Cohabitation of Independent Sets and Dominating Sets in Trees

Christian Löwenstein and Dieter Rautenbach
Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany, emails: {christian.loewenstein, dieter.rautenbach}@tu-ilmenau.de

Abstract. We give a constructive characterization of trees that have a maximum independent set and a minimum dominating set which are disjoint and show that the corresponding decision problem is NP-complete for general graphs.

Keywords. domination; independence; inverse domination
AMS subject classification. 05C69

1 Introduction

We consider finite, undirected and simple graphs $G = (V, E)$ with vertex set $V$ and edge set $E$. A set $I \subseteq V$ of vertices is an independent set of $G$, if no two vertices from $I$ are adjacent in $G$. The maximum cardinality of an independent set of $G$ is the independence number $\alpha(G)$ of $G$. A set $D \subseteq V$ of vertices is a dominating set of $G$, if every vertex in $V \setminus D$ has a neighbour in $D$. The minimum cardinality of a dominating set of $G$ is the domination number $\gamma(G)$ of $G$.

Minimum independent and maximum dominating sets are among the most fundamental and well-studied graph theoretic concepts [7]. As early as 1978 Bange, Barkauskas, and Slater [1] and Slater [10] characterized trees which have two disjoint minimum dominating sets and two disjoint maximum independent sets, respectively. In [2, 4, 6] the problem of finding two minimum dominating sets of minimum intersection is studied while in [8] trees with two disjoint minimum independent dominating sets are characterized. In [3, 5, 9] the minimum cardinality of a dominating set which lies in the complement of a minimum dominating set is studied.

Complementing this previous research we consider graphs $G = (V, E)$ that have a maximum independent set $I$ and a minimum dominating set $D$ which are disjoint. We call such a pair of sets $(I, D)$ an $(\alpha, \gamma)$-pair of $G$. Intuitively, two independent sets or two dominating sets compete for similar types of vertices while an independent set and a dominating set seem easier to reconcile. After proving that the decision problem whether a given graph has an $(\alpha, \gamma)$-pair is NP-complete, we give a constructive characterization of trees with an $(\alpha, \gamma)$-pair.

Theorem 1 The problem to decide whether an input graph has an $(\alpha, \gamma)$-pair is NP-complete.
Proof: For a 3SAT instance $C$ with $n$ variables and $m$ clauses, we will describe a graph $G = (V, E)$ of order polynomial in $n$ and $m$ such that $C$ is satisfiable if and only if $G$ has an $(\alpha, \gamma)$-pair.

For every boolean variable $x$ of $C$, the graph $G$ contains a copy $H_x$ of the gadget shown in Figure 1 with two specified vertices $x$ and $\overline{x}$.

For every clause $C$, the graph $G$ contains $3n + 1$ disjoint paths of length three $P^C_1, P^C_2, \ldots, P^C_{3n+1}$.

In each of these paths $P^C_i$ we specify one endvertex $x^C_i$. If $C$ contains the literal $y$, then $G$ contains the edges $yx^C_i$ for $1 \leq i \leq 3n + 1$. The graph $G$ contains no further vertices or edges.

Clearly, every independent set of $G$ contains at most three vertices from every of the gadgets $H_x$ and at most two vertices from every of the paths $P^C_i$, i.e. $\alpha(G) \leq 3n + 2m(3n + 1)$. Since choosing three independent vertices from every of the gadgets $H_x$ and the vertices at distance one and three from $x^C_i$ from every of the paths $P^C_i$ yields an independent set of order $3n + 2m(3n + 1)$, we have $\alpha(G) = 3n + 2m(3n + 1)$.

Clearly, every dominating set of $G$ contains at least two vertices from every of the gadgets $H_x$ and at least one vertex from every of the paths $P^C_i$. Hence $\gamma(G) \geq 2n + m(3n + 1)$. Furthermore, since choosing $x$, $\overline{x}$ and the neighbour of the endvertex from every of the gadgets $H_x$ and the vertex at distance two from $x^C_i$ from every of the paths $P^C_i$ yields a dominating set of order $3n + m(3n + 1)$, we have $\gamma(G) \leq 3n + m(3n + 1)$.

