Remarks about Disjoint Dominating Sets

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Abstract

We solve a number of problems posed by Hedetniemi, Hedetniemi, Laskar, Markus,
and Slater concerning pairs of disjoint sets in graphs which are dominating or independent and dominating.

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1 Introduction

We consider finite, simple and undirected graphs \( G = (V, E) \) with vertex set \( V \) and edge set \( E \). A set of vertices \( D \subseteq V \) of \( G \) is dominating, if every vertex in \( V \setminus D \) has a neighbour in \( D \). The minimum cardinality of a dominating set is the domination number \( \gamma(G) \) of \( G \). A set of vertices \( I \subseteq V \) of \( G \) is independent, if no two vertices in \( I \) are adjacent. The maximum cardinality of an independent set is the independence number \( \alpha(G) \) of \( G \).

Dominating and independent sets are among the most well-studied graph sets. The literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [10, 11]. While much of the related research is devoted to \( \gamma(G) \) and \( \alpha(G) \), the problem of partitioning the vertex set into dominating sets [3, 7, 4] and even more the problem of partitioning the vertex set into independent sets, i.e. vertex colourings, have been extensively studied.

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Quite recently several authors have studied pairs of disjoint dominating sets. Kulli and Sigarkanti [14] introduced the inverse domination number $\gamma^{-1}(G)$ of a graph $G$ as the minimum cardinality of a dominating set whose complement contains a minimum dominating set of $G$. Motivated by a false proof for the inequality $\gamma^{-1}(G) \leq \alpha(G)$ that appeared in [14], several authors [5, 8] studied this parameter. A classical result in domination theory due to Ore [15] is that if $D$ is a minimal dominating set of a graph $G$ with no isolated vertex, then $V \setminus D$ is also a dominating set of $G$. Thus every such graph $G$ contains two disjoint dominating sets. In [13] Hedetniemi et al. initiate the study of the minimum cardinality $\gamma(G) = |D_1| + |D_2|$ of the union of two disjoint dominating sets $D_1$ and $D_2$ of a graph $G$ with no isolated vertex. Similarly, they defined $\gamma_i(G)$ as the minimum cardinality $|D_1| + |I_2|$ of the union of two disjoint dominating sets $D_1$ and $I_2$ of $G$ for which $I_2$ is independent and they define $\gamma_{ii}(G)$ as the minimum cardinality $|I_1| + |I_2|$ of the union of two disjoint independent dominating sets $I_1$ and $I_2$ of $G$. Various graph theoretic and algorithmic properties of these parameters are presented in [13].

For notation and graph theory terminology we in general follow [10]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n = |V|$ and edge set $E$ of size $m = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of $v$ is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$. For a set $S$ of vertices, the closed neighborhood of $S$ is defined by $N_G[S] = \cup_{v \in S} N_G(v)$. If $X, Y \subseteq V$, then the set $X$ is said to dominate the set $Y$ if $Y \subseteq N_G[X]$. In particular if $X$ dominates $V$, then $X$ is a dominating set of $G$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$.

2 Nine Problems posed in [13]

In this section, we list nine problems posed by Hedetniemi et al. in [13].

A) Characterize the graphs $G$ for which $\gamma \gamma(G) = 2\gamma(G)$, i.e., characterize the graphs which have two disjoint minimum dominating sets. (Problem 1 in [13].)

B) Under what conditions does $\gamma_{ii}(G)$ exist? (Problem 10 in [13].)

C) When is $\gamma \gamma(G) = \gamma i(G)$? (Problem 11 in [13].)

D) When is $\gamma i(G) = \gamma_{ii}(G)$? (Problem 12 in [13].)

E) Is the calculation of $\gamma \gamma(G)$ NP-complete for bipartite graphs? (Problem 17 in [13].)

F) What is the complexity of the decision problem corresponding to $\gamma i(G)$? (Problem 13 in [13].)

G) For which class of trees $T$ of order $n \geq 2$ is $\gamma \gamma(T) = 2(n + 1)/3$? (Problem 8 in [13]. Note that it is shown in [13] that $\gamma \gamma(T) \geq 2(n + 1)/3$ for all trees $T$ of order $n \geq 2$.)

