefore $\mathcal{H}^\circ = \operatorname{span}\{(A-\lambda_0)^{-1}h_j : j = 1, \ldots, n\} \subset \operatorname{dom} A$. If $u \in \operatorname{dom} A^+$, there exists a $v \in \mathcal{H}$ such that $[Ax, u] = [x, v]$ for all $x \in \operatorname{dom} A$. Thus, for all $x \in \operatorname{dom} A$ we have

$$\begin{aligned} &[u - (A-\lambda_0)^{-1}(v - \lambda_0 u), (A-\overline{\lambda_0})x] \\ &= [u, Ax] - [u, \overline{\lambda_0}x] - [v - \lambda_0 u, x] = 0. \end{aligned}$$

But this implies $u - (A-\lambda_0)^{-1}(v-\lambda_0 u) \in \mathcal{H}^\circ$, which yields $u \in \operatorname{dom} A$. Hence

$$\operatorname{dom} A = \operatorname{dom} A^+$$

and (ii) holds. □

Corollary 4.6. *Let A be a densely defined operator in $\mathcal{H}$ satisfying one of the conditions* (i)-(iv) *of Proposition* 4.5. *Then the following statements are equivalent:*

(i) *A is closed;*
(ii) *A is closable;*
(iii) *A_{12} is $\widetilde{A}$-bounded.*

Proof. Let A be closable. Let $(x_n) \subset \operatorname{dom} A$ be a sequence with $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. Then for $u \in \operatorname{dom} A$ we have

$$[Au, x] = \lim_{n\to\infty} [Au, x_n] = \lim_{n\to\infty} [u, Ax_n] = [u, y],$$

which implies $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in A^+$. Therefore, by Theorem 4.5, $x \in \operatorname{dom} A^+ = \operatorname{dom} A$. As A is closable we have $Ax = y$, i.e. A is closed and (i) is equivalent to (ii).

Let λ be a complex number in $\rho(\widetilde{A}) \cap \rho(A_0)$. Then it is easy to see that $A - \lambda$ is bijective with

$$(A-\lambda)^{-1} = \begin{pmatrix} (A_0-\lambda)^{-1} & -(A_0-\lambda)^{-1}A_{12}(\widetilde{A}-\lambda)^{-1} \\ 0 & (\widetilde{A}-\lambda)^{-1} \end{pmatrix}$$

and that A_{12} is $\widetilde{A}$-bounded if and only if $A_{12}(\widetilde{A}-\lambda)^{-1}$ is bounded. If A is closed then $\lambda \in \rho(A)$ which implies that $(A_0-\lambda)^{-1}A_{12}(\widetilde{A}-\lambda)^{-1}$ is bounded. Hence, A_{12} is $\widetilde{A}$-bounded. If $A_{12}(\widetilde{A}-\lambda)^{-1}$ is bounded, then obviously $(A-\lambda)^{-1}$ is bounded and thus $\lambda \in \rho(A)$ which shows that A is closed. □

Note that in general the operator A_{12} in representation (4.3) need not be closed or closable.

Corollary 4.7. *The spectrum of a G-selfadjoint operator A in $\mathcal{H}$ is the union of the spectrum of a selfadjoint operator in a Pontryagin space and a finite set in $\mathbb{C}$. Moreover, there exists a maximal non-positive* (*and hence κ-dimensional*) *subspace $\mathcal{L} \subset \operatorname{dom} A$ which is invariant under A.*

Proof. The first statement of the corollary follows from representation (4.3) and the closedness of A (see (4.4)). By a theorem of Pontryagin there exists a maximal non-positive (in Π) subspace $\widetilde{\mathcal{L}} \subset \operatorname{dom} \widetilde{A}$ which is invariant under $\widetilde{A}$ and it is easily seen that the subspace $\mathcal{L} := \widetilde{\mathcal{L}} \dotplus \mathcal{H}^\circ \subset \operatorname{dom} A$ is maximal non-positive (in $\mathcal{H}$) and invariant under A. □

Remark 4.8. Let A be a densely defined operator in $\mathcal{H}$ with $GA = (GA)^*$ and let $\Pi, \Pi' \subset \mathcal{H}$ be two subspaces such that $\mathcal{H} = \mathcal{H}^\circ[\dotplus]\Pi = \mathcal{H}^\circ[\dotplus]\Pi'$. Then, by Theorem 4.5, A has the representations

$$A = \begin{pmatrix} A_0 & A_{12} \\ 0 & \widetilde{A} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_0 & A'_{12} \\ 0 & \widetilde{A}' \end{pmatrix}$$

with respect to the particular decompositions. Let P and P' be the projections in $\mathcal{H}$ with $\operatorname{ran} P = \Pi$, $\operatorname{ran} P' = \Pi'$ and $\ker P = \ker P' = \mathcal{H}^\circ$ and set $U := P' \upharpoonright \Pi$. Then U is a bounded operator. It is easily seen that U, $U : \Pi \to \Pi'$, is bijective and $U^{-1} = U^+ = P \upharpoonright \Pi'$. We have

$$\widetilde{A}' = U\widetilde{A}U^+ \quad \text{and} \quad A'_{12} = (AP' - P'A)U^+.$$