If $C$ has a satisfying truth assignment, then choosing three independent vertices containing the false literal among $x$ and $\overline{x}$ from every of the gadgets $H_x$ and the vertices at distance one and three from $x^C_i$ from every of the paths $P^C_i$ yields a maximum independent set $I$. Furthermore, choosing the true literal among $x$ and $\overline{x}$ and the neighbour of the endvertex from every of the gadgets $H_x$ and the vertex at distance two from $x^C_i$ from every of the paths $P^C_i$ yields an dominating set $D$ of order $2n + m(3n + 1)$. Hence $(I, D)$ is an $(\alpha, \gamma)$-pair of $G$.

Conversely, if $G$ has an $(\alpha, \gamma)$-pair $(I, D)$, then we may assume that $D$ contains exactly one of the two vertices $x$ and $\overline{x}$ from every of the gadgets $H_x$. If one of the vertices $x^C_i$ from some path $P^C_i$ is not dominated by a vertex from one of the gadgets $H_x$, then $D$ must contain at least two vertices from every of the $3n + 1$ paths $P^C_i$ and at least one vertex from every of the remaining paths. Hence $|D| \geq 3n + 1 + m(3n + 1)$ which is a contradiction. Therefore, all of the vertices $x^C_i$ from every of the paths $P^C_i$ are dominated by a vertex
from one of the gadgets $H_x$. Hence the literals contained in $D$ define a satisfying truth assignment for $C$ and the proof is complete. □

2 Trees with an $(\alpha, \gamma)$-pair

In this section we will describe a polynomial time procedure to decide whether a given tree has an $(\alpha, \gamma)$-pair. We describe suitable reductions and explain how these reductions yield a constructive characterization of trees with an $(\alpha, \gamma)$-pair.

The first lemma deals with some small trees.

Lemma 2 (i) For $2 \leq n \leq 6$ the path $P_n : u_1 u_2 \ldots u_n$ has the following $(\alpha, \gamma)$-pair $(I_n, D_n)$:

$$(I_2, D_2) = (\{u_1\}, \{u_2\})$$

$$(I_3, D_3) = (\{u_1, u_3\}, \{u_2\})$$

$$(I_4, D_4) = (\{u_1, u_4\}, \{u_2, u_3\})$$

$$(I_5, D_5) = (\{u_1, u_3, u_5\}, \{u_2, u_4\})$$

$$(I_6, D_6) = (\{u_1, u_3, u_6\}, \{u_2, u_5\}).$$

(ii) The tree $T^* = (V^*, E^*)$ with

$V^* = \{u_0, u_1, v_0, v_1, v_2, w_0, w_1, w_2, w_3, x\}$

$E^* = \{u_0 u_1, u_1 x, v_0 v_1, v_1 v_2, v_2 x, w_0 w_1, w_1 w_2, w_2 w_3, w_3 x\}$

has the $(\alpha, \gamma)$-pair

$$(\{u_0, v_0, w_0, v_2, w_2\}, \{u_1, v_1, w_1, x\}).$$

Proof: It is very easy to check that the given sets are maximum independent sets and minimum dominating sets which are disjoint. □

Figure 2 The trees $P_2, P_3, \ldots, P_6$ and $T^*$.  

3
Lemma 3 Let $T$ contain a path $P : u_0u_1 \ldots u_5$ such that $d_T(u_0) = 1$ and $d_T(u_1) = d_T(u_2) = d_T(u_3) = d_T(u_4) = 2$.

(i) $\alpha(T') + 2 \leq \alpha(T) \leq \alpha(T') + 3$ for $T' = T - \{u_0, u_1, \ldots, u_4\}$.

(ii) If $\alpha(T') = \alpha(T'') + 3$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T'' = T - \{u_0, u_1\}$ has an $(\alpha, \gamma)$-pair, $\alpha(T) = \alpha(T'') + 1$ and $\gamma(T) = \gamma(T'') + 1$.