H) Conjecture. A tree $T$ satisfies $\gamma \gamma(T) = 2\gamma(T)$ if and only if no vertex of $T$ belongs to every minimum dominating set of $T$. (Problem 7 in [13].)
I) Does every tree of order \( n \geq 2 \) have a minimum dominating set whose complement contains an independent dominating set of \( T \)? (Problem 21 in [13].)

3 Results

Our aim in this paper is to solve the nine problems listed in Section 2.

3.1 Problem A

While trees with two disjoint minimum dominating sets were constructively characterized in [1] (cf. also [2, 6, 9, 12]), we give a somewhat negative ‘solution’ to Problem A by showing that the corresponding decision problem is NP-hard. We do not know whether this problem is actually in NP.

**Theorem 1** It is NP-hard to decide whether a given graph has two disjoint minimum dominating sets.

**Proof.** Given a 3Sat instance \( \mathcal{C} \) we will construct a graph \( G \) whose order is polynomially bounded in the size of \( \mathcal{C} \) such that \( \mathcal{C} \) is satisfiable if and only if \( G \) has two disjoint minimum dominating sets.

For every boolean variable \( x \) occurring in \( \mathcal{C} \) we introduce a copy \( G_x \) of the gadget shown in the left part of Figure 1 which contains two specified vertices \( x \) and \( \bar{x} \). Furthermore, for every clause \( C \) of \( \mathcal{C} \) we introduce a copy \( G_C \) of the gadget shown in the right part of Figure 1 which contains one specified vertex \( C \).

![Figure 1. The gadgets \( G_x \) and \( G_C \).](image)

If the literal \( x \) occurs in clause \( C \) we connect the specified vertex \( x \) in \( G_x \) with the specified vertex \( C \) in \( G_C \). (For an example see Figure 2 where \( \mathcal{C} = \{x \lor \bar{y} \lor z, x \lor z \lor u\} \).) Let \( G \) denote the resulting graph.
Let $\mathcal{C}$ use $n$ boolean variables and contain $m$ clauses. Note that the order of $G$ is $6n+4m$. Every dominating set of $G$ contains at least two vertices from every gadget $G_x$ and at least one vertex from every gadget $G_C$. Conversely, choosing the two vertices at distance 1 and 3 from the endvertex in every gadget $G_x$ and the dominating vertex in every gadget $G_C$ yields a dominating set of $G$. This implies that $\gamma(G) = 2n + m$.

If $\mathcal{C}$ is satisfiable, then we consider a satisfying truth assignment for $\mathcal{C}$. The set of vertices corresponding to the true literals together with the neighbour of the endvertex in every gadget $G_x$ and one of the two vertices of degree 2 in every gadget $G_C$ yields a minimum dominating set $D$ of $G$. Furthermore, choosing the two vertices at distance 0 and 3 from the endvertex in every gadget $G_x$ and the dominating vertex in every gadget $G_C$ yields a minimum dominating set of $G$ which is disjoint from $D$.

Conversely, we assume now that $G$ has two disjoint minimum dominating sets $D_1$ and $D_2$. By the above reasoning, each of $D_1$ and $D_2$ contains exactly one vertex from each gadget $G_C$. This implies that for every gadget $G_C$ the specified vertex $C$ must be dominated within one of $D_1$ and $D_2$ by a vertex not contained in $G_C$. Furthermore, for every gadget $G_x$ the set $D_1 \cup D_2$ contains at most one of the two specified vertices $x$ and $\bar{x}$. Therefore, the vertices in $D_1 \cup D_2$ corresponding to literals indicate a satisfying truth assignment for $\mathcal{C}$. Note that the truth value of a variable $x$ for which neither $x$ nor $\bar{x}$ is in $D_1 \cup D_2$ can be set arbitrarily. (The two minimum dominating sets indicated in Figure 2 correspond to setting $x$, $y$ and $z$ false and $u$ true.) This completes the proof. $\square$

3.2 Problem B

As with Problem A, our ‘solution’ to Problem B is a hardness result.
**Theorem 2** It is NP-complete to decide whether a given graph has two disjoint independent dominating sets.

**Proof.** The given decision problem is clearly in NP. Given a 3Sat instance $C$ we will construct a graph $G$ whose order is polynomially bounded in the size of $C$ such that $C$ is satisfiable if and only if $G$ has two disjoint independent dominating sets.