Example 4.9. In contrast to the case that $(\mathcal{H}, [\cdot\,,\cdot])$ is a Pontryagin space, a densely defined operator A in an Almost Pontryagin space with $GA = (GA)^*$ may not even be closable. As an example, let $(\Pi, [\cdot\,,\cdot]_\Pi)$ be a Pontryagin space with fundamental symmetry J, scalar product $(\cdot\,,\cdot)_\Pi = [J\cdot,\cdot]_\Pi$ and associated norm $\|\cdot\|_\Pi$. Suppose that $\widetilde{A}$ is a bounded selfadjoint operator in $(\Pi, [\cdot\,,\cdot]_\Pi)$ and set $\mathcal{H} := \mathbb{C} \oplus \Pi$. Furthermore, define the inner products $[\cdot\,,\cdot]$ and $(\cdot\,,\cdot)$ on $\mathcal{H}$ by

$$\left[\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} y \\ w \end{pmatrix}\right] := [x,y]_\Pi \quad \text{and} \quad \left(\begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} y \\ w \end{pmatrix}\right) := (x,y)_\Pi + z\overline{w}$$

for $x, y \in \Pi$ and $z, w \in \mathbb{C}$. Then $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ is an Almost Pontryagin space (where $\mathcal{O}$ is the topology induced by $(\cdot\,,\cdot)$), and the operator

$$G := \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$$

is the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$ in $\mathcal{H}$. If we choose a linear functional $\Phi : \Pi \to \mathbb{C}$ which is not bounded, it is easily seen that the operator A, defined by

$$A := \begin{pmatrix} 0 & \Phi \\ 0 & \widetilde{A} \end{pmatrix},$$

satisfies $GA = (GA)^*$.

Now, let $(\widetilde{x}_n) \subset \Pi$ be a sequence with the properties $\lim_{n\to\infty} \|\widetilde{x}_n\|_\Pi = 0$ and $\liminf_{n\to\infty} |\Phi\widetilde{x}_n| > 0$. Without loss of generality we may assume $\Phi\widetilde{x}_n \to z \in \mathbb{C} \setminus \{0\}$ as $n \to \infty$. The sequence $\binom{0}{\widetilde{x}_n}$ then tends to zero in $\mathcal{H} = \Pi \oplus \mathbb{C}$ and $A\binom{0}{\widetilde{x}_n}$ to $\binom{z}{0}$ in $\mathcal{H}$ as $n \to \infty$, which implies that A is not closable.

5. The Spectral Function of a G-Selfadjoint operator

As in the previous section, let $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$ be an Almost Pontryagin space with finite rank of non-positivity. Furthermore, let $(\cdot\,,\cdot)$ be a Hilbert space inner product inducing $\mathcal{O}$ and let G be the Gram operator of $[\cdot\,,\cdot]$ with respect to $(\cdot\,,\cdot)$. The results of this section also hold for Almost Pontryagin spaces with finite rank of non-negativity.

With the help of the spectral function of a selfadjoint operator in a Pontryagin space we will construct the spectral function for a G-selfadjoint operator A in an Almost Pontryagin space. By $L(\mathcal{H})$ we denote the set of all bounded linear operators in $\mathcal{H}$.

Theorem 5.1. *Let A be a G-selfadjoint operator in the Almost Pontryagin space $(\mathcal{H}, \mathcal{O}, [\cdot\,,\cdot])$. Then A possesses a spectral function with finitely many singularities, i.e. there exists a finite set $c(A) \subset \mathbb{R}$ and a mapping $E : \mathcal{R}(A) \to L(\mathcal{H})$ (here, $\mathcal{R}(A)$ is the collection of all bounded Borel-sets $M \subset \mathbb{R}$ with $\partial M \cap c(A) = \varnothing$ and their complements in $\overline{\mathbb{R}}$) with the following properties ($\Delta, \Delta_1, \Delta_2, \dots \in \mathcal{R}(A)$):*

(i) *$E(\Delta)$ is a G-selfadjoint projection,*
(ii) *$E(\varnothing) = 0$,*
(iii) *$E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2)$,*
(iv) *if $\Delta_1, \Delta_2, \dots$ are pairwise disjoint and if $\Delta = \bigcup_{i=1}^{\infty} \Delta_i$ is in $\mathcal{R}(A)$, then*

$$E(\Delta) = \sum_{i=1}^{\infty} E(\Delta_i),$$

where the series converges in the strong operator topology,
(v) *$E(\Delta)$ is in the double commutant of the resolvent of A,*
(vi) *if Δ is bounded then $A \upharpoonright E(\Delta)\mathcal{H}$ is bounded,*
(vii) *$\sigma(A \upharpoonright E(\Delta)\mathcal{H}) \subset \overline{\Delta}$.*
(viii) *If $\Delta \cap c(A) = \varnothing$, then $(E(\Delta)\mathcal{H}, [\cdot\,,\cdot])$ is a Hilbert space.*

Proof. We fix a fundamental decomposition $\mathcal{H} = \mathcal{H}^\circ [\dotplus] \Pi$ of $\mathcal{H}$ with a Pontryagin space Π as in (3.2). By Proposition 3.5 we may assume without loss of generality that

$$\Pi = (\mathcal{H}^\circ)^\perp, \tag{5.1}$$

where the orthogonal complement is understood with respect to the Hilbert space inner product $(\cdot\,,\cdot)$ inducing $\mathcal{O}$. Then with Theorem 4.5 we have a block operator representation

$$A = \begin{pmatrix} A_0 & A_{12} \\ 0 & \widetilde{A} \end{pmatrix}, \quad \operatorname{dom} A = \mathcal{H}^\circ \dotplus (\operatorname{dom} A \cap \Pi)$$

of A where $\widetilde{A}$ is selfadjoint in $(\Pi, [\cdot\,,\cdot])$. For $\lambda \in \rho(A)$ we have

$$(A-\lambda)^{-1} = \begin{pmatrix} (A_0-\lambda)^{-1} & -(A_0-\lambda)^{-1}A_{12}(\widetilde{A}-\lambda)^{-1} \\ 0 & (\widetilde{A}-\lambda)^{-1} \end{pmatrix}.$$