(iii) If $\alpha(T) = \alpha(T'') + 2$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T''' = T - \{u_0, u_1, u_2\}$ has an $(\alpha, \gamma)$-pair, $\alpha(T) = \alpha(T''') + 1$ and $\gamma(T) = \gamma(T''') + 1$.

\[ \begin{array}{cccc}
T' & T' & T' & T' \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
T'' & T'' & T'' & T'' \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
T''' & T''' & T''' & T''' \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array} \]

$\alpha(T) = \alpha(T') + 3$

$\alpha(T) = \alpha(T') + 2$

\textbf{Figure 3} The trees $T$, $T'$, $T''$ and $T'''$. 

\textbf{Proof}: (i) The first inequality follows, since for every independent set $I'$ of $T'$ the set $I' \cup \{u_0, u_2\}$ is an independent set of $T$. The second inequality follows, since every independent set $I$ of $T$ contains at most three of the vertices in $\{u_0, u_1, \ldots, u_4\}$ and $I \setminus \{u_0, u_1, \ldots, u_4\}$ is an independent set of $T'$.

(ii) Let $\alpha(T) = \alpha(T') + 3$. Note that this implies that every maximum independent set of $T$ contains $u_0, u_2$ and $u_4$. Therefore, if $T$ has an $(\alpha, \gamma)$-pair $(I, D)$, then $u_0, u_2, u_4 \in I$ and hence $u_1, u_3 \in D$. Clearly, $\alpha(T'') \leq \alpha(T') + 2$. Since $I \setminus \{u_0\}$ is an independent set in $T''$, we have $\alpha(T'') \geq \alpha(T) - 1 = \alpha(T') + 2$ and thus $\alpha(T) = \alpha(T') + 3 = \alpha(T'') + 1$. Clearly, $\gamma(T) \leq \gamma(T'') + 1$. Since $D \setminus \{u_1\}$ is a dominating set in $T''$, we have $\gamma(T'') \leq \gamma(T) - 1$ and thus $\gamma(T) = \gamma(T'') + 1$. Now $(I \setminus \{u_0\}, D \setminus \{u_1\})$ is an $(\alpha, \gamma)$-pair of $T''$.

Conversely, if $T''$ has an $(\alpha, \gamma)$-pair $(I'', D'')$, $\alpha(T) = \alpha(T'') + 1$ and $\gamma(T) = \gamma(T'') + 1$, then $(I'' \cup \{u_0\}, D'' \cup \{u_1\})$ is an $(\alpha, \gamma)$-pair of $T$.

(iii) Let $\alpha(T) = \alpha(T') + 2$. If $T$ has an $(\alpha, \gamma)$-pair $(I, D)$, then we may assume without loss of generality that $u_0, u_3 \in I$ and $u_1, u_4 \in D$. Clearly, $\alpha(T'') \leq \alpha(T') + 1$. Since $I \setminus \{u_0\}$ is an independent set in $T''$, we have $\alpha(T'') \geq \alpha(T) - 1 = \alpha(T') + 1$ and thus $\alpha(T) = \alpha(T') + 2 = \alpha(T'') + 1$. Clearly, $\gamma(T) \leq \gamma(T'') + 1$. Since $D \setminus \{u_1\}$ is a dominating
set in $T''$, we have $\gamma(T'') \leq \gamma(T) - 1$ and thus $\gamma(T) = \gamma(T'') + 1$. Now $(I \setminus \{u_0\}, D \setminus \{u_1\})$ is an $(\alpha, \gamma)$-pair of $T''$.

Conversely, if $T''$ has an $(\alpha, \gamma)$-pair $(I'', D'')$, $\alpha(T) = \alpha(T'') + 1$ and $\gamma(T) = \gamma(T'') + 1$, then $(I'' \cup \{u_0\}, D'' \cup \{u_1\})$ is an $(\alpha, \gamma)$-pair of $T$. $\square$

Combining Lemma 2 (i) with Lemma 3 it is easy to check that the only paths $P_n$ with an $(\alpha, \gamma)$-pair satisfy $n \in \{2, 3, 4, 5, 6, 7, 8, 10\}$.

