Spectral Points of Definite Type and Type π for Linear Operators and Relations in Krein Spaces

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Abstract

Spectral points of type π_+ and type π_- for closed linear operators and relations in Krein spaces are introduced with the help of approximative eigensequences. It turns out that these spectral points are stable under compact perturbations and perturbations small in the gap metric.

Key words: Krein space, approximative eigensequences, spectral points of positive and negative type, spectral points of type π_+ and type π_-, compact perturbation, gap metric, linear relation, definitizable operator

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1 Introduction

Let $(\mathcal{H}, [\cdot,\cdot])$ be a Krein space and let A be a bounded or unbounded linear operator in $\mathcal{H}$ which is selfadjoint with respect to the Krein space inner product $[\cdot,\cdot]$. The spectral properties of selfadjoint operators in Krein spaces differ essentially from the spectral properties of selfadjoint operators in Hilbert spaces, e.g., the spectrum $\sigma(A)$ of A is in general not real and even $\sigma(A) = \mathbb{C}$ may happen. If, besides selfadjointness, further assumptions on A are imposed the situation becomes more interesting from a spectral theoretic point of view.

Let, e.g., A be a $[\cdot,\cdot]$-nonnegative selfadjoint operator in $\mathcal{H}$ with a nonempty resolvent set. Then $\sigma(A) \subset \mathbb{R}$ holds and the spectral points of A in $(0,\infty)$ and $(-\infty,0)$ are of *positive type* and *negative type*, respectively, i.e., each point in $\sigma(A) \cap (0,\infty)$ $(\sigma(A) \cap (-\infty,0))$ belongs to the approximate point spectrum of A and for every normed approximative eigensequence (x_n) the accumulation points of the sequence $([x_n, x_n])$ are positive (resp. negative), cf. Definition 3.1. These spectral points were introduced and studied by P. Lancaster, H. Langer, A. Markus and V. Matsaev in [26,28] for arbitrary bounded selfadjoint operators.

Not surprisingly, spectral points of positive and negative type are in general not stable under finite rank and compact perturbations. But, if the nonnegative selfadjoint operator A from above is perturbed by a finite rank operator F such that the resulting operator $B = A + F$ is selfadjoint, then the hermitian form $[B\cdot,\cdot]$ is only nonnegative on the complement of a finite dimensional subspace. In contrast to nonnegative selfadjoint operators the spectral points in $(0,\infty)$ $((-\infty,0))$ are not all of positive type (resp. negative type) with respect to B. However, if (x_n) is an approximative eigensequence corresponding to $\lambda \in \sigma(B) \cap (0,\infty)$ $(\lambda \in \sigma(B) \cap (-\infty,0))$ and all x_n belong to a suitable linear manifold of finite codimension, then the accumulation points of the sequence $([x_n, x_n])$ are again positive (resp. negative). These spectral points are called of *type* π_+ and *type* π_-, respectively, and were introduced for arbitrary selfadjoint operators in Krein spaces in [4].

The concept of spectral points of positive/negative type and type π_+/π_- can be regarded as a localization of the spectral properties of (selfadjoint) operators in Hilbert and Pontryagin spaces, respectively. In particular, these types of spectral points appear in the analysis of definitizable and locally definitizable selfadjoint operators. Vice versa the notion of spectral points of positive/negative type offers a convenient way to define and describe locally definitizable operators, cf. [24,28].

Spectral points of positive/negative type and type π_+/π_- for selfadjoint operators and their behaviour under different types of selfadjoint perturbations play an important role in many situations, see, e.g., [17,18,21,22,29]. However, since the definitions and basic properties of these spectral points do not essentially depend on selfadjointness it seems artificial and inconvenient to restrict spectral theoretic investigations and perturbation problems to the selfadjoint case; we mention only [2,3,6,7,14,31] for problems involving normal or dissipative operators in Krein spaces.

It is the aim of this paper to introduce and study spectral points of positive/negative type and type π_+/π_- for closed linear operators in Krein spaces and to develop a comprehensive perturbation theory. For us it is natural to regard bounded and unbounded linear operators via their graphs as linear subspaces and therefore we find it easier and convenient to present our observations for the slightly more general case of linear relations.

The paper is organized as follows. After some preparations in Section 2 we introduce spectral points of positive/negative type and type π_+/π_- for closed linear operators and relations in Krein spaces in Section 3. We generalize various earlier results from [4,28] on spectral sets consisting of these points.

The main objective of this paper is the investigation of stability properties of spectral points of positive and negative type, and type π_+ and type π_- in the non-selfadjoint case under various kinds of perturbations in Section 4. Many of the perturbations results proved there are also new for the special case of bounded or unbounded selfadjoint operators in Krein spaces. Let us sketch the main results. In Theorem 4.1 it is shown that spectral points of type π_+ and type π_- of closed linear operators and relations are stable under compact perturbations. As a corollary we obtain a variant of [8, Theorem 2.4], [28, Theorem 5.1] and [4, Theorem 29], see also Theorem 5.4. Section 4.2 is devoted to perturbations which are small in the gap metric. We verify first that spectral points of positive and negative type are stable under sufficiently small perturbations and extend this result to spectral points of type π_+ and type π_-. The behaviour of spectral points of positive and negative type of fundamentally reducible closed linear operators and relations under perturbations small in gap is studied in Theorem 4.10. This can be viewed as a natural generalization of a result for bounded selfadjoint operators in [28, Theorem 4.1].

Finally, in Section 5 we consider the special case of selfadjoint operators and relations in Krein spaces. In Theorem 5.1 it is shown that a real spectral point of type π_+ (type π_-) of a selfadjoint relation A, which is not an interior point of $\sigma(A)$, has a deleted neighbourhood consisting only of spectral points of positive type (resp. negative type) or of regular points of A. This implies also local definitizability of A in a neighbourhood of a point of type π_+ or type π_-, cf. Definition 5.2 and Theorem 5.3. As a consequence A possesses a local spectral function on subsets of $\mathbb{R}$ consisting of spectral points of type π_+

or type π_- and regular points.

2 Preliminaries

Throughout this paper $(\mathcal{H}, [\cdot,\cdot])$ denotes a Krein space. In the following all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that the indefinite inner product $[\cdot,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent, see, e.g., [27, Proposition 1.2].

We study closed linear relations in $\mathcal{H}$, that is, linear subspaces of the Cartesian product $\mathcal{H} \times \mathcal{H}$. We emphasize that a subspace is always assumed to be a closed linear manifold. A closed linear relation A will usually be viewed as a multivalued mapping and the elements $\hat{x} \in A$ will be written as column vectors, $\hat{x} = \left(\begin{smallmatrix} x \\ x' \end{smallmatrix}\right)$. Linear operators are always identified with linear relations via their graphs. The linear space of bounded linear operators defined on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. For the usual definitions of the linear operations with relations, the inverse etc., we refer to [1,12], and to the monographs [11] and [19]. We denote the sum of linear manifolds and subspaces by $\widehat{+}$, if this sum is direct we shall mention it explicitely. Sometimes it is convenient for us to make use of the so-called transformer of a linear relation (see [13,32]): If $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ is a 2×2-matrix and A is a closed linear relation in $\mathcal{H}$ we define MA by

$$MA := \left\{ \begin{pmatrix} \alpha x + \beta y \\ \gamma x + \delta y \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in A \right\}. \tag{2.1}$$

Clearly, MA is a closed linear relation in $\mathcal{H}$. We assign to every regular matrix $M = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ the fractional linear mapping Φ_M of $\overline{\mathbb{C}}$ onto itself defined by

$$\Phi_M(\lambda) := \frac{\delta\,\lambda + \gamma}{\beta\,\lambda + \alpha}, \quad \Phi_M\left(-\frac{\alpha}{\beta}\right) := \infty, \quad \Phi_M(\infty) := \frac{\delta}{\beta}. \tag{2.2}$$

Observe that $\Phi_{M_1 M_2} = \Phi_{M_1} \circ \Phi_{M_2}$ holds for regular 2×2-matrices M_1 and M_2.