The Riesz-Dunford projection P_0 corresponding to the non-real spectrum of A satisfies for $x = x_1 + x_2$, $y = y_1 + y_2$ with $x_1, y_1 \in \mathcal{H}^\circ$ and $x_2, y_2 \in \Pi$

$$\begin{aligned}[P_0 x, y] &= -\frac{1}{2\pi i}\int_{\mathcal{C}} \left[\begin{pmatrix} (A_0-\lambda)^{-1}x_1 - (A_0-\lambda)^{-1}A_{12}(\widetilde{A}-\lambda)^{-1}x_2 \\ (\widetilde{A}-\lambda)^{-1}x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right] d\lambda \\ &= -\frac{1}{2\pi i}\int_{\mathcal{C}} \left[(\widetilde{A}-\lambda)^{-1}x_2, y_2 \right] d\lambda = \left[\widetilde{E}_0 x_2, y_2 \right],\end{aligned}$$

where $\mathcal{C}$ is a closed, smooth curve enclosing the non-real spectrum of A and $\widetilde{E}_0$ is the Riesz-Dunford projection corresponding to the non-real spectrum of $\widetilde{A}$. Hence, P_0 is a bounded, G-symmetric operator, that is, P_0 is G-selfadjoint in $\mathcal{H}$. The subspace $(I-P_0)\mathcal{H}$ is an Almost Pontryagin space (Corollary 3.9) and the restriction of A to $(I-P_0)\mathcal{H}$ is also G-selfadjoint, see Theorem 4.5 (vi). Therefore we will assume that $\sigma(A)$ is real.

Let $\lambda_1, \dots, \lambda_n$ be the (real) eigenvalues of A_0 ($\lambda_i \neq \lambda_j$ for $i \neq j$), $m := \dim \mathcal{H}^\circ$, and let $\widetilde{E}$ be the spectral function of $\widetilde{A}$ in Π with the set of critical points $c(\widetilde{A})$ (see [L1, L2]). We set

$$c(A) := \{\lambda_1, \dots, \lambda_n\} \cup c(\widetilde{A}).$$

It follows from Remark 4.8 that $c(A)$ does not depend on the chosen decomposition (3.2) of the Almost Pontryagin space $\mathcal{H}$. By $E_0^{(j)}$ denote the Riesz-Dunford projection of A_0 in $\mathcal{H}^\circ$ corresponding to the eigenvalue λ_j ($j = 1, \dots, n$) of A_0. Let $\Delta \in \mathcal{R}(A)$. For $j \in \{1, \dots, n\}$ we define $\widetilde{A}_\Delta := \widetilde{A} \upharpoonright \widetilde{E}(\Delta)\Pi$ and, if $\lambda_j \notin \Delta$,

$$E_{12}^{(j)}(\Delta) := E_0^{(j)} \left(\sum_{k=1}^{m} (A_0-\lambda_j)^{k-1} A_{12} (\widetilde{A}_\Delta - \lambda_j)^{-k} \right) \widetilde{E}(\Delta).$$

Otherwise ($\lambda_j \in \Delta$) we set

$$E_{12}^{(j)}(\Delta) := -E_{12}^{(j)}(\Delta^c), \quad \text{where } \Delta^c := \overline{\mathbb{R}} \setminus \Delta.$$

Note that $E_{12}^{(j)}(\Delta)$ is well defined since $\lambda_j \notin \partial\Delta$ and that it is a bounded operator from Π to $\mathcal{H}^\circ$.

Now we define

$$E_{12}(\Delta) := \sum_{j=1}^{n} E_{12}^{(j)}(\Delta), \quad E_0(\Delta) := \sum_{\lambda_j \in \Delta} E_0^{(j)} \quad \text{and}$$

$$E(\Delta) := \begin{pmatrix} E_0(\Delta) & E_{12}(\Delta) \\ 0 & \widetilde{E}(\Delta) \end{pmatrix}.$$

Obviously, $E(\Delta)$ is a bounded and, by Theorem 4.5, a G-selfadjoint operator in $\mathcal{H}$ for every $\Delta \in \mathcal{R}(A)$.

In the following, we will show (iii), which also implies that $E(\Delta)$ is a projection. To this end it suffices to show that for $\Delta_1, \Delta_2 \in \mathcal{R}(A)$ and $j \in \{1, \ldots, n\}$ we have

$$E_{12}^{(j)}(\Delta_1 \cap \Delta_2) = E_0(\Delta_1)E_{12}^{(j)}(\Delta_2) + E_{12}^{(j)}(\Delta_1)\widetilde{E}(\Delta_2). \tag{5.2}$$

For the sake of simplicity we assume $\lambda_j = 0$, which is no restriction. If $0 \notin \Delta_1$, then

$$\begin{aligned} E_{12}^{(j)}(\Delta_1 \cap \Delta_2) &= \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_1 \cap \Delta_2}^{-k} \widetilde{E}(\Delta_1 \cap \Delta_2) \\ &= \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_1}^{-k} \widetilde{E}(\Delta_1)\widetilde{E}(\Delta_2) \\ &= E_{12}^{(j)}(\Delta_1)\widetilde{E}(\Delta_2). \end{aligned}$$

We have $E_0(\Delta_1)E_0^{(j)} = 0$ and, therefore, $E_0(\Delta_1)E_{12}^{(j)}(\Delta_2) = 0$, hence, (5.2) holds.

In the case $0 \in \Delta_1 \setminus \Delta_2$ we have

$$\begin{aligned} E_{12}^{(j)}(\Delta_1 \cap \Delta_2) &= \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_1 \cap \Delta_2}^{-k} \widetilde{E}(\Delta_1 \cap \Delta_2) \\ &= E_{12}^{(j)}(\Delta_2)\widetilde{E}(\Delta_1) = E_{12}^{(j)}(\Delta_2) - E_{12}^{(j)}(\Delta_2)\widetilde{E}(\Delta_1^c) \\ &= E_{12}^{(j)}(\Delta_2) - \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_2}^{-k} \widetilde{E}(\Delta_2)\widetilde{E}(\Delta_1^c) \\ &= E_{12}^{(j)}(\Delta_2) - \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_1^c}^{-k} \widetilde{E}(\Delta_1^c)\widetilde{E}(\Delta_2) \\ &= E_{12}^{(j)}(\Delta_2) + E_{12}^{(j)}(\Delta_1)\widetilde{E}(\Delta_2) \\ &= E_0(\Delta_1)E_{12}^{(j)}(\Delta_2) + E_{12}^{(j)}(\Delta_1)\widetilde{E}(\Delta_2), \end{aligned}$$

as $0 \in \Delta_1$. Now, let $0 \in \Delta_1 \cap \Delta_2$. First of all we observe that

$$\begin{aligned} \widetilde{E}((\Delta_1 \cap \Delta_2)^c) &= I - \widetilde{E}(\Delta_1 \cap \Delta_2) = \widetilde{E}(\Delta_2) + \widetilde{E}(\Delta_2^c) - \widetilde{E}(\Delta_1)\widetilde{E}(\Delta_2) \\ &= (I - \widetilde{E}(\Delta_1))\widetilde{E}(\Delta_2) + \widetilde{E}(\Delta_2^c) = \widetilde{E}(\Delta_1^c)\widetilde{E}(\Delta_2) + \widetilde{E}(\Delta_2^c). \end{aligned}$$