**Lemma 4** Let $T$ contain a path $P : u_0u_1 \ldots u_rwv_s\ldots v_0$ with $r, s \geq 0$ such that $d_T(u_0) = d_T(v_0) = 1$, $d_T(u_i) = 2$ for $1 \leq i \leq r$ and $d_T(v_j) = 2$ for $1 \leq j \leq s$.

![Figure 4](image.png)

\textbf{Figure 4} The path $P : u_0u_1 \ldots u_rwv_s\ldots v_0$.

(i) If $r = 2k$ and $s = 2l$ for some $0 \leq k, l \leq 1$ with $k \geq l$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_i \mid 0 \leq i \leq 2k\}$ has an $(\alpha, \gamma)$-pair, $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + k$.

(ii) If $r = 2k + 1$ and $s = 0$ for some $0 \leq k \leq 1$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_i \mid 0 \leq i \leq 2k + 1\}$ has an $(\alpha, \gamma)$-pair, $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + 1$.

(iii) If $r = s = 1$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_0, u_1\}$ has an $(\alpha, \gamma)$-pair.

(iv) If $r = s = 3$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_0, u_1, u_2, v_0, v_1, v_2\}$ has an $(\alpha, \gamma)$-pair and $\alpha(T) = \alpha(T') + 2$.

(v) If $r = 1$, $s = 2$ and $d_T(w) = 3$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - V(P)$ has an $(\alpha, \gamma)$-pair.

(vi) If $r = 1$, $s = 3$ and $d_T(w) = 3$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_0, u_1\}$ has an $(\alpha, \gamma)$-pair.

(vii) If $r = 2$, $s = 3$ and $d_T(w) = 3$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_0, u_1, v_0, v_1, v_2, v_3\}$ has an $(\alpha, \gamma)$-pair.

\textbf{Proof:} (i) Note that every maximum independent set $I$ of $T$ satisfies $I \cap V(P) = \{u_{2i} \mid 0 \leq i \leq k\} \cup \{v_{2j} \mid 0 \leq j \leq l\}$.

Therefore, if $T$ has an $(\alpha, \gamma)$-pair $(I, D)$, then $u_{2i} \in I$ for $0 \leq i \leq k$, $v_{2j} \in I$ for $0 \leq j \leq l$, $u_{2i+1} \in D$ for $0 \leq i \leq k - 1$ and $v_{2j+1} \in D$ for $0 \leq j \leq l - 1$. Clearly, $\alpha(T) \leq \alpha(T') + k + 1$. 

5
Since $I \setminus \{u_{2i} \mid 0 \leq i \leq k\}$ is an independent set in $T'$, we have $\alpha(T') \leq \alpha(T) - (k + 1)$ and thus $\alpha(T) = \alpha(T') + k + 1$. Clearly, $\gamma(T) \leq \gamma(T') + k$ — note that $k = 0$ implies $l = 0$ and $w \in D$. Since $D \setminus \{u_{2i+1} \mid 0 \leq i \leq k-1\}$ is a dominating set in $T'$, we have $\gamma(T') \leq \gamma(T) - k$ and thus $\gamma(T) = \gamma(T') + k$. Now $(I \setminus \{u_{2i} \mid 0 \leq i \leq k\}, D \setminus \{u_{2i+1} \mid 0 \leq i \leq k-1\})$ is an $(\alpha, \gamma)$-pair of $T$.

Conversely, if $T'$ has an $(\alpha, \gamma)$-pair $(I', D')$, $\alpha(T) = \alpha(T') + k + 1$ and $\gamma(T) = \gamma(T') + k$, then in view of $l \leq 1$ we may assume that $v_{2i} \in I'$. Hence $w \not\in I'$ and $(I' \cup \{u_{2i} \mid 0 \leq i \leq k\}, D' \cup \{u_{2i+1} \mid 0 \leq i \leq k-1\})$ is an $(\alpha, \gamma)$-pair of $T$.