Let A be a closed linear relation in $\mathcal{H}$. The *resolvent set* $\rho(A)$ of A is the set of all $\lambda \in \mathbb{C}$ such that $(A - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$, the *spectrum* $\sigma(A)$ of A is the complement of $\rho(A)$ in $\mathbb{C}$. The *extended spectrum* $\widetilde{\sigma}(A)$ of A is defined by $\widetilde{\sigma}(A) = \sigma(A)$ if $A \in \mathcal{L}(\mathcal{H})$ and $\widetilde{\sigma}(A) = \sigma(A) \cup \{\infty\}$ otherwise. The *extended resolvent set* $\widetilde{\rho}(A)$ of A is defined by $\overline{\mathbb{C}} \setminus \widetilde{\sigma}(A)$. A point $\lambda \in \mathbb{C}$ is an *eigenvalue* of A if $\ker(A - \lambda) \neq \{0\}$; we write $\lambda \in \sigma_p(A)$.

We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum* of a closed linear relation A, denoted by $\sigma_{ap}(A)$, if there exists a sequence $\left(\begin{smallmatrix} x_n \\ \tilde{x}_n \end{smallmatrix}\right) \in A$,

$n = 1, 2, \dots$, such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{x}_n - \lambda x_n\| = 0.$$

The *extended approximate point spectrum* $\widetilde{\sigma}_{ap}(A)$ *of* A is defined by

$$\widetilde{\sigma}_{ap}(A) := \begin{cases} \sigma_{ap}(A) \cup \{\infty\} & \text{if} \quad 0 \in \sigma_{ap}(A^{-1}), \\ \sigma_{ap}(A) & \text{if} \quad 0 \notin \sigma_{ap}(A^{-1}). \end{cases}$$

We set

$$r(A) := \mathbb{C} \backslash \sigma_{ap}(A) \quad \text{and} \quad \tilde{r}(A) := \overline{\mathbb{C}} \backslash \widetilde{\sigma}_{ap}(A).$$

A point $\mu \in r(A)$ is called of *regular type* of A.

A proof of the following useful lemma can be found in [13,24].

Lemma 2.1 *Let A be a closed linear relation in $\mathcal{H}$, let M be a regular 2×2-matrix, and define MA and Φ_M as in (2.1) and (2.2), respectively. Then we have*

$$\widetilde{\sigma}_{ap}(MA) = \Phi_M(\widetilde{\sigma}_{ap}(A)), \quad \widetilde{r}(MA) = \Phi_M(\widetilde{r}(A)), \quad \widetilde{\rho}(MA) = \Phi_M(\widetilde{\rho}(A)).$$

Since A is closed it follows that for every $\mu \in r(A)$ the range of $A - \mu$ is closed and $\ker(A - \mu) = \{0\}$ holds, i.e., $(A - \mu)^{-1}$ is a bounded (in general not everywhere defined) operator. Similarly, if $\infty \in \widetilde{r}(A)$, then A is a bounded (in general not everywhere defined) operator.

Lemma 2.2 *Let A be a closed linear relation in $\mathcal{H}$. Then the following holds.*

(i) *The boundary points of $\widetilde{\sigma}(A)$ in $\overline{\mathbb{C}}$ belong to $\widetilde{\sigma}_{ap}(A)$.*

(ii) *For every $\lambda_0 \in \widetilde{r}(A)$ there exist an open neighbourhood $\mathfrak{U}_{\lambda_0}$ in $\overline{\mathbb{C}}$ of λ_0 and $k_{\lambda_0} > 0$ such that for all $\lambda \in \mathfrak{U}_{\lambda_0} \setminus \{\infty\}$ and all $\binom{x}{\tilde{x}} \in A$*

$$\|\tilde{x} - \lambda x\| \geq k_{\lambda_0} \|x\| \tag{2.3}$$

holds. In particular, the sets $r(A)$ and $\widetilde{r}(A)$ are open in $\mathbb{C}$ and $\overline{\mathbb{C}}$, respectively.

Proof. (i) The statement is well-known for bounded and closed linear operators, see, e.g., [15, §IV 1.10]. The general case will be reduced to this as follows. If λ_0 is a boundary point of $\widetilde{\sigma}(A)$, then $\rho(A)$ is non-empty. Choose $\mu \in \rho(A)$ and $M = \begin{pmatrix} -\mu & 1 \\ 1 & 0 \end{pmatrix}$. Then by (2.1) we have $MA = (A - \mu)^{-1}$ and (2.2) becomes $\Phi_M(\lambda) = (\lambda - \mu)^{-1}$ for $\lambda \in \mathbb{C}$, $\Phi_M(\infty) = 0$ and $\Phi_M(\mu) = \infty$. According to Lemma 2.1 we have

$$\Phi_M(\widetilde{\rho}(A)) = \widetilde{\rho}((A - \mu)^{-1}) \quad \text{and} \quad \Phi_M(\widetilde{\sigma}_{ap}(A)) = \widetilde{\sigma}_{ap}((A - \mu)^{-1}).$$

Therefore $\Phi_M(\lambda_0)$ is a boundary point of $\widetilde{\sigma}((A-\mu)^{-1})$ and, as $(A-\mu)^{-1}$ is a bounded operator, $\Phi_M(\lambda_0) \in \widetilde{\sigma}_{ap}((A-\mu)^{-1})$. Hence Lemma 2.1 implies $\lambda_0 \in \widetilde{\sigma}_{ap}(A)$.

(ii) For $\lambda = \lambda_0 \in r(A)$ it follows from the definition of the set $r(A)$ that (2.3) holds and from this it is easy to see that for all λ in an open neighbourhood $\mathfrak{U}_{\lambda_0}$ of λ_0 (2.3) is still true. A similar argument applies for $\infty \in \widetilde{r}(A)$. □

3 Spectral points of definite type and type π for closed linear relations

We first recall the notion of spectral points of positive and negative type of closed linear operators and relations in Krein spaces from [24]. For bounded selfadjoint operators this definition can already be found in [28]. Equivalent descriptions of the spectral points of positive and negative type in the selfadjoint case were obtained in [24, Theorem 3.18]. In the following $\mathcal{H}$ is always assumed to be a Krein space with an indefinite inner product denoted by $[\cdot,\cdot]$.