This implies

$$\begin{aligned} E_{12}^{(j)}(\Delta_1 \cap \Delta_2) &= -\sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_1^c \cup \Delta_2^c}^{-k} \big(\widetilde{E}(\Delta_2^c) + \widetilde{E}(\Delta_1^c)\widetilde{E}(\Delta_2)\big) \\ &= -\sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \big(\widetilde{A}_{\Delta_2^c}^{-k} \widetilde{E}(\Delta_2^c) + \widetilde{A}_{\Delta_1^c}^{-k} \widetilde{E}(\Delta_1^c)\widetilde{E}(\Delta_2)\big) \\ &= E_{12}^{(j)}(\Delta_2) + E_{12}^{(j)}(\Delta_1)\widetilde{E}(\Delta_2) \\ &= E_0(\Delta_1)E_{12}^{(j)}(\Delta_2) + E_{12}^{(j)}(\Delta_1)\widetilde{E}(\Delta_2), \end{aligned}$$

as $0 \in \Delta_1$ and the proof of (5.2) is complete.

Let $\Delta, \Delta_1, \Delta_2, \dots \in \mathcal{R}(A)$ be as in (iv). In order to prove (iv) it suffices to show that $E_{12}^{(j)}(\Delta)x = \sum_{i=1}^{\infty} E_{12}^{(j)}(\Delta_i)x$ holds for all $x \in \Pi$ and all $j \in \{1, \dots, n\}$. If $\lambda_j = 0$ and $0 \notin \Delta$, then, in fact, we have

$$\begin{aligned} \sum_{i=1}^{N} E_{12}^{(j)}(\Delta_i) &= \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \sum_{i=1}^{N} \widetilde{A}_{\Delta_i}^{-k} \widetilde{E}(\Delta_i) \\ &= \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta}^{-k} \sum_{i=1}^{N} \widetilde{E}(\Delta_i). \end{aligned}$$

As $\sum_{i=1}^{N} \widetilde{E}(\Delta_i)$ converges strongly to $\widetilde{E}(\Delta)$, see [L2], the above sum converges strongly to $E_{12}^{(j)}(\Delta)$ as $N \to \infty$.

Let $0 \in \Delta$. Without loss of generality we may assume $0 \in \Delta_1$. Then $0 \notin \widetilde{\Delta} := \bigcup_{i=2}^{\infty} \Delta_i$ and by

$$\widetilde{E}(\Delta_1^c) = \widetilde{E}(\widetilde{\Delta}) + \widetilde{E}(\Delta^c)$$

we see that

$$E_{12}^{(j)}(\Delta_1) + E_{12}^{(j)}(\widetilde{\Delta}) = \sum_{k=1}^{m} E_0^{(j)} A_0^{k-1} A_{12} \widetilde{A}_{\Delta_1^c \cup \widetilde{\Delta}}^{-k} (\widetilde{E}(\widetilde{\Delta}) - \widetilde{E}(\Delta_1^c)) = E_{12}^{(j)}(\Delta)$$

holds. Thus, (iv) follows from the proof of the case $0 \notin \Delta$.

It suffices to show (v) only for bounded and closed intervals $\Delta \in \mathcal{R}(A)$. Let $\Delta = [a, b]$ be such an interval, and let $\eta > 0$ be a fixed number. For $\varepsilon > 0$ let $\mathcal{C}_{\Delta}^{\varepsilon}$ be the positively oriented and piecewise linear, closed curve which parameterizes the boundary of the set

$$\{z \in \mathbb{C} : a - \varepsilon \leq \mathrm{Re} z \leq b + \varepsilon,\ |\mathrm{Im} z| \leq \eta\}.$$

For δ with $0 < \delta < \eta$ we denote by $\mathcal{C}_{\Delta}^{\varepsilon,\delta}$ the curve $\mathcal{C}_{\Delta}^{\varepsilon}$ without the line segment connecting the points $a - \varepsilon + i\delta$ and $a - \varepsilon - i\delta$ and without the line segment connecting the points $b + \varepsilon - i\delta$ and $b + \varepsilon + i\delta$. We will show

$$E(\Delta) = \lim_{\varepsilon \downarrow 0} -\frac{1}{2\pi i} \int_{\mathcal{C}_{\Delta}^{\varepsilon}}' (A - \lambda)^{-1}\, d\lambda := -\frac{1}{2\pi i} \cdot \lim_{\varepsilon \downarrow 0} \lim_{\delta \downarrow 0} \int_{\mathcal{C}_{\Delta}^{\varepsilon,\delta}} (A - \lambda)^{-1}\, d\lambda,$$

where the limits exist in the strong operator topology. It follows from the properties of the spectral functions E_0 and $\widetilde{E}$ that we only have to prove

$$E_{12}^{(j)}(\Delta) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathcal{C}_{\Delta}^{\varepsilon}}' E_0^{(j)} (A_0 - \lambda)^{-1} A_{12} (\widetilde{A} - \lambda)^{-1}\, d\lambda \tag{5.3}$$

for $j = 1, \dots, n$. Let $j \in \{1, \dots, n\}$ be given. Considering the Jordan structure of A_0, we obtain for $\lambda \in \rho(A)$

$$E_0^{(j)} (A_0 - \lambda)^{-1} = -E_0^{(j)} \sum_{k=1}^{m} (\lambda - \lambda_j)^{-k} (A_0 - \lambda_j)^{k-1}.$$