For every boolean variable $x$ occurring in $C$ we introduce a copy $G_x$ of the gadget shown in the left part of Figure 3 which contains two specified vertices $x$ and $\bar{x}$. Furthermore, for every clause $C$ of $C$ we introduce a copy $G_C$ of the gadget shown in the right part of Figure 3 which contains one specified vertex $C$.

![Figure 3. The gadgets $G_x$ and $G_C$.](image)

If the literal $x$ occurs in clause $C$ we connect the specified vertex $x$ in $G_x$ with the specified vertex $C$ in $G_C$. (For an example see Figure 4 where $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$.) Let $G$ denote the resulting graph.

![Figure 4. The graph $G$ for $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$.](image)

Let $C$ use $n$ boolean variables and contain $m$ clauses. Note that the order of $G$ is $4n + 6m$. 


If \( C \) is satisfiable, then we consider a satisfying truth assignment for \( C \). Choosing in every gadget \( G_x \) the endvertex and the vertex corresponding to the true literal and choosing in every gadget \( G_C \) an endvertex different from \( C \) and the neighbour of the other endvertex different from \( C \) yields an independent dominating set \( I \) of \( G \). Furthermore, choosing in every gadget \( G_x \) the neighbour of the endvertex and choosing in every gadget \( G_C \) the vertex \( C \) and the two vertices not adjacent to \( C \) or contained in \( I \) yields an independent dominating set of \( G \) disjoint from \( I \).

Conversely, we assume now that \( G \) has two disjoint independent dominating sets \( I_1 \) and \( I_2 \). Since in every gadget \( G_C \) the two vertices at distance two from \( C \) are necessarily in \( I_1 \cup I_2 \), the neighbour of \( C \) in \( G_C \) is not in \( I_1 \cup I_2 \). This implies that \( C \) is dominated within one of the two sets \( I_1 \) or \( I_2 \) by a vertex not contained in \( G_C \). Clearly, at most one of the two vertices \( x \) and \( \bar{x} \) in every gadget \( G_x \) can be in \( I_1 \cup I_2 \). Therefore, the vertices in \( I_1 \cup I_2 \) corresponding to literals indicate a satisfying truth assignment for \( C \). Again, the truth value of a variable \( x \) for which neither \( x \) nor \( \bar{x} \) is in \( I_1 \cup I_2 \) can be set arbitrarily. (The two independent dominating sets indicated in Figure 4 correspond to setting \( x, y \) and \( z \) false and \( u \) true.) This completes the proof. □

### 3.3 Problems C and D

As with Problems A and B, yet further hardness results.

**Theorem 3** Given a graph \( G \) the following two problems are NP-hard.

(i) Decide whether \( G \) satisfies \( \gamma \gamma (G) = \gamma i(G) \).

(ii) Decide whether \( G \) satisfies \( \gamma i(G) = ii(G) \).

**Proof.** Given a 3Sat instance \( C \) we will construct two graphs \( G \) and \( G' \) whose order is polynomially bounded in the size of \( C \) such that \( C \) is satisfiable if and only if \( \gamma \gamma (G) = \gamma i(G) \) if and only if \( \gamma i(G') = ii(G') \).

For the construction of \( G \) we proceed as follows. For every boolean variable \( x \) occurring in \( C \) we introduce a copy \( G_x \) of the gadget shown in the left part of Figure 5 which contains two specified vertices \( x \) and \( \bar{x} \). Furthermore, for every clause \( C \) of \( C \) we introduce a copy \( G_C \) of the gadget shown in the middle part of Figure 5 which contains one specified vertex \( C \).
If the literal $x$ occurs in clause $C$ we connect the specified vertex $x$ in $G_x$ with the specified vertex $C$ in $G_C$. (For an example see Figure 6 where $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$.)

For the graph $G'$ we proceed exactly as above using the gadget $G_C'$ shown in the right part of Figure 5 instead of $G_C$.

Let $C$ use $n$ boolean variables and contain $m$ clauses. Note that the orders of $G$ and $G'$ are $6n + 8m$. Every dominating set of $G$ contains at least two vertices from every gadget $G_x$ and at least two vertices from every gadget $G_C$. Conversely, choosing in every gadget the vertices as indicated in Figure 5 yields two disjoint minimum dominating sets, i.e., $\gamma(G) = 2\gamma(G) = 4n + 4m$. Similarly, $\gamma(G') = 2\gamma(G') = 4n + 4m$.