(ii) If $T$ has an $(\alpha, \gamma)$-pair, then it has an $(\alpha, \gamma)$-pair $(I, D)$ such that $v_0 \in I$, $w \in D$, $|I \setminus \{u_i \mid 0 \leq i \leq 2k + 1\}| = k + 1$ and $|D \setminus \{u_i \mid 0 \leq i \leq 2k + 1\}| = 1$. Similarly, if $T'$ has an $(\alpha, \gamma)$-pair, then it has an $(\alpha, \gamma)$-pair $(I', D')$ such that $v_0 \in I$ and $w \in D$. This easily implies that $\alpha(T) = \alpha(T') + k + 1$, $\gamma(T) = \gamma(T') + 1$ and that $T$ has an $(\alpha, \gamma)$-pair if and only if $T'$ has an $(\alpha, \gamma)$-pair.

(iii) If $T$ has an $(\alpha, \gamma)$-pair, then it has an $(\alpha, \gamma)$-pair $(I, D)$ such that $v_0 \in I$ and $v_1 \in D$. Similarly, if $T'$ has an $(\alpha, \gamma)$-pair, then it has an $(\alpha, \gamma)$-pair $(I', D')$ such that $v_0 \in I$ and $v_1 \in D$. This easily implies that $\alpha(T) = \alpha(T') + 1$, $\gamma(T) = \gamma(T') + 1$ and that $T$ has an $(\alpha, \gamma)$-pair if and only if $T'$ has an $(\alpha, \gamma)$-pair.

(iv) Note that every minimum dominating set of $T$ contains $w, u_1$ and $v_1$. Similarly every minimum dominating set of $T'$ contains $w$. This easily implies that $\alpha(T) = \alpha(T') + 2$, $\gamma(T) = \gamma(T') + 2$ and that $T$ has an $(\alpha, \gamma)$-pair if and only if $T'$ has an $(\alpha, \gamma)$-pair.

(v) It is easy to see that $\alpha(T) = \alpha(T') + 3$ and $\gamma(T) = \gamma(T') + 2$. If $T$ has an $(\alpha, \gamma)$-pair, then it has an $(\alpha, \gamma)$-pair $(I, D)$ such that $u_0, v_0, v_2 \in I$ and $u_1, v_1 \in D$. This easily implies that $T$ has an $(\alpha, \gamma)$-pair if and only if $T'$ has an $(\alpha, \gamma)$-pair.

(vi) It is easy to see that $\alpha(T) = \alpha(T') + 1$. Similarly, since $T'$ has a minimum dominating set containing $w$, we have $\gamma(T) = \gamma(T') + 1$ which again implies the desired result.

(vii) Note that $T$ has a maximum independent set containing $u_2$ and a maximum dominating set containing $w$. This easily implies that $\alpha(T) = \alpha(T') + 3$ and $\gamma(T) = \gamma(T') + 2$ which again implies the desired result. □

**Lemma 5** Let $T$ contain three internally vertex disjoint paths $P: u_0u_1x$, $Q: v_0v_1v_2x$ and $R: w_0w_1w_2w_3x$ such that $d_T(u_0) = d_T(v_0) = d_T(w_0) = 1$, $d_T(u_1) = d_T(v_1) = d_T(v_2) = d_T(w_1) = d_T(w_2) = d_T(w_3) = 2$ and $d_T(x) = 4$, then $T$ has an $(\alpha, \gamma)$-pair if and only if $T' = T - \{u_0, u_1, v_0, v_1, w_0, w_1, w_2, w_3\}$ has an $(\alpha, \gamma)$-pair.