Hence, as A_{12} is $\widetilde{A}$-bounded (see Corollary 4.6), (5.3) reduces to

$$E_{12}^{(j)}(\Delta) = E_0^{(j)} \sum_{k=1}^{m} (A_0 - \lambda_j)^{k-1} A_{12} \left(\lim_{\varepsilon \downarrow 0} \; -\frac{1}{2\pi i} \int_{C_\Delta^\varepsilon}' (\lambda - \lambda_j)^{-k} (\widetilde{A} - \lambda)^{-1} d\lambda \right).$$

But this holds since by [L2, p. 33] we have

$$\lim_{\varepsilon \downarrow 0} -\frac{1}{2\pi i} \int_{\mathcal{C}_\Delta^\varepsilon}' (\lambda - \lambda_j)^{-k} (\widetilde{A} - \lambda)^{-1} \, d\lambda = \begin{cases} (\widetilde{A}_\Delta - \lambda_j)^{-k} \widetilde{E}(\Delta), & \text{if } \lambda_j \notin \Delta \\ -(\widetilde{A}_{\Delta^c} - \lambda_j)^{-k} \widetilde{E}(\Delta^c), & \text{if } \lambda_j \in \Delta. \end{cases}$$

Now, (v) follows immediately. We note that an operator $B \in L(\mathcal{H})$ commutes with the resolvent of A if and only if B commutes with A in the following sense: if $x \in \operatorname{dom} A$ then $Bx \in \operatorname{dom} A$ and $ABx = BAx$. Therefore, (v) implies in particular that all the operators $E(\Delta)$, $\Delta \in \mathcal{R}(A)$, commute with A. This leads to the identity

$$A_0 E_{12}(\Delta)x + A_{12}\widetilde{E}(\Delta)x = E_0(\Delta)A_{12}x + E_{12}(\Delta)\widetilde{A}x, \; x \in \operatorname{dom} A, \tag{5.4}$$

which we will need below. If $\Delta \in \mathcal{R}(A)$ is bounded, then $\widetilde{E}(\Delta)\Pi \subset \operatorname{dom} \widetilde{A} = \operatorname{dom} A_{12}$. Hence $E(\Delta)\mathcal{H} \subset \operatorname{dom} A$ which implies (vi) as A is closed.

We show (vii). Let $\Delta \in \mathcal{R}(A)$ and $\lambda \notin \overline{\Delta}$. We have

$$E(\Delta)\mathcal{H} = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{H} : \; u = E_0(\Delta)u + E_{12}(\Delta)v, \;\; v = \widetilde{E}(\Delta)v \right\}. \tag{5.5}$$

Let $x = \binom{u}{v} \in (E(\Delta)\mathcal{H}) \cap \operatorname{dom} A$ with $(A - \lambda)x = 0$. Then $(\widetilde{A}_\Delta - \lambda)v = 0$ implies $v = 0$. Thus, $u = E_0(\Delta)u$ and $(A_0 - \lambda)u = 0$. Then $\lambda \notin \overline{\Delta}$ implies $u = 0$. Thus, $x = 0$, and $(A \restriction E(\Delta)\mathcal{H}) - \lambda$ is injective. In order to show that this operator is also surjective, choose some $y = \binom{w}{z} \in E(\Delta)\mathcal{H}$ and set $x := \binom{u}{v}$, where

$$\begin{aligned} v &:= (\widetilde{A}_\Delta - \lambda)^{-1} z \;\text{ and} \\ u &:= E_{12}(\Delta)v + \big((A_0 \restriction E_0(\Delta)\mathcal{H}^\circ) - \lambda\big)^{-1} E_0(\Delta)(w - A_{12}v). \end{aligned}$$

Obviously, $x \in \operatorname{dom} A$. The relations (5.2) and $E_{12}(\Delta) = -E_{12}(\Delta^c)$ imply

$$(I - E_0(\Delta))u = E_0(\Delta^c)E_{12}(\Delta)v = E_{12}(\Delta)\widetilde{E}(\Delta)v = E_{12}(\Delta)v.$$

Hence $x \in E(\Delta)\mathcal{H}$. Finally, by (5.4) we have

$$\begin{aligned} (A - \lambda)x &= \begin{pmatrix} (A_0 - \lambda)E_{12}(\Delta)v + E_0(\Delta)(w - A_{12}v) + A_{12}v \\ z \end{pmatrix} \\ &= \begin{pmatrix} E_{12}(\Delta)(\widetilde{A} - \lambda)v - A_{12}\widetilde{E}(\Delta)v + E_0(\Delta)w + A_{12}v \\ z \end{pmatrix} \\ &= \begin{pmatrix} E_{12}(\Delta)z + E_0(\Delta)w \\ z \end{pmatrix} = \begin{pmatrix} w \\ z \end{pmatrix} = y, \end{aligned}$$

which shows (vii).