If $C$ is satisfiable, then we consider a satisfying truth assignment for $C$. We choose the two disjoint minimum dominating sets described above such that from every gadget $G_x$ the
vertex corresponding to the true literal is in one of the two sets. Furthermore, in every
gadget $G_C$ we choose vertices as indicated in Figure 6. This yields two disjoint minimum
dominating sets one of which is independent, i.e., $\gamma\gamma(G) = \gamma i(G)$. Similar arguments yield
$\gamma i (G') = \gamma i (G')$.

Conversely, we assume now that $G$ satisfies $\gamma \gamma (G) = \gamma i (G)$. Let $D_1$ and $I_2$ be two disjoint
dominating sets such that $I_2$ is independent and $|D_1| + |I_2| = \gamma \gamma (G) = \gamma i (G) = 2 \gamma (G)$,
i.e., $D_1$ and $I_2$ are both minimum dominating. By the above reasoning, each of $D_1$ and
$I_2$ contains exactly two vertices from each gadget $G_C$. This easily implies that in every
gadget $G_C$ the specified vertex $C$ is dominated within one of $D_1$ and $I_2$ by a vertex not
contained in $G_C$. Furthermore, for every gadget $G_x$ the set $D_1 \cup I_2$ contains at most one
of the two specified vertices $x$ and $\bar{x}$. Therefore, the vertices in $D_1 \cup I_2$ corresponding to
literals indicate a satisfying truth assignment for $C$. (The two minimum dominating sets
indicated in Figure 6 correspond to setting $x$, $y$ and $z$ false and $u$ true.) Again, if we assume
that $G'$ satisfies $\gamma i (G') = \gamma i (G')$, then the same train of thought implies that $C$ is satisfiable. This completes the proof. \(\square\)

3.4 Problems E and F

In [13] it is shown that the calculation of $\gamma \gamma (G)$ is NP-hard even when restricted to chordal
graphs. In Problem E, the authors in [13] ask about the complexity for the class of bipartite
graphs, while in Problem F they ask about the complexity of the decision problem corre-
sponding to $\gamma i (G)$. We prove that the corresponding decision problems are NP-complete.
Note that Theorem 2 and the statement made about $\gamma i (G)$ in Theorem 4 that follows do
not imply each other.

**Theorem 4** Given a bipartite graph $G$ and given an integer $k$ the following three problems
are NP-complete.

(i) Decide whether $G$ has two disjoint dominating sets $D_1$ and $D_2$ with $|D_1| + |D_2| \leq k$.

(ii) Decide whether $G$ has two disjoint dominating sets $D_1$ and $D_2$ with $|D_1| + |D_2| \leq k$
such that $D_2$ is independent.

(iii) Decide whether $G$ has two disjoint independent dominating sets $D_1$ and $D_2$ with $|D_1| +
|D_2| \leq k$.

**Proof.** The three decision problems are clearly in NP. Given a 3Sat instance $C$ we will
construct a graph $G$ whose order is polynomially bounded in the size of $C$ and specify
an integer $k$ also polynomially bounded in the size of $C$ such that if $C$ is satisfiable, then
$\gamma i (G) \leq k$ and if $\gamma \gamma (G) \leq k$, then $C$ is satisfiable. This clearly implies the desired statements.

For every boolean variable $x$ occurring in $C$ we introduce a copy $G_x$ of the gadget shown
in the left part of Figure 7 which contains two specified vertices $x$ and $\bar{x}$. Furthermore,
for every clause $C$ of $C$ we introduce a copy $G_C$ of the gadget shown in the right part of
Figure 7 which contains two specified vertices $C$ and $\bar{C}$.
If the (unnegated) variable $x$ occurs in clause $C$ we connect the specified vertex $x$ in $G_x$ with the specified vertex $C$ in $G_C$. Similarly, if the negated variable $\bar{x}$ occurs in clause $C$ we connect the specified vertex $\bar{x}$ in $G_x$ with the specified vertex $\bar{C}$ in $G_C$. Note that this way of adding edges to the disjoint union of the bipartite gadgets results in a bipartite graph. (For an example see Figure 8 where $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$.) Let $G$ denote the resulting graph.

Let $C$ use $n$ boolean variables and contain $m$ clauses. Note that the order of $G$ is $12n + 8m$. Let $k = 8n + 5m$.