**Proof:** Note that $T$ has a maximum independent set $I$ such that $I \cap (V(P) \cup V(Q) \cup V(R)) = \{u_0, v_0, w_0, v_2, w_2\}$ and a minimum dominating set $D$ such that $D \cap (V(P) \cup V(Q) \cup V(R)) = \{u_1, v_1, w_1, x\}$. This easily implies that $\alpha(T) = \alpha(T') + 4$ and $\gamma(T) = \gamma(T') + 3$.

If $T$ has an $(\alpha, \gamma)$-pair, then $T$ has an $(\alpha, \gamma)$-pair $(I, D)$ such that $I \cap (V(P) \cup V(Q) \cup V(R)) = \{u_0, v_0, w_0, v_2, w_2\}$ and $D \cap (V(P) \cup V(Q) \cup V(R)) = \{u_1, v_1, w_1, x\}$. In this case $(I \setminus \{v_0, u_0, w_0, w_2\}, D \setminus \{u_1, v_1, w_1\})$ is an $(\alpha, \gamma)$-pair of $T'$. Conversely, if $T'$ has an $(\alpha, \gamma)$-pair, then $T'$ has an $(\alpha, \gamma)$-pair $(I', D')$ such that $v_2 \in I'$ and $x \in D'$. In this case $(I' \cup \{u_0, v_0, w_0, w_2\}, D' \cup \{u_1, v_1, w_1\})$ is an $(\alpha, \gamma)$-pair of $T$ which completes the proof. □
For integers $k \geq 1$ and $d_1 \geq d_2 \geq \ldots \geq d_k \geq 1$ a tree $T$ is said to have a $(d_1, d_2, \ldots, d_k)$-tinsel $(P_1, P_2, \ldots, P_k)$ pending on $v$ if $P_1, P_2, \ldots, P_k$ are $k$ internally vertex disjoint paths in $T$ such that

$$P_i : u_{i,0}u_{i,1} \ldots u_{i,d_i-1}v,$$

$d_T(u_{i,0}) = 1$ and $d_T(u_{i,j}) = 2$ for $1 \leq i \leq k$ and $1 \leq j \leq d_i - 1$ and $d_T(v) = k + 1$. For integers $\partial d_1, \partial d_2, \ldots, \partial d_k$ with $0 \leq \partial d_i \leq d_i$ for $1 \leq i \leq k$, the tree

$$T - \bigcup_{i=1}^{k} \bigcup_{j=0}^{\partial d_i-1} \{u_{i,j}\}$$

is said to arise from the tree $T$ by $(\partial d_1, \partial d_2, \ldots, \partial d_k)$-cutting the $(d_1, d_2, \ldots, d_k)$-tinsel $(P_1, P_2, \ldots, P_k)$. Note that a tree $T$ which is not a path and is rooted at an endvertex of a longest path has a tinsel $(P_1, P_2, \ldots, P_k)$ pending on some vertex $v$ such that $k \geq 2$ and all vertices of the paths $P_i$ are either $v$ or descendants of $v$.

The next result summarizes the reductions captured by Lemmas 3 through 5 and yields a constructive characterization of trees having an $(\alpha, \gamma)$-pair.

**Theorem 6** Let $T = (V, E)$ be a tree which is not a path and different from the tree $T^*$. Let $(P_1, P_2, \ldots, P_k)$ be a $(d_1, d_2, \ldots, d_k)$-tinsel pending on $v$ with $k \geq 2$.

The tree $T$ has an $(\alpha, \gamma)$-pair if and only if the tree $T'$ which arises from the tree $T$ by $(\partial d_1, \partial d_2, \ldots, \partial d_k)$-cutting the $(d_1, d_2, \ldots, d_k)$-tinsel $(P_1, P_2, \ldots, P_k)$ has an $(\alpha, \gamma)$-pair and $(\alpha(T) - \alpha(T'), \gamma(T) - \gamma(T')) = (\partial \alpha, \partial \gamma)$ where

(i) if $d_1 \geq 5$ and $\alpha(T) = \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) + 3$, then $(\partial d_1, \partial d_2, \ldots, \partial d_k) = (2, 0, \ldots, 0)$ and $(\partial \alpha, \partial \gamma) = (1, 1)$.