Definition 3.1 *Let A be a closed linear relation in $\mathcal{H}$. A point $\lambda \in \sigma_{ap}(A)$ is said to be of* positive type *(*negative type*) with respect to A, if for every sequence $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2 \ldots,$ with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|\tilde{x}_n - \lambda x_n\| = 0$ we have*

$$\liminf_{n\to\infty}\, [x_n, x_n] > 0 \quad (\textit{resp. } \limsup_{n\to\infty}\, [x_n, x_n] < 0).$$

If $\infty \in \widetilde{\sigma}_{ap}(A)$, then ∞ is said to be of positive type *(*negative type*) with respect to A, if for every sequence $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2 \ldots,$ with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|\tilde{x}_n\| = 1$ we have*

$$\liminf_{n\to\infty}\, [\tilde{x}_n, \tilde{x}_n] > 0 \quad (\textit{resp. } \limsup_{n\to\infty}\, [\tilde{x}_n, \tilde{x}_n] < 0).$$

The set of all points of $\widetilde{\sigma}(A)$ of positive type (negative type) with respect to A will be denoted by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$). A point from $\sigma_{++}(A) \cup \sigma_{--}(A)$ is said to be of definite type*.*

The spectral points of positive and negative type of a closed linear relation A transform in the same way as the points in $\widetilde{\sigma}_{ap}(A)$, $\widetilde{r}(A)$ and $\widetilde{\rho}(A)$, cf. Lemma 2.1. More precisely, if M is a regular 2×2-matrix, MA and Φ_M are as in (2.1) and (2.2), respectively, then we have

$$\sigma_{++}(MA) = \Phi_M(\sigma_{++}(A)) \quad \text{and} \quad \sigma_{--}(MA) = \Phi_M(\sigma_{--}(A)), \tag{3.1}$$

cf. [24, Lemma 2.4].

The following statement is a straightforward generalization of [4, Lemma 2], see also [28, §1]. The proof is a simple modification of the proof of [4, Lemma 2] and is left to the reader.

Lemma 3.2 *Let A be a closed linear relation in $\mathcal{H}$ and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{++}(A) \cup \widetilde{r}(A)$ ($\mathfrak{F} \subset \sigma_{--}(A) \cup \widetilde{r}(A)$). Then there exists an open neighbourhood $\mathfrak{U}$ in $\overline{\mathbb{C}}$ of $\mathfrak{F}$ such that the following holds.*

(i) *There exist numbers $\varepsilon > 0$, $\delta > 0$ such that*

$$\lambda \in \mathfrak{U}\backslash\{\infty\}, \qquad \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \in A, \quad \|x\| = 1, \quad \textit{and} \quad \|\tilde{x} - \lambda x\| \le \varepsilon$$

implies

$$[x, x] \ge \delta \qquad (\textit{resp. } [x, x] \le -\delta).$$

(ii) $\mathfrak{U} \subset \sigma_{++}(A) \cup \widetilde{r}(A)$ (*resp.* $\mathfrak{U} \subset \sigma_{--}(A) \cup \widetilde{r}(A)$)

Let A be a closed linear relation in $\mathcal{H}$ and let $S \subset A$ be a linear manifold. If, for a finite dimensional subspace F, we have $A = S \widehat{+} F$, where $\widehat{+}$ denotes the sum of linear manifolds, then we will write

$$\text{codim}_A S < \infty.$$

In the next definition we generalize the notion of spectral points of positive and negative type.

Definition 3.3 *Let A be a closed linear relation in $\mathcal{H}$. A point $\lambda_0 \in \sigma_{ap}(A)$ is said to be of* type π_+ *(*type π_-*) with respect to A, if there exists a linear relation $S \subset A$ with* $\text{codim}_A S < \infty$ *such that for every sequence $\binom{x_n}{\tilde{x}_n} \in S$, $n = 1, 2 \ldots$, with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|\tilde{x}_n - \lambda_0 x_n\| = 0$ we have*

$$\liminf_{n\to\infty} \, [x_n, x_n] > 0 \quad (\textit{resp. } \limsup_{n\to\infty} \, [x_n, x_n] < 0). \tag{3.2}$$

If $\infty \in \widetilde{\sigma}_{ap}(A)$, then ∞ is said to be of type π_+ *(*type π_-*) with respect to A, if there exists a linear relation $S \subset A$ with* $\text{codim}_A S < \infty$ *such that for every sequence $\binom{x_n}{\tilde{x}_n} \in S$, $n = 1, 2 \ldots$, with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|\tilde{x}_n\| = 1$ we have*

$$\liminf_{n\to\infty} \, [\tilde{x}_n, \tilde{x}_n] > 0 \quad (\textit{resp. } \limsup_{n\to\infty} \, [\tilde{x}_n, \tilde{x}_n] < 0). \tag{3.3}$$

The set of all points in $\widetilde{\sigma}(A)$ of type π_+ (type π_-) with respect to A will be denoted by $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$). A point from $\sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)$ is said to be *of* type π.

We remark that for selfadjoint operators the spectral points of type π_+ and type π_- were introduced in a slightly different way in [4]. However, in the selfadjoint case is not difficult to check that the definition in [4] coincides with the definition above. Note also that in Definition 3.3 it is not assumed that the linear relation S is closed. As it is more convenient in some situations to work with a closed linear relation S we formulate the following proposition.

Proposition 3.4 *Let A be a closed linear relation in $\mathcal{H}$ and suppose that $\lambda_0 \in \widetilde{\sigma}_{ap}(A)$ belongs to $\sigma_{\pi_+}(A)$ $(\sigma_{\pi_-}(A))$. Then there exists a closed linear relation $S \subset A$ with $\operatorname{codim}_A S < \infty$ such that every sequence $\binom{x_n}{\tilde{x}_n} \in S$, $n = 1, 2 \ldots$, satisfies* (3.2) *if $\lambda_0 \neq \infty$ and* (3.3) *if $\lambda_0 = \infty$, respectively.*

Proof. Let S be as in Definition 3.3. Then the closure $\overline{S}$ of S is a closed linear relation with finite codimension in A. For every sequence $\binom{x_n}{\tilde{x}_n} \in \overline{S}$, $n = 1, 2 \ldots$, there exists a sequence $\binom{y_n}{\tilde{y}_n} \in S$, $n = 1, 2 \ldots$, with

$$\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} (\tilde{x}_n - \tilde{y}_n) = 0.$$

Now it is easily seen that every sequence $\binom{x_n}{\tilde{x}_n} \in \overline{S}$, $n = 1, 2 \ldots$, with $\|x_n\| = 1$ and $\lim_{n\to\infty} \|\tilde{x}_n - \lambda_0 x_n\| = 0$ has the property

$$\liminf_{n\to\infty} \, [x_n, x_n] = \liminf_{n\to\infty} \, [y_n, y_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} \, [x_n, x_n] < 0)$$

or, if $\lambda_0 = \infty$, then every sequence with $\|\tilde{x}_n\| = 1$ and $\lim_{n\to\infty} \|x_n\| = 0$ has the property

$$\liminf_{n\to\infty} \, [\tilde{x}_n, \tilde{x}_n] = \liminf_{n\to\infty} \, [\tilde{y}_n, \tilde{y}_n] > 0 \quad (\text{resp. } \limsup_{n\to\infty} \, [\tilde{x}_n, \tilde{x}_n] < 0).$$

□

Definition 3.3 and Proposition 3.4 imply the following corollary.