First we assume that $C$ is satisfiable and describe how to obtain two disjoint dominating sets $D_1$ and $D_2$ of $G$ with $|D_1| + |D_2| \leq k$. Consider a satisfying truth assignment for $C$. We choose in every gadget $G_x$ the vertices for the sets $D_1$ and $D_2$ as indicated in the left part of Figure 7 or its mirror image such that $D_1$ contains the vertex corresponding to the true literal among $x$ or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices $C$ or $\bar{C}$ in every gadget $G_C$ is dominated in $D_1$ by a vertex not contained in $V(G_C)$. This implies that the two sets $D_1$ and $D_2$ can be extended as indicated in Figure 8 using a total of five vertices in each of the gadgets $G_C$. Hence, $|D_1| + |D_2| = k$. 

**Figure 7.** The gadgets $G_x$ and $G_C$.

**Figure 8.** The graph $G$ for $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$. 

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First we assume that $C$ is satisfiable and describe how to obtain two disjoint dominating sets $D_1$ and $D_2$ of $G$ with $|D_1| + |D_2| \leq k$. Consider a satisfying truth assignment for $C$. We choose in every gadget $G_x$ the vertices for the sets $D_1$ and $D_2$ as indicated in the left part of Figure 7 or its mirror image such that $D_1$ contains the vertex corresponding to the true literal among $x$ or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices $C$ or $\bar{C}$ in every gadget $G_C$ is dominated in $D_1$ by a vertex not contained in $V(G_C)$. This implies that the two sets $D_1$ and $D_2$ can be extended as indicated in Figure 8 using a total of five vertices in each of the gadgets $G_C$. Hence, $|D_1| + |D_2| = k$. 

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**Figure 8.** The graph $G$ for $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$. 

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First we assume that $C$ is satisfiable and describe how to obtain two disjoint dominating sets $D_1$ and $D_2$ of $G$ with $|D_1| + |D_2| \leq k$. Consider a satisfying truth assignment for $C$. We choose in every gadget $G_x$ the vertices for the sets $D_1$ and $D_2$ as indicated in the left part of Figure 7 or its mirror image such that $D_1$ contains the vertex corresponding to the true literal among $x$ or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices $C$ or $\bar{C}$ in every gadget $G_C$ is dominated in $D_1$ by a vertex not contained in $V(G_C)$. This implies that the two sets $D_1$ and $D_2$ can be extended as indicated in Figure 8 using a total of five vertices in each of the gadgets $G_C$. Hence, $|D_1| + |D_2| = k$. 

**Figure 7.** The gadgets $G_x$ and $G_C$.

**Figure 8.** The graph $G$ for $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$. 

Let $C$ use $n$ boolean variables and contain $m$ clauses. Note that the order of $G$ is $12n + 8m$. Let $k = 8n + 5m$.

First we assume that $C$ is satisfiable and describe how to obtain two disjoint dominating sets $D_1$ and $D_2$ of $G$ with $|D_1| + |D_2| \leq k$. Consider a satisfying truth assignment for $C$. We choose in every gadget $G_x$ the vertices for the sets $D_1$ and $D_2$ as indicated in the left part of Figure 7 or its mirror image such that $D_1$ contains the vertex corresponding to the true literal among $x$ or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices $C$ or $\bar{C}$ in every gadget $G_C$ is dominated in $D_1$ by a vertex not contained in $V(G_C)$. This implies that the two sets $D_1$ and $D_2$ can be extended as indicated in Figure 8 using a total of five vertices in each of the gadgets $G_C$. Hence, $|D_1| + |D_2| = k$. 

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**Figure 8.** The graph $G$ for $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$. 

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First we assume that $C$ is satisfiable and describe how to obtain two disjoint dominating sets $D_1$ and $D_2$ of $G$ with $|D_1| + |D_2| \leq k$. Consider a satisfying truth assignment for $C$. We choose in every gadget $G_x$ the vertices for the sets $D_1$ and $D_2$ as indicated in the left part of Figure 7 or its mirror image such that $D_1$ contains the vertex corresponding to the true literal among $x$ or $\bar{x}$. Since the truth assignment is satisfying, at least one of the vertices $C$ or $\bar{C}$ in every gadget $G_C$ is dominated in $D_1$ by a vertex not contained in $V(G_C)$. This implies that the two sets $D_1$ and $D_2$ can be extended as indicated in Figure 8 using a total of five vertices in each of the gadgets $G_C$. Hence, $|D_1| + |D_2| = k$. 