Let $\Delta \in \mathcal{R}(A)$ with $\Delta \cap c(A) = \varnothing$. In order to show (viii) we prove the existence of $\delta > 0$ with

$$[x,x] \geq \delta \|x\|^2 \quad \text{for all } x \in E(\Delta)\mathcal{H}. \tag{5.6}$$

By [L2], $(\widetilde{E}(\Delta)\Pi, [\cdot\,,\cdot])$ is a Hilbert space. Hence, there exists some $\delta_1 > 0$ such that $[x_1, x_1] \geq \delta_1 \|x_1\|^2$ for all $x_1 \in \widetilde{E}(\Delta)\Pi$. Set $\delta := \delta_1(1 + \|E_{12}(\Delta)\|^2)^{-1}$. By $E_0(\Delta) = 0$ and (5.5) we have for $x = \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in E(\Delta)\mathcal{H}$, $u \in \mathcal{H}^\circ$, $v \in \Pi$:

$$u = E_{12}(\Delta)v \quad \text{and} \quad v = \widetilde{E}(\Delta)v.$$

Therefore,

$$[x,x] = [v,v] \geq \delta_1 \|v\|^2 = \delta(1 + \|E_{12}(\Delta)\|^2)\|v\|^2 \geq \delta\|v\|^2 + \delta\|E_{12}(\Delta)v\|^2 = \delta\|x\|^2,$$

where the last equality follows from (5.1), and thus (5.6) holds. □

6. Application to the Klein-Gordon Equation

The motion of a relativistic spinless particle of mass m and charge e in an electrostatic field with potential q is described by the Klein-Gordon equation

$$\left(\left(\frac{\partial}{\partial t} - ieq\right)^2 - \Delta + m^2\right)\psi = 0, \tag{6.1}$$

where ψ is a function of $t \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$. As in [LNT] we obtain an abstract model of Equation (6.1) if we replace the expression $-\Delta + m^2$ by a strictly positive selfadjoint operator H_0 in a Hilbert space $(\mathcal{H}, (\cdot\,,\cdot))$ such that $(H_0 x, x) \geq m^2\|x\|^2$ for all $x \in \operatorname{dom} H_0$ and if we replace the operator of multiplication by the function eq by a symmetric operator V in $\mathcal{H}$:

$$\left(\left(\frac{\mathrm{d}}{\mathrm{d}t} - iV\right)^2 + H_0\right)u = 0,$$

where u is a function of t with values in $\mathcal{H}$. By substituting

$$x = u \quad \text{and} \quad y = -i\dot{u}$$

we obtain a first order differential equation for $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = i\hat{A}\mathbf{x}, \quad \hat{A} = \begin{pmatrix} 0 & I \\ H_0 - V^2 & 2V \end{pmatrix}. \tag{6.2}$$

In order to associate a well-defined operator to $\hat{A}$, we make the following assumptions (cf. [LNT]):

(i) $\operatorname{dom} H_0^{1/2} \subset \operatorname{dom} V$ and
(ii) there exist $S_0, S_1 \in \mathcal{L}(\mathcal{H})$ with $\|S_0\| < 1$ and S_1 compact such that

$$VH_0^{-1/2} = S_0 + S_1.$$

We emphasize that the condition $1 \in \rho(S^*S)$ from [LNT] is omitted here. Assumption (i) assures that the operator

$$S := VH_0^{-1/2}$$

is bounded. Assumption (ii) implies $I - S^*S = (I - S_0^*S_0) + K$ with a compact operator K in $\mathcal{H}$. Moreover, $I - S_0^*S_0$ is strictly positive. Thus, there is an $\varepsilon > 0$ such that $\sigma(I - S^*S) \cap (-\infty, \varepsilon)$ consists of at most finitely many eigenvalues with finite multiplicities. In particular, $\operatorname{ran}(I - S^*S)$ is closed and $\dim\ker(I - S^*S) < \infty$.

By $\mathcal{H}_{1/2}$ we denote the Hilbert space $\big(\operatorname{dom} H_0^{1/2}, (\cdot\,,\cdot)_{1/2}\big)$, where

$$(x, y)_{1/2} := \left(H_0^{1/2}x, H_0^{1/2}y\right) \quad \text{for} \quad x, y \in \operatorname{dom} H_0^{1/2}.$$

Now, we define the operator $H : \mathcal{H}_{1/2} \to \mathcal{H}$ by

$$H := H_0^{1/2}(I - S^*S)H_0^{1/2}, \ \operatorname{dom} H := \left\{x \in \mathcal{H}_{1/2} : (I - S^*S)H_0^{1/2}x \in \operatorname{dom} H_0^{1/2}\right\}.$$

Lemma 6.1. *The operator H, $H : \mathcal{H}_{1/2} \to \mathcal{H}$, is densely defined and closed.*

Proof. The operator $(I - S^*S)H_0^{1/2} : \mathcal{H}_{1/2} \to \mathcal{H}$ is bounded and $H_0^{1/2}$ is a closed operator in $\mathcal{H}$. This shows the closedness of H. Since $H_0^{1/2}$ (from $\mathcal{H}_{1/2}$ to $\mathcal{H}$) is bounded and boundedly invertible, H is densely defined if and only if the linear manifold

$$M := \left\{y \in \mathcal{H} : (I - S^*S)y \in \operatorname{dom} H_0^{1/2}\right\}$$

is dense in $\mathcal{H}$. We have

$$\mathcal{H} = \ker(I - S^*S) \oplus \operatorname{ran}(I - S^*S).$$

Choose $y = y_0 + \widetilde{y} \in \mathcal{H}$ with $y_0 \in \ker(I - S^*S)$ and $\widetilde{y} \in \operatorname{ran}(I - S^*S)$. As $\operatorname{codim}\operatorname{ran}(I - S^*S) < \infty$, the linear manifold $\operatorname{dom} H_0^{1/2} \cap \operatorname{ran}(I - S^*S)$ is dense in $\operatorname{ran}(I - S^*S)$. Thus, there exists a sequence $(x_n) \subset \operatorname{dom} H_0^{1/2} \cap \operatorname{ran}(I - S^*S)$ with $x_n \to (I - S^*S)\widetilde{y}$ as $n \to \infty$. Observe that the restriction of $I - S^*S$ to $\operatorname{ran}(I - S^*S)$ is a bounded and boundedly invertible operator in $\operatorname{ran}(I - S^*S)$. Therefore

$$\left((I - S^*S) \restriction \operatorname{ran}(I - S^*S)\right)^{-1} x_n \to \widetilde{y} \quad \text{as } n \to \infty.$$

That is, $\widetilde{y}$ belongs to the closure of M and, as $y_0 \in M$, the lemma is proved. □