Corollary 3.5 *Let A be a closed linear relation in $\mathcal{H}$. A point $\lambda_0 \in \widetilde{\sigma}_{ap}(A)$ belongs to $\sigma_{\pi_+}(A)$ $(\sigma_{\pi_-}(A))$ if and only if there exists a closed linear relation $S \subset A$ with $\operatorname{codim}_A S < \infty$ such that*

$$\lambda_0 \in \sigma_{++}(S) \cup \widetilde{r}(S) \qquad (\textit{resp. } \lambda_0 \in \sigma_{--}(S) \cup \widetilde{r}(S)).$$

Remark 3.6 *Definition* 3.3 *can be viewed as a localization of the spectral properties of closed linear relations in Pontryagin spaces. Indeed, if $\mathcal{H}$ is a Pontryagin space with finite rank of negativity and if $\mathcal{H} = \mathcal{H}_+ [\widehat{+}] \mathcal{H}_-$, direct sum, is a fundamental decomposition of $\mathcal{H}$, $\dim \mathcal{H}_- < \infty$, and A is a closed linear relation in $\mathcal{H}$, then with $S := A \cap (\mathcal{H}_+)^2$ it follows $\widetilde{\sigma}_{ap}(A) = \sigma_{\pi_+}(A)$.*

Next we verify that spectral points of type π_+ and type π_- transform in the same way as spectral points of positive and negative type in (3.1).

Lemma 3.7 *Let A be a closed linear relation in $\mathcal{H}$, let M be a regular 2×2-matrix, and define MA and Φ_M as in* (2.1) *and* (2.2)*, respectively. Then we have*

$$\sigma_{\pi_+}(MA) = \Phi_M(\sigma_{\pi_+}(A)) \quad \textit{and} \quad \sigma_{\pi_-}(MA) = \Phi_M(\sigma_{\pi_-}(A)).$$

Proof. We verify $\sigma_{\pi_+}(MA) = \Phi_M(\sigma_{\pi_+}(A))$. The proof of the equality $\sigma_{\pi_-}(MA) = \Phi_M(\sigma_{\pi_-}(A))$ is completely analogous. Let us first check the inclusion

$$\Phi_M(\sigma_{\pi_+}(A)) \subset \sigma_{\pi_+}(MA). \tag{3.4}$$

For this, let $\lambda_0 \in \sigma_{\pi_+}(A)$. Then, by Corollary 3.5, there exists a closed linear relation $S \subset A$ with $\operatorname{codim}_A S < \infty$ such that $\lambda_0 \in \sigma_{++}(S) \cup \widetilde{r}(S)$. Let $F \subset A$ be a finite dimensional subspace of A with $A = S \widehat{+} F$, direct sum. This gives

$$MA = MS \,\widehat{+}\, MF, \qquad \text{direct sum},$$

and $\dim MF = \dim F$. By Lemma 2.1 and (3.1) the point $\Phi_M(\lambda_0)$ belongs to $\sigma_{++}(MS) \cup \widetilde{r}(MS)$, hence, by Corollary 3.5, $\Phi_M(\lambda_0) \in \sigma_{\pi_+}(MA)$. Thus we have (3.4) and therefore also $\Phi_{M^{-1}}(\sigma_{\pi_+}(MA)) \subset \sigma_{\pi_+}(A)$. This implies

$$\sigma_{\pi_+}(MA) = (\Phi_M \circ \Phi_{M^{-1}})(\sigma_{\pi_+}(MA)) \subset \Phi_M(\sigma_{\pi_+}(A)),$$

so that $\sigma_{\pi_+}(MA) = \Phi_M(\sigma_{\pi_+}(A))$ holds. □

The next result parallels Lemma 3.2.

Proposition 3.8 *Let A be a closed linear relation in $\mathcal{H}$ and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$ ($\mathfrak{F} \subset \sigma_{\pi_-}(A) \cup \widetilde{r}(A)$). Then there exists an open neighbourhood $\mathfrak{U}$ in $\overline{\mathbb{C}}$ of $\mathfrak{F}$ such that the following holds.*

(i) *There exists a closed linear relation $S \subset A$ with $\operatorname{codim}_A S < \infty$ and numbers $\varepsilon > 0$, $\delta > 0$ such that*

$$\lambda \in \mathfrak{U}\backslash\{\infty\}, \quad \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \in S, \quad \|x\| = 1, \quad \textit{and} \quad \|\tilde{x} - \lambda x\| \le \varepsilon$$

imply

$$[x, x] \ge \delta \qquad (\textit{resp.}\ [x,x] \le -\delta).$$

(ii) $\mathfrak{U} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$ (*resp.* $\mathfrak{U} \subset \sigma_{\pi_-}(A) \cup \widetilde{r}(A)$)

Proof. We prove the statements for $\mathfrak{F} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$. If $\mathfrak{F} \subset \sigma_{\pi_-}(A) \cup \widetilde{r}(A)$ the proof is completely analogous.

Either $\lambda_0 \in \mathfrak{F}$ belongs to $\widetilde{r}(A)$, then there exist an open neighbourhood $\mathfrak{U}_{\lambda_0}$ in $\overline{\mathbb{C}}$ of λ_0 and $k_{\lambda_0} > 0$ such that for all $\lambda \in \mathfrak{U}_{\lambda_0} \setminus \{\infty\}$ and all $\binom{x}{\tilde{x}} \in A$

$$\|\tilde{x} - \lambda x\| \geq k_{\lambda_0}\|x\|$$

holds, cf. Lemma 2.2. Or $\lambda_0 \in \mathfrak{F}$ belongs to $\sigma_{\pi_+}(A)$, then we choose a closed linear relation $S_{\lambda_0} \subset A$ with $\operatorname{codim}_A S_{\lambda_0} < \infty$ such that λ_0 belongs to $\sigma_{++}(S_{\lambda_0})$ or to $\widetilde{r}(S_{\lambda_0})$, see Corollary 3.5. In the latter case there exist an open neighbourhood $\mathfrak{U}_{\lambda_0}$ in $\overline{\mathbb{C}}$ of λ_0 and $k_{\lambda_0} > 0$ such that for all $\lambda \in \mathfrak{U}_{\lambda_0} \setminus \{\infty\}$ and all $\binom{x}{\tilde{x}} \in S_{\lambda_0}$

$$\|\tilde{x} - \lambda x\| \geq k_{\lambda_0}\|x\|$$

holds. If λ_0 belongs to $\sigma_{++}(S_{\lambda_0})$, then, by Theorem 3.2, there exist an open neighbourhood $\mathfrak{U}_{\lambda_0}$ of λ_0 in $\overline{\mathbb{C}}$ and numbers $\varepsilon_{\lambda_0}, \delta_{\lambda_0} > 0$ such that

$$\lambda \in \mathfrak{U}_{\lambda_0}\backslash\{\infty\}, \quad \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \in S_{\lambda_0}, \quad \|x\| = 1, \quad \|\tilde{x} - \lambda x\| \leq \varepsilon_{\lambda_0}$$

implies

$$[x, x] \geq \delta_{\lambda_0}.$$

Therefore, for each $\lambda \in \mathfrak{F}$ there exist an open neighbourhood $\mathfrak{U}_\lambda$ in $\overline{\mathbb{C}}$ of λ, a closed linear relation $S_\lambda \subset A$ with $\operatorname{codim}_A S_\lambda < \infty$ and numbers $\varepsilon_\lambda, \delta_\lambda > 0$ such that $\lambda' \in \mathfrak{U}_\lambda\backslash\{\infty\}$, $\binom{x}{\tilde{x}} \in S_\lambda$, $\|x\| = 1$ and $\|\tilde{x} - \lambda' x\| \leq \varepsilon_\lambda$ implies

$$[x, x] \geq \delta_\lambda.$$

Since $\mathfrak{F}$ is a compact set, there exist finitely many points $\lambda_1, \ldots, \lambda_k \in \mathfrak{F}$ such that $\mathfrak{F} \subset \bigcup_{j=1}^{k} \mathfrak{U}_{\lambda_j}$. With

$$\varepsilon := \min_{j=1,\ldots,k} \varepsilon_{\lambda_j}, \quad \delta := \min_{j=1,\ldots,k} \delta_{\lambda_j} \quad \text{and} \quad S := \bigcap_{j=1,\ldots,k} S_{\lambda_j}$$

statement (i) in Proposition 3.8 is valid. Assertion (ii) is a direct consequence of (i) and the definition of spectral points of type π_+. □