**Figure 7.** The gadgets $G_x$ and $G_C$.

**Figure 8.** The graph $G$ for $C = \{x \lor \bar{y} \lor z, x \lor z \lor u\}$.
Next, we assume that \( G \) has two disjoint dominating sets \( D_1 \) and \( D_2 \) such that \(|D_1| + |D_2| \leq k\). In every gadget \( G_x \), the set \( V(G_x) \cap (D_1 \cup D_2) \) contains at least eight vertices in order to dominate the ten vertices on the path \( G_x - \{x, \bar{x}\} \). Furthermore, if \( V(G_x) \cap (D_1 \cup D_2) \) contains exactly eight vertices, then at least one of \( x \) and \( \bar{x} \) is not contained in \( D_1 \cup D_2 \).

If for some gadget \( G_C \) neither \( C \) nor \( \bar{C} \) are dominated by a vertex in \( D_1 \cup D_2 \) not contained in \( V(G_C) \), then \( V(G_C) \cap (D_1 \cup D_2) \) contains at least six vertices. (One possible configuration is shown in the right part of Figure 7.) Furthermore, if for some gadget \( G_C \) one or both of \( C \) and \( \bar{C} \) are dominated by vertices in \( D_1 \cup D_2 \) not contained in \( V(G_C) \), then \( V(G_C) \cap (D_1 \cup D_2) \) contains at least five vertices.

Since \(|D_1| + |D_2| \leq 8n + 5m\), we obtain that for every gadget \( G_x \) at most one of \( x \) and \( \bar{x} \) is contained in \( D_1 \cup D_2 \) and for every gadget \( G_C \) one of \( C \) and \( \bar{C} \) is dominated by a vertex in \( D_1 \cup D_2 \) not contained in \( V(G_C) \). This implies that the vertices contained in \( D_1 \cup D_2 \) corresponding to literals indicate a satisfying truth assignment for \( C \) and the proof is complete. \( \square \)

### 3.5 Problem G

As remark earlier, it is shown in [13] that \( \gamma_\gamma(T) \geq 2(n+1)/3 \) for all trees \( T \) of order \( n \geq 2 \). In Problem G, the authors ask for a characterization of the trees achieving equality in this bound.

**Theorem 5** If \( T = (V, E) \) is a tree of order \( n \), then \( \gamma_\gamma(T) \geq 2(n+1)/3 \) with equality if and only if \( V \) can be partitioned into two sets \( D \) and \( R \) such that \( D \) induces a perfect matching and \( R \) is an independent set all vertices of which have degree 2 in \( T \).

**Proof.** Let \( T \) be a tree of order \( n \) and let \( D_1 \) and \( D_2 \) be two disjoint dominating sets of \( T \) such that \( \gamma_\gamma(T) = |D_1| + |D_2| \). We assume that \(|D_1| \geq |D_2|\). Let \( D = D_1 \cup D_2 \) and let \( R = V \setminus D \). Since every vertex in \( R \) has a neighbour in \( D_1 \) and a neighbour in \( D_2 \) and every vertex in \( D_1 \) has a neighbour in \( D_2 \), counting the edges of \( T \) yields

\[
    n - 1 \geq 2|R| + |D_1| \geq 2|R| + |D|/2 = 2(n - \gamma_\gamma(T)) + \gamma_\gamma(T)/2,
\]

which implies \( \gamma_\gamma(T) \geq 2(n + 1)/3 \).

If \( \gamma_\gamma(T) = 2(n + 1)/3 \), then equality holds throughout the above inequality chain. This implies that \(|D_1| = |D_2|\), every vertex in \( R \) has exactly one neighbour in \( D_1 \) and one neighbour in \( D_2 \), every vertex from \( D_1 \) has exactly one neighbour in \( D_2 \) and the three sets \( D_1, D_2 \) and \( R \) are independent. Since every vertex of \( D_2 \) has at least one neighbour in \( D_1 \), the set \( D \) induces a perfect matching and the structure of \( T \) is as described in the statement of the result.