Now, we define the state space

$$\mathcal{G} := \mathcal{H}_{1/2} \times \mathcal{H}.$$

Let $\mathcal{O}$ be the natural Hilbert space topology on $\mathcal{G}$ induced by the Hilbert space inner products $(\cdot\,,\cdot)$ and $(\cdot\,,\cdot)_{1/2}$. With the bounded and selfadjoint operator

$$G := \begin{pmatrix} H_0^{-1/2}(I - S^*S)H_0^{1/2} & 0 \\ 0 & I \end{pmatrix}$$

in $\mathcal{G}$ the so-called energy inner product is defined as follows: For $x, x' \in \mathcal{H}_{1/2}$ and $y, y' \in \mathcal{H}$ we set

$$\begin{aligned}\left[\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right] &:= \left(G\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right)_{\mathcal{G}} \\ &= ((I - S^*S)H_0^{1/2}x, H_0^{1/2}x') + (y, y') \\ &= (x, x')_{1/2} + (y, y') - (Vx, Vx').\end{aligned}$$

With the formal matrix in (6.2) we associate (see [LNT]) the operator A in $\mathcal{G}$ defined by

$$A := \begin{pmatrix} 0 & I \\ H & 2V \end{pmatrix}, \quad \operatorname{dom} A := \operatorname{dom} H \times \operatorname{dom} H_0^{1/2} \subset \mathcal{G}.$$

As V, considered as an operator from $\mathcal{H}_{1/2}$ into $\mathcal{H}$, is bounded and as H is closed (Lemma 6.1), it follows that A is a closed and densely defined operator in $\mathcal{G}$.

Recall that a closed and densely defined linear operator B in a Banach space is called *Fredholm* if the dimension of the kernel of B and the codimension of the range of B are finite. In particular, a Fredholm operator has a closed range. The set

$$\sigma_{ess}(B) := \{\lambda \in \mathbb{C} \mid B - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of B. The main result of this section is the following

Theorem 6.2. *Under the assumptions* (i) *and* (ii) *the following assertions hold:*

(a) *The triplet $\mathcal{K} := (\mathcal{G}, \mathcal{O}, [\cdot\,,\cdot])$ is an Almost Pontryagin space with finite rank of non-positivity κ, where κ is the dimension of the spectral subspace corresponding to the non-positive eigenvalues of the operator $I - S^*S$;*

(b) *A is G-selfadjoint in $\mathcal{K}$;*

(c) *A has a spectral function with at most finitely many critical points.*

(d) *The non-real spectrum of A consists of at most finitely many eigenvalues with finite multiplicities.*

(e) *The essential spectrum $\sigma_{\text{ess}}(A)$ is real and*

$$\sigma_{\text{ess}}(A) \cap (-\alpha, \alpha) = \varnothing,$$

where $\alpha := (1 - \|S_0\|)m$.

(f) *The operator iA is the generator of a strongly continuous group $\left(e^{itA}\right)_{t\in\mathbb{R}}$ in $\mathcal{G}$. Hence, the Cauchy problem*

$$\frac{d\mathbf{x}}{dt} = iA\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution $\mathbf{x}(t) = e^{itA}\mathbf{x}_0$, $t \in \mathbb{R}$, for all initial values $\mathbf{x}_0 \in \mathcal{G}$.

Proof of Theorem 6.2. We have $\sigma(G) = \sigma(I - S^*S) \cup \{1\}$. Thus, (a) follows from Proposition 3.5. In the following we show that the operator

$$GA = \begin{pmatrix} 0 & H_0^{-1/2}(I - S^*S)H_0^{1/2} \\ H & 2V \end{pmatrix}, \quad \operatorname{dom} GA = \operatorname{dom} A,$$

is selfadjoint in $(\mathcal{G}, (\cdot\,,\cdot)_{\mathcal{G}})$. Then (b) follows. For $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \operatorname{dom} A$ we have

$$\begin{aligned} \left(GA\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) &= \left[\begin{pmatrix} H_0^{-1/2}(I - S^*S)H_0^{1/2}y \\ Hx + 2Vy \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right] \\ &= \big((I - S^*S)H_0^{1/2}y, H_0^{1/2}x\big) + (Hx + 2Vy, y) \\ &= (y, Hx) + (Hx, y) + 2(Vy, y), \end{aligned}$$

and this is a real value. Thus, $GA \subset (GA)^*$. Now, let $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \operatorname{dom}((GA)^*)$. Then the linear functional

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \left(GA\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = (H_0^{1/2}y, (I - S^*S)H_0^{1/2}u) + 2(Vy, v) + (Hx, v)$$

is continuous on $\operatorname{dom} A = \operatorname{dom} H \times \operatorname{dom} H_0^{1/2}$. In particular, the linear functionals

$$\begin{aligned} &f : x \mapsto (Hx, v), \quad x \in \operatorname{dom} H \subset \mathcal{H}_{1/2}, \quad \text{and} \\ &g : y \mapsto \big(H_0^{1/2}y, (I - S^*S)H_0^{1/2}u\big) + 2(Vy, v), \quad y \in \operatorname{dom} H_0^{1/2} \subset \mathcal{H} \end{aligned}$$

are continuous, where f is continuous with respect to the norm in $\mathcal{H}_{1/2}$ and g is continuous with respect to the norm in $\mathcal{H}$. We show that the continuity of f implies $v \in \operatorname{dom} H_0^{1/2}$. There exists a finite-dimensional subspace $X \subset \operatorname{dom} H_0^{1/2}$ such that

$$\mathcal{H} = X \dotplus \operatorname{ran}(I - S^*S) \;\text{ and }\; \operatorname{dom} H_0^{1/2} = X \dotplus \big(\operatorname{dom} H_0^{1/2} \cap \operatorname{ran}(I - S^*S)\big).$$