In the next theorem we find a useful criterion for spectral points not belonging to $\sigma_{\pi_+}(A)$ or $\sigma_{\pi_-}(A)$. For selfadjoint operators Theorem 3.9 reduces to [4, Theorem 13].

Theorem 3.9 *Let A be a closed linear relation in $\mathcal{H}$ and let $\lambda_0 \in \widetilde{\sigma}_{ap}(A)$.*

(i) *If $\lambda_0 \neq \infty$, then $\lambda_0 \notin \sigma_{\pi_+}(A)$ ($\lambda_0 \notin \sigma_{\pi_-}(A)$) if and only if there exists a sequence $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2 \ldots$, with $\|x_n\| = 1$, $\lim_{n\to\infty} \|\tilde{x}_n - \lambda_0 x_n\| = 0$*

such that (x_n) *converges weakly to zero and*

$$\liminf_{n\to\infty}\,[x_n,x_n]\le 0 \quad (\textit{resp.}\ \limsup_{n\to\infty}\,[x_n,x_n]\ge 0).$$

(ii) *If* $\lambda_0=\infty$, *then* $\lambda_0\notin\sigma_{\pi_+}(A)$ $(\lambda_0\notin\sigma_{\pi_-}(A))$ *if and only if there exists a sequence* $\binom{x_n}{\tilde{x}_n}\in A$, $n=1,2\ldots$, *with* $\|\tilde{x}_n\|=1$, $\lim_{n\to\infty}\|x_n\|=0$ *such that* $(\tilde{x}_n)$ *converges weakly to zero and*

$$\liminf_{n\to\infty}\,[\tilde{x}_n,\tilde{x}_n]\le 0 \quad (\textit{resp.}\ \limsup_{n\to\infty}\,[\tilde{x}_n,\tilde{x}_n]\ge 0).$$

Proof. We prove the assertions only for $\lambda_0\in\sigma_{\pi_+}(A)$. A similar reasoning applies for $\lambda_0\in\sigma_{\pi_-}(A)$.

We will prove assertion (i) first, therefore we assume $\lambda_0\ne\infty$ and $\lambda_0\notin\sigma_{\pi_+}(A)$. Then for any closed linear relation $S\subset A$ with $\operatorname{codim}_A S<\infty$ there exists a sequence $\binom{x_n}{\tilde{x}_n}\in S$ such that $\|x_n\|=1$, $\|\tilde{x}_n-\lambda_0x_n\|\to 0$ as $n\to\infty$ and $\liminf_{n\to\infty}[x_n,x_n]\le 0$. Let us choose

$$\begin{pmatrix}x_1\\ \tilde{x}_1\end{pmatrix}\in A,\quad \|x_1\|=1\quad\text{with}\quad \|\tilde{x}_1-\lambda_0x_1\|\le 1\quad\text{and}\quad [x_1,x_1]\le 1,$$

and denote by $\perp_A$ the orthogonal complement in A with respect to the usual Hilbert scalar product in $\mathcal{H}\oplus\mathcal{H}$. Then there exists an element

$$\begin{pmatrix}x_2\\ \tilde{x}_2\end{pmatrix}\in\left\{\begin{pmatrix}x_1\\ \tilde{x}_1\end{pmatrix}\right\}^{\perp_A}$$

such that $\|x_2\|=1$, $\|\tilde{x}_2-\lambda_0x_2\|\le\frac12$ and $[x_2,x_2]\le\frac12$. Next we choose

$$\begin{pmatrix}x_3\\ \tilde{x}_3\end{pmatrix}\in\left\{\begin{pmatrix}x_1\\ \tilde{x}_1\end{pmatrix},\begin{pmatrix}x_2\\ \tilde{x}_2\end{pmatrix}\right\}^{\perp_A},\ \|x_3\|=1,\ \|\tilde{x}_3-\lambda_0x_3\|\le\tfrac13\ \text{and}\ [x_3,x_3]\le\tfrac13.$$

Repeating this consideration we find a sequence $\binom{x_n}{\tilde{x}_n}\in A$ with $\|x_n\|=1$, $\|\tilde{x}_n-\lambda_0x_n\|\to 0$ as $n\to\infty$, $\liminf_{n\to\infty}[x_n,x_n]\le 0$ and

$$\begin{pmatrix}x_n\\ \tilde{x}_n\end{pmatrix}\perp_A\begin{pmatrix}x_m\\ \tilde{x}_m\end{pmatrix},\qquad n\ne m.$$

Therefore the sequences $\binom{x_n}{\tilde{x}_n}$ and (x_n) converge weakly to zero.

For the converse, let $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2 \ldots$, be a sequence with $\|x_n\| = 1$, $\lim_{n\to\infty} \|\tilde{x}_n - \lambda_0 x_n\| = 0$ such that (x_n) converges weakly to zero and

$$\liminf_{n\to\infty} \,[x_n, x_n] \leq 0. \tag{3.5}$$

Let $S \subset A$ be a closed linear relation with $\operatorname{codim}_A S < \infty$ and let F be some finite dimensional subspace $F \subset A$ such that $A = S \hat{+} F$, direct sum. We write

$$\begin{pmatrix} x_n \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} y_n \\ \tilde{y}_n \end{pmatrix} + \begin{pmatrix} z_n \\ \tilde{z}_n \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} y_n \\ \tilde{y}_n \end{pmatrix} \in S \;\text{ and }\; \begin{pmatrix} z_n \\ \tilde{z}_n \end{pmatrix} \in F.$$

As $\binom{x_n}{\tilde{x}_n - \lambda_0 x_n} \in A - \lambda_0$ converges weakly to zero $\binom{z_n}{\tilde{z}_n - \lambda_0 z_n} \in F - \lambda_0$ satisfies

$$\lim_{n\to\infty} \left\| \begin{pmatrix} z_n \\ \tilde{z}_n - \lambda_0 z_n \end{pmatrix} \right\| = 0.$$

Hence $\lim_{n\to\infty} \|y_n\| = 1$, $\lim_{n\to\infty} \|\tilde{y}_n - \lambda_0 y_n\| = 0$ and from (3.5) we obtain

$$\liminf_{n\to\infty} \,[y_n, y_n] \leq 0. \tag{3.6}$$

We have shown that for every closed linear relation $S \subset A$ there exists a sequence $\binom{y_n}{\tilde{y}_n} \in S$ with $\lim_{n\to\infty} \|y_n\| = 1$, $\lim_{n\to\infty} \|\tilde{y}_n - \lambda_0 y_n\| = 0$ and (3.6), i.e., $\lambda_0 \notin \sigma_{\pi_+}(A)$, cf. Definition 3.3 and Proposition 3.4.

Let $\lambda_0 = \infty$ and assume $\infty \in \widetilde{\sigma}_{ap}(A)$. Then $0 \in \sigma_{ap}(A^{-1})$. We have, by assertion (i), that $0 \notin \sigma_{\pi_+}(A^{-1})$ if and only if there is a sequence $\binom{\tilde{x}_n}{x_n} \in A^{-1}$, $n = 1, 2 \ldots$, with $\|\tilde{x}_n\| = 1$, $\lim_{n\to\infty} \|x_n\| = 0$ such that $(\tilde{x}_n)$ converges weakly to zero and

$$\liminf_{n\to\infty} \,[\tilde{x}_n, \tilde{x}_n] \leq 0.$$

As $\binom{x_n}{\tilde{x}_n} \in A$, $n = 1, 2 \ldots$, assertion (ii) is true. □

With the help of Theorem 3.9 we describe the spectral points belonging to $\sigma_{\pi_+}(A) \backslash \sigma_{++}(A)$ and $\sigma_{\pi_-}(A) \backslash \sigma_{--}(A)$ in the next theorem.

Theorem 3.10 *Let A be a closed linear relation in $\mathcal{H}$ and let $\lambda_0 \in \mathbb{C}$. If $\lambda_0 \in \sigma_{\pi_+}(A) \backslash \sigma_{++}(A)$ ($\lambda_0 \in \sigma_{\pi_-}(A) \backslash \sigma_{--}(A)$), then λ_0 is an eigenvalue of A with a corresponding nonpositive* (*resp. nonnegative*) *eigenvector. If $\infty \in \sigma_{\pi_+}(A) \backslash \sigma_{++}(A)$ ($\infty \in \sigma_{\pi_-}(A) \backslash \sigma_{--}(A)$), then the multivalued part of A contains a nonpositive* (*resp. nonnegative*) *vector.*

Proof. Let $\lambda_0 \in \sigma_{\pi_+}(A) \backslash \sigma_{++}(A)$ and assume that $\lambda_0 \neq \infty$. Then there exists a sequence $\binom{x_n}{\tilde{x}_n} \in A$, $\|x_n\| = 1$, $\|\tilde{x}_n - \lambda_0 x_n\| \to 0$ and $\liminf_{n\to\infty} [x_n, x_n] \leq 0$.

Moreover, there exists a subsequence (x_{n_k}) of (x_n) which converges weakly to some x_0 and by Theorem 3.9 we have $x_0 \neq 0$. As A is assumed to be closed A and $A - \lambda_0$ are also closed in the weak topology, see, e.g. [34, §V.1 Theorem 10]. Therefore we have $\begin{pmatrix} x_0 \\ 0 \end{pmatrix} \in A - \lambda_0$ and it remains to show

$$[x_0, x_0] \leq 0. \tag{3.7}$$

Let $y_k := x_{n_k} - x_0$ and $\tilde{y}_k := \tilde{x}_{n_k} - \lambda_0 x_0$, $k = 1, 2, \ldots$. If (y_k) contains a subsequence converging to zero, then $[x_0, x_0] \leq 0$ follows from $\lim_{n\to\infty}[x_n, x_n] \leq 0$. Assume therefore that $\liminf_{k\to\infty} \|y_k\| > 0$. As $\begin{pmatrix} y_k \\ \tilde{y}_k \end{pmatrix} \in A$,

$$\|\tilde{y}_k - \lambda_0 y_k\| = \|\tilde{x}_{n_k} - \lambda_0 x_{n_k}\| \to 0 \quad \text{as } k \to \infty$$

and since the sequence (y_k) converges weakly to zero, Theorem 3.9 implies $\liminf_{k\to\infty}[y_k, y_k] > 0$. From

$$\liminf_{k\to\infty}[y_k, y_k] = \liminf_{k\to\infty}[x_{n_k}, x_{n_k}] - [x_0, x_0]$$

and $\liminf_{k\to\infty}[x_k, x_k] \leq 0$ we obtain (3.7). A similar reasoning applies in the case $\lambda_0 \in \sigma_{\pi_-}(A)\backslash\sigma_{--}(A)$. The case $\lambda_0 = \infty$ follows from Lemma 3.7 with $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in (2.1). □

For selfadjoint operators the next corollary reduces to [4, Lemma 10].

Corollary 3.11 *Let A be a closed linear operator in $\mathcal{H}$ with $\infty \in \widetilde{\sigma}_{ap}(A)$. Then $\infty \in \sigma_{\pi_+}(A)$ ($\infty \in \sigma_{\pi_-}(A)$) implies $\infty \in \sigma_{++}(A)$ (resp. $\infty \in \sigma_{--}(A)$).*

Proof. Assume that $\infty \in \sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$. Then, with $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in (2.1) and Lemma 3.7 we have

$$0 \in \sigma_{\pi_+}(A^{-1}) \setminus \sigma_{++}(A^{-1}).$$

Theorem 3.10 implies $0 \in \sigma_p(A^{-1})$, a contradiction to the assumption that A is an operator. □

4 Stability properties of spectral points of definite type and type π under perturbations

In this section we consider the behaviour of spectral points of positive and negative type, and type π_+ and type π_- of closed linear relations in Krein spaces under different perturbations.

Let A and B be closed linear relations in the Krein space $(\mathcal{H}, [\cdot,\cdot])$ and let P_A and P_B be the orthogonal projections in the Hilbert space $\mathcal{H}\oplus\mathcal{H}$ onto A and B, respectively. We shall say that A is a *compact (finite rank) perturbation of* B if the difference $P_A - P_B \in \mathcal{L}(\mathcal{H}\oplus\mathcal{H})$ is a compact (resp. finite rank) operator on $\mathcal{H}\oplus\mathcal{H}$. These notions are natural generalizations of the notions of compact and finite rank perturbations of bounded operators or unbounded operators with common points in their resolvent sets, cf. [5]. We remark, that the projections P_A and P_B can be expressed as block operator matrices in terms of A and B with the so-called Stone-de Snoo formula, see, e.g., [16,20,33].

Theorem 4.1 *Let A and B be closed linear relations in $\mathcal{H}$ and suppose that A is a compact perturbation of B. Then we have*

$$\sigma_{\pi_+}(A)\cup\widetilde{r}(A) = \sigma_{\pi_+}(B)\cup\widetilde{r}(B) \quad \textit{and} \quad \sigma_{\pi_-}(A)\cup\widetilde{r}(A) = \sigma_{\pi_-}(B)\cup\widetilde{r}(B). \tag{4.1}$$

Proof. For the first assertion in (4.1) it is sufficient to verify the inclusion $\sigma_{\pi_+}(A)\cup\widetilde{r}(A) \subset \sigma_{\pi_+}(B)\cup\widetilde{r}(B)$. The proof of the second equality in (4.1) is completely analogous and will therefore be omitted.