(ii) if $d_1 \geq 5$ and $\alpha(T) = \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) + 2$, then $(\partial d_1, \partial d_2, \ldots, \partial d_k) = (3, 0, \ldots, 0)$ and $(\partial \alpha, \partial \gamma) = (1, 1)$.

(iii) if there are two indices $1 \leq i < j \leq k$ such that $d_i, d_j \in \{1, 3\}$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial \alpha, \partial \gamma) = \left(\frac{d_i+1}{2}, \frac{d_j-1}{2}\right)$.

(iv) if $d_k = 1$ and there is an index $1 \leq i < k$ such that $d_i \in \{2, 4\}$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial \alpha, \partial \gamma) = \left(\frac{d_i}{2}, 1\right)$.

(v) if there are two indices $1 \leq i < j \leq k$ such that $d_i = d_j = 2$, then $\partial d_i = d_i$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \neq i$ and $(\partial \alpha, \partial \gamma) = (1, 1)$.

(vi) if there are two indices $1 \leq i < j \leq k$ such that $d_i = d_j = 4$, then $\partial d_i = \partial d_j = 3$, $\partial d_r = 0$ for $1 \leq r \leq k$ with $r \notin \{i, j\}$ and $\partial \alpha = 2$.

(vii) if $k = 2$ and $(d_1, d_2) = (3, 2)$, then $T' = T - (V(P_1) \cup V(P_2))$.

(viii) if $k = 2$ and $(d_1, d_2) = (4, 2)$, then $(\partial d_1, \partial d_2) = (0, 2)$.

(ix) if $k = 2$ and $(d_1, d_2) = (4, 3)$, then $(\partial d_1, \partial d_2) = (4, 2)$. 

7
(x) if $k = 3$ and $(d_1, d_2) = (4, 3, 2)$, then $(\partial d_1, \partial d_2, \partial d_3) = (4, 2, 2)$.

Furthermore, one of the cases (i)-(x) occurs.

Proof: If $d_1 \geq 5$, then, by Lemma 3 (i), $2 \leq \alpha(T) - \alpha(T - \{u_{1,0}, u_{1,1}, \ldots, u_{1,4}\}) \leq 3$. Now, by Lemma 3 (ii) and (iii), either (i) or (ii) occurs. Hence we may assume that $d_1 \leq 4$, i.e. all $d_i$ are at most 4. If there are two odd $d_i$’s, then, by Lemma 4 (i), the case (iii) occurs. Hence we may assume that at most one of the $d_i$ is odd. If $d_k = 1$, then, by Lemma 4 (ii), the case (iv) occurs. Hence we may assume that all $d_i$ are either 2, 3 or 4. If there are two $d_i$’s equal to 2, then, by Lemma 4 (iii), the case (v) occurs. Hence we may assume that at most one of the $d_i$ is 2. If there are two $d_i$’s equal to 4, then, by Lemma 4 (iv), the case (vi) occurs. Hence we may assume that at most one of the $d_i$ is 4. If $k \geq 3$, then $k = 3$, $(d_1, d_2, d_3) = (4, 3, 2)$ and, by Lemma 5, the case (x) occurs. Hence we may assume $k = 2$ and, by Lemma 4 (v) through (vii), one of the cases (vii) through (ix) occurs. This completes the proof. □

Corollary 7 It is possible to decide in polynomial time whether a given tree of order at least 2 has an $(\alpha, \gamma)$-pair.

Proof: If $T$ is a path of order at most 6 or the tree $T^*$, then, by Lemma 2, $T$ has an $(\alpha, \gamma)$-pair. If $T$ is a path of order at least 7, then Lemma 3 allows to reduce the decision problem to a smaller tree in polynomial time. If $T$ is not a path not the tree $T^*$, then Theorem 6 allows to reduce the decision problem to a smaller tree in polynomial time. □

References