Let P_1 be the projection in $\mathcal{H}$ onto X along $\operatorname{ran}(I - S^*S)$ and set $P_2 := I - P_1$. Then $H_0^{1/2}P_1$ is a bounded operator in $\mathcal{H}$. By T denote the bounded and boundedly invertible restriction of $I - S^*S$ to $\operatorname{ran}(I - S^*S) = P_2\mathcal{H}$. For $z \in \operatorname{dom} H_0^{1/2}$ we have

$$\begin{aligned} \big(H_0^{1/2}z, v\big) &= \big(H_0^{1/2}P_1z, v\big) + \big(H_0^{1/2}(I - S^*S)H_0^{1/2}H_0^{-1/2}T^{-1}P_2z, v\big) \\ &= \big(H_0^{1/2}P_1z, v\big) + f\big(H_0^{-1/2}T^{-1}P_2z\big), \end{aligned}$$

which implies $v \in \operatorname{dom}\big(\big(H_0^{1/2}\big)^*\big)$ as $H_0^{-1/2}T^{-1}P_2$ is a bounded operator from $\mathcal{H}$ to $\mathcal{H}_{1/2}$. But $H_0^{1/2}$ is selfadjoint in $\mathcal{H}$, and therefore $v \in \operatorname{dom} H_0^{1/2}$. By (i), this yields $v \in \operatorname{dom} V$ and thus, as g is continuous on $\operatorname{dom} H_0^{1/2}$, it follows that $(I - S^*S)H_0^{1/2}u \in \operatorname{dom} H_0^{1/2}$, hence $\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \in \operatorname{dom} A$ and (b) is proved.

Statements (c) and (d) hold due to the results of Sections 4 and 5 while (f) is also an easy consequence of Section 5, Theorem 5.1, since with the help of the spectral function the operator A can be written as a direct sum $A_1[\dotplus]A_2$ of a bounded operator in an Almost Pontryagin space and a selfadjoint operator in a

Hilbert space. Obviously, $e^{itA} := e^{itA_1} \,[\dotplus]\, e^{itA_2}$ is a strongly continuous group of bounded operators in $\mathcal{G}$, and it is easily seen that iA is its generator.

Let us prove (e). To this end we define the quadratic pencils

$$L(\lambda) := I - S^*S + 2\lambda H_0^{-1/2}S - \lambda^2 H_0^{-1} \quad\text{and}$$
$$\widetilde{L}(\lambda) := I - S_0^*S_0 + 2\lambda H_0^{-1/2}S_0 - \lambda^2 H_0^{-1}$$

of bounded operators in $\mathcal{H}$, $\lambda \in \mathbb{C}$. If $\lambda \in \mathbb{R}$, $|\lambda| < (1 - \|S_0\|)m$, then

$$\begin{aligned}&\|S_0^*S_0 - 2\lambda H_0^{-1/2}S_0 + \lambda^2 H_0^{-1}\|\\ &< \|S_0\|^2 + 2(1-\|S_0\|)m\frac{1}{m}\|S_0\| + (1-\|S_0\|)^2 m^2 \frac{1}{m^2} = 1.\end{aligned}$$

This implies that $\widetilde{L}(\lambda)$ is boundedly invertible for any $\lambda \in (-\alpha, \alpha)$. And as $L(\lambda)$ is a compact perturbation of $\widetilde{L}(\lambda)$ it follows that $L(\lambda)$ is a (bounded) Fredholm operator for all $\lambda \in (-\alpha, \alpha)$ (see [Ka, Theorem IV.5.26]). We will now show that $A - \lambda$ is a Fredholm operator for $\lambda \in (-\alpha, \alpha)$ which then completes the proof of (e). Let $\lambda \in (-\alpha, \alpha)$. For $\binom{x}{y} \in \operatorname{dom} A$ we have

$$(A-\lambda)\binom{x}{y} = \begin{pmatrix} y - \lambda x \\ H_0^{1/2}L(\lambda)H_0^{1/2}x + (2SH_0^{1/2} - \lambda)(y - \lambda x)\end{pmatrix}. \tag{6.3}$$

From this it is immediately seen that $\ker(A-\lambda)$ is finite-dimensional since $\ker L(\lambda)$ is finite-dimensional. We show that $\operatorname{ran}(A-\lambda)$ is closed. In order to see this let $(\binom{x_n}{y_n})$ be a sequence in $\operatorname{dom} A$ such that $(A-\lambda)\binom{x_n}{y_n}$ converges in $\mathcal{G}$ to some $\binom{w}{z} \in \mathcal{G}$. Then from (6.3) it follows that $H_0^{1/2}L(\lambda)H_0^{1/2}x_n$ converges in $\mathcal{H}$ to $z - (2SH_0^{1/2} - \lambda)w$. As $L(\lambda)$ is a Fredholm operator, we find some $x \in \operatorname{dom} H$ such that

$$L(\lambda)H_0^{1/2}x = H_0^{-1/2}\left(z - (2SH_0^{1/2} - \lambda)w\right).$$

With $y := w + \lambda x$ we have $\binom{x}{y} \in \operatorname{dom} A$ and (see (6.3))

$$(A-\lambda)\binom{x}{y} = \begin{pmatrix} w \\ H_0^{1/2}H_0^{-1/2}\left(z - (2SH_0^{1/2} - \lambda)w\right) + (2SH_0^{1/2} - \lambda)w\end{pmatrix} = \begin{pmatrix} w \\ z\end{pmatrix},$$

and hence $\binom{w}{z} \in \operatorname{ran}(A-\lambda)$.

We have just shown that for all $\lambda \in (-\alpha, \alpha)$ the operator $A - \lambda$ is a Φ_+-operator. By Corollary 4.7 and [Ka, IV §5.6] the operator $A-\lambda$ is even a Fredholm operator of index 0 for all $\lambda \in (-\alpha, \alpha)$. This completes the proof of the theorem. □

Remark 6.3. In [LNT, Remark 5.4] it was already mentioned that if (i) and (ii) are fulfilled, the operator A induces a selfadjoint operator $\widetilde{A}$ in the factor space $\widetilde{\mathcal{G}} := \mathcal{G}/\mathcal{G}^\circ$, which is a Pontryagin space. This operator corresponds to the $\widetilde{A}$ in Proposition 4.5.

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