Let $\lambda_0 \neq \infty$ and let $\lambda_0 \in \sigma_{\pi_+}(A)\cup r(A)$. Assume $\lambda_0 \in \sigma_{ap}(B)\setminus\sigma_{\pi_+}(B)$. By Theorem 3.9 there exists a sequence $\binom{x_n}{\tilde{x}_n}$ in B with $\|x_n\| = 1$, $n = 1,2\ldots$, and $\lim_{n\to\infty}\|\tilde{x}_n - \lambda_0 x_n\| = 0$ such that (x_n) converges weakly to zero and

$$\liminf_{n\to\infty}\,[x_n, x_n] \leq 0.$$

Therefore $\binom{x_n}{\tilde{x}_n}$ converges in $\mathcal{H}\times\mathcal{H}$ weakly to zero and since $P_A - P_B$ is compact this implies

$$\lim_{n\to\infty}\left\|(P_B - P_A)\binom{x_n}{\tilde{x}_n}\right\| = 0.$$

We set $\binom{y_n}{\tilde{y}_n} := P_A\binom{x_n}{\tilde{x}_n}$, $n = 1,2\ldots$. As $P_B\binom{x_n}{\tilde{x}_n} = \binom{x_n}{\tilde{x}_n}$, $n = 1,2\ldots$, we have

$$\begin{pmatrix} x_n \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} y_n \\ \tilde{y}_n \end{pmatrix} + (P_B - P_A)\begin{pmatrix} x_n \\ \tilde{x}_n \end{pmatrix}$$

and hence $\lim_{n\to\infty}\|x_n - y_n\| = \lim_{n\to\infty}\|\tilde{x}_n - \tilde{y}_n\| = 0$. Therefore, (y_n) converges weakly to zero,

$$\lim_{n\to\infty}\|y_n\| = 1, \quad \lim_{n\to\infty}\|\tilde{y}_n - \lambda_0 y_n\| = 0 \quad \text{and} \quad \liminf_{n\to\infty}\,[y_n, y_n] \leq 0.$$

Then $\lambda_0 \notin r(A)$ follows and Theorem 3.9 gives $\lambda_0 \notin \sigma_{\pi_+}(A)$, a contradiction. Hence, λ_0 belongs to $\sigma_{\pi_+}(B)\cup r(B)$.

Let $\infty \in \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$. Then Lemma 2.1 and Lemma 3.7 imply $0 \in \sigma_{\pi_+}(A^{-1}) \cup \widetilde{r}(A^{-1})$. Denote by $P_{A^{-1}}$ and $P_{B^{-1}}$ the orthogonal projections in $\mathcal{H} \oplus \mathcal{H}$ onto A^{-1} and B^{-1}, respectively. Observe that P_A is connected with the orthogonal projection $P_{A^{-1}}$ onto A^{-1} in the following manner: For $h, \tilde{h} \in \mathcal{H}$ we have

$$P_A \begin{pmatrix} h \\ \tilde{h} \end{pmatrix} = \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \in A \text{ if and only if } P_{A^{-1}} \begin{pmatrix} \tilde{h} \\ h \end{pmatrix} = \begin{pmatrix} \tilde{x} \\ x \end{pmatrix} \in A^{-1}.$$

The projections P_B and $P_{B^{-1}}$ are connected in the same way. Therefore, since the compact operator $P_A - P_B$ maps bounded sequences onto sequences with a convergent subsequence, the same is true for $P_{A^{-1}} - P_{B^{-1}}$, and hence also this operator is compact. The reasoning above implies $0 \in \sigma_{\pi_+}(B^{-1}) \cup \widetilde{r}(B^{-1})$, hence, by Lemma 2.1 and Lemma 3.7, $\infty \in \sigma_{\pi_+}(B) \cup \widetilde{r}(B)$. □

Remark 4.2 *We mention that the sets $\sigma_{\pi_+}(A)$ and $\sigma_{\pi_-}(A)$ in Theorem 4.1 can not be replaced by $\sigma_{++}(A)$ and $\sigma_{--}(A)$, so that, roughly speaking, spectral points of type π are stable under compact perturbations (and, in particular, finite rank perturbations) but spectral points of definite type are not, see also [28, Theorem 5.1] and [9, Theorem 5.1].*

If A and B are closed linear relations in $\mathcal{H}$ with $\rho(A) \cap \rho(B) \neq \emptyset$, then according to [5, Corollary 4.5] A is a compact perturbation of B if and only if $(A - \lambda)^{-1} - (B - \lambda)^{-1}$ is a compact operator for some, and hence for all, $\lambda \in \rho(A) \cap \rho(B)$. Together with Theorem 4.1 this implies the following corollary.

Corollary 4.3 *Let A and B be closed linear relations in $\mathcal{H}$ with $\rho(A) \cap \rho(B) \neq \emptyset$ and assume that*

$$(A - \lambda)^{-1} - (B - \lambda)^{-1}, \qquad \lambda \in \rho(A) \cap \rho(B),$$

is compact. Then

$$\sigma_{\pi_+}(A) \cup \widetilde{r}(A) = \sigma_{\pi_+}(B) \cup \widetilde{r}(B) \quad \text{and} \quad \sigma_{\pi_-}(A) \cup \widetilde{r}(A) = \sigma_{\pi_-}(B) \cup \widetilde{r}(B).$$

In the following proposition we consider a special case of finite rank perturbations. Recall that $\widehat{+}$ stands for the sum of linear manifolds and subspaces.

Proposition 4.4 *Let A be a closed linear relation in $\mathcal{H}$ and let F be a finite dimensional subspace of $\mathcal{H} \times \mathcal{H}$. Then*

$$\sigma_{\pi_+}(A) = \sigma_{\pi_+}(A \widehat{+} F) \cap \widetilde{\sigma}_{ap}(A), \quad \sigma_{\pi_-}(A) = \sigma_{\pi_-}(A \widehat{+} F) \cap \widetilde{\sigma}_{ap}(A) \tag{4.2}$$

and

$$\widetilde{r}(A) \subset \widetilde{r}(A \widehat{+} F) \cup \Big(\sigma_{\pi_+}(A \widehat{+} F) \cap \sigma_{\pi_-}(A \widehat{+} F)\Big). \tag{4.3}$$

Proof. As A is closed also the linear relation $A\widehat{+}F$ is a closed. Denote by P_A and $P_{A\widehat{+}F}$ the orthogonal projections in $\mathcal{H}\oplus\mathcal{H}$ onto A and $A\widehat{+}F$, respectively. The operator $P_{A\widehat{+}F}-P_A$ is finite rank and hence, in particular, compact. Then (4.2) follows from (4.1) and the fact that $\widetilde{r}(A\widehat{+}F)\subset\widetilde{r}(A)$.

In order to prove (4.3) let $\lambda_0\neq\infty$ be a point in $r(A)\setminus r(A\widehat{+}F)$ and assume $\lambda_0\notin\sigma_{\pi_+}(A\widehat{+}F)\cap\sigma_{\pi_-}(A\widehat{+}F)$. Then $\lambda_0\notin\sigma_{\pi_+}(A\widehat{+}F)$ or $\lambda_0\notin\sigma_{\pi_-}(A\widehat{+}F)$. According to Theorem 3.9 in both cases there exists a sequence $\binom{x_n}{\tilde{x}_n}\in A\widehat{+}F$, $n=1,2\ldots$, with $\|x_n\|=1$, $\lim_{n\to\infty}\|\tilde{x}_n-\lambda_0x_n\|=0$ such that (x_n) converges weakly to zero. We choose a finite dimensional subspace $F'\subset F$ such that $A\cap F'=\{0\}$ and $A\widehat{+}F$ coincides with $A\widehat{+}F'$, i.e. the sum in $A\widehat{+}F'$ is direct. Denote the projections in $\mathcal{H}\oplus\mathcal{H}$ on $A-\lambda_0$ and $F'-\lambda_0$ corresponding to the decomposition

$$(A\widehat{+}F)-\lambda_0=(A-\lambda_0)\widehat{+}(F'-\lambda_0)$$

by P_0 and P_1, respectively. Then, by $\dim(F'-\lambda_0)<\infty$, $P_1\binom{x_n}{\tilde{x}_n-\lambda_0x_n}$ converges strongly to zero. If $\binom{x'_n}{\tilde{x}'_n-\lambda_0x'_n}:=P_0\binom{x_n}{\tilde{x}_n-\lambda_0x_n}$, where $\binom{x'_n}{\tilde{x}'_n}\in A$, $n=1,2\ldots$, then $\lim_{n\to\infty}\|x'_n\|=1$ and $\lim_{n\to\infty}\|\tilde{x}'_n-\lambda_0x'_n\|=0$ which implies $\lambda_0\notin r(A)$, a contradiction.

It remains to consider $\lambda_0=\infty$. Let $\infty\in\widetilde{r}(A)$, that is $0\in r(A^{-1})$. Then, by the reasoning above, we have

$$0\in r(A^{-1}\widehat{+}F^{-1})\cup\Big(\sigma_{\pi_+}(A^{-1}\widehat{+}F^{-1})\cap\sigma_{\pi_-}(A^{-1}\widehat{+}F^{-1})\Big),$$

as F^{-1} is finite dimensional. Moreover, we have $A^{-1}\widehat{+}F^{-1}=(A\widehat{+}F)^{-1}$ and (4.3) follows from Lemma 2.1 and Lemma 3.7. □

Proposition 3.8 and Proposition 4.4 imply the following corollary.

Corollary 4.5 *Let A be a closed linear relation in $\mathcal{H}$ and let Let $\mathfrak{F}\subset\overline{\mathbb{C}}$ be a compact set. Then $\mathfrak{F}\subset\sigma_{\pi_+}(A)\cup\widetilde{r}(A)$ ($\mathfrak{F}\subset\sigma_{\pi_-}(A)\cup\widetilde{r}(A)$) if and only if there exists a closed linear relation $S\subset A$ with $\operatorname{codim}_A S<\infty$ and*

$$\mathfrak{F}\subset\sigma_{++}(S)\cup\widetilde{r}(S)\quad(\textit{resp. }\mathfrak{F}\subset\sigma_{--}(S)\cup\widetilde{r}(S)).$$

4.2 Perturbations small in gap

We consider now the behaviour of spectral points of definite type and type π under perturbations which are small with respect to the gap metric. The *gap* between two subspaces $\mathcal{L}$ and $\mathcal{M}$ of some Hilbert space $\mathcal{G}$ is defined by

$$\hat{\delta}(\mathcal{L},\mathcal{M}):=\max\Big\{\sup_{u\in\mathcal{L},\|u\|=1}\operatorname{dist}(u,\mathcal{M}),\sup_{v\in\mathcal{M},\|v\|=1}\operatorname{dist}(v,\mathcal{L})\Big\},$$

cf. [25]. Recall that a subspace is always assumed to be a closed linear manifold. If $P_{\mathcal{L}}$ and $P_{\mathcal{M}}$ denote the orthogonal projections in $\mathcal{G} \oplus \mathcal{G}$ onto $\mathcal{L}$ and $\mathcal{M}$, respectively, then the gap between $\mathcal{L}$ and $\mathcal{M}$ is

$$\hat{\delta}(\mathcal{L}, \mathcal{M}) = \|P_{\mathcal{L}} - P_{\mathcal{M}}\|.$$

In the next theorem we show, roughly speaking, that spectral points of positive and negative type are stable under perturbations small in the gap metric.

Theorem 4.6 *Let A be a closed linear relation in $\mathcal{H}$ and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{++}(A) \cup \widetilde{r}(A)$ ($\mathfrak{F} \subset \sigma_{--}(A) \cup \widetilde{r}(A)$). Then there exists a constant $\gamma \in (0,1)$ such that for all closed linear relations B with $\hat{\delta}(A,B) < \gamma$ we have*

$$\mathfrak{F} \subset \sigma_{++}(B) \cup \widetilde{r}(B) \quad (\textit{resp. } \mathfrak{F} \subset \sigma_{--}(B) \cup \widetilde{r}(B)). \tag{4.4}$$

Proof. Assume first that $\infty \notin \mathfrak{F}$ and let $\mathfrak{F} \subset \sigma_{++}(A) \cup r(A)$. We choose ϵ and δ as in Lemma 3.2. It is no restriction to assume $\epsilon < 1$ and $\delta < 1$. We set

$$M := 1 + \epsilon + \max_{\lambda \in \mathfrak{F}} |\lambda| \quad \text{and} \quad \gamma := \frac{\min\{\epsilon, \delta\}}{6M^2}.$$

Let B be a closed linear relation with $\hat{\delta}(A,B) < \gamma$. For $\lambda \in \mathfrak{F} \setminus r(B)$ choose $\binom{y}{\tilde{y}} \in B$ with $\|y\| = 1$ and $\|\tilde{y} - \lambda y\| < \frac{5}{6}\epsilon$. Then it follows

$$\left\| \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} \right\| \leq M \quad \text{and} \quad \operatorname{dist}\left(\begin{pmatrix} y \\ \tilde{y} \end{pmatrix}, A \right) < M\gamma,$$

and there exists $\binom{x}{\tilde{x}} \in A$ with

$$\|x - y\| \leq M\gamma \quad \text{and} \quad \|\tilde{x} - \tilde{y}\| \leq M\gamma.$$

This implies $1 - M\gamma \leq \|x\| \leq 1 + M\gamma$ and

$$\begin{aligned} \left\| \frac{\tilde{x}}{\|x\|} - \lambda \frac{x}{\|x\|} \right\| &\leq \frac{\|\tilde{y} - \lambda y\| + \|\tilde{x} - \tilde{y}\| + \|\lambda y - \lambda x\|}{\|x\|} \leq \\ &\leq \frac{\frac{5}{6}\epsilon + M\gamma(1 + \max_{\lambda \in \mathfrak{F}} |\lambda|)}{1 - M\gamma} \leq \\ &\leq \frac{\frac{5}{6}\epsilon + M\gamma(M - \epsilon)}{1 - M\gamma} \leq \frac{\epsilon - M\gamma\epsilon}{1 - M\gamma} = \epsilon, \end{aligned}$$

hence

$$[x,x] \geq \delta \|x\|^2.$$

Moreover, we have

$$[y,y] = [x,x] + [y-x,x] + [x,y-x] + [y-x,y-x]$$

and therefore

$$[y,y] \geq \delta\|x\|^2 - 2M\gamma\|x\| - M^2\gamma^2 \geq$$
$$\geq \delta(1-M\gamma