Conversely, we assume now that \( V \) can be partitioned into two sets \( D \) and \( R \) such that \( D \) induces a perfect matching and \( R \) is an independent set all vertices of which have degree 2 in \( T \). We will prove by induction on the order \( n \) of \( T \) that \( \gamma_\gamma(T) = 2(n + 1)/3 \). More
specifically, we prove that $D$ can be partitioned into two independents sets $D_1$ and $D_2$ which are both dominating. Note that, by the assumptions, such sets $D_1$ and $D_2$ satisfy $|D_1| + |D_2| = 2(n + 1)/3$. If $n = 2$, then the statement is trivial. Hence, we may assume that $n \geq 3$. Let $uv$ be an edge which corresponds to an endvertex of the tree which arises from $T$ by contracting all edges of the perfect matching induced by $D$. Note that after these contractions all vertices in $R$ are still of degree 2. This implies that we may assume that $u$ is an endvertex of $T$ and $v$ has degree 2 in $T$. Let $w$ be the neighbour of $v$ different from $u$. Clearly, $w \in R$. The vertex set $V \setminus \{u, v, w\}$ of the tree $T' = T - \{u, v, w\}$ can be partitioned into two sets $D' = D \setminus \{u, v\}$ and $R' = R \setminus \{w\}$ such that $D'$ induces a perfect matching and $R'$ is an independent set all vertices of which have degree 2 in $T'$. Hence, by induction, $D'$ can be partitioned into two independent sets $D'_1$ and $D'_2$ both of which are dominating in $T'$. We may assume that the neighbour of $w$ different from $v$ belongs to $D'_1$. Now the two sets $D_1 = D'_1 \cup \{u\}$ and $D_2 = D'_2 \cup \{v\}$ are independent and dominating in $T$ and partition $D$ which completes the proof. □

3.6 Problem H

In Problem H the authors conjecture that for a tree $T$ the equality $\gamma(T) = 2\gamma(T)$ is equivalent to the property that no vertex of $T$ belongs to every minimum dominating set of $T$. While this property is obviously necessary, we describe an example disproving the conjecture.

**Observation 6** There are trees $T$ for which no vertex belongs to every minimum dominating set of $T$ and which do not have two disjoint minimum dominating sets, i.e., $\gamma(T) > 2\gamma(T)$.

**Proof.** The tree two copies of which are shown in Figure 9 has domination number 7 and the two indicated minimum dominating sets show that no vertex belongs to every minimum dominating set of $T$. On the other hand it is easy to see that the union of every two disjoint dominating sets of $T$ contains at least five vertices in each of the indicated rectangular boxes which implies that one of the sets cannot be minimum. □

![Figure 9](image-url) A counterexample to the conjecture posed in Problem H.
3.7 Problem I

In Problem I, it is asked whether every tree of order $n$ has a minimum dominating set whose complement contains an independent dominating set. We answer this question in the affirmative. For this purpose, given a rooted tree $T$, a set $D$ of vertices of $T$ and a vertex $v \in D$, we define an external $D$-private child of $v$ in $T$ to be a child of $v$ in $N_T(v) \setminus N_T[D \setminus \{v\}]$. Hence if $u$ is an external $D$-private child of $v$ in $T$, then $u \notin D$, $u$ is a child of $v$ in $T$, and $N_T(u) \cap D = \{v\}$.

**Theorem 7** Every tree of order at least two has a minimum dominating set and an independent dominating set which are disjoint.

**Proof.** Let $u$ be an endvertex of $T$. Let $D$ be a minimum dominating set containing a neighbour $r$ of $u$ such that

$$f(D) := \sum_{v \in D} \text{dist}_T(v, r)$$

is minimum. Root $T$ at $r$. Note that $u$ is an external $D$-private child of $r$ in $T$. If some vertex $v \in D \setminus \{r\}$ has no external $D$-private child in $T$, then the parent $w$ of $v$ is not in $D$. Now the set $D' = (D \setminus \{v\}) \cup \{w\}$ is a minimum dominating set of $T$ containing $r$ with $f(D') = f(D) - 1$, which is a contradiction. Hence all vertices in $D$ have external $D$-private children in $T$. Clearly, a set $I$ containing exactly one external $D$-private child of every vertex in $D$ is an independent set and a maximal independent subset of $V \setminus D$ which contains $I$ is a dominating set of $T$. This completes the proof. □

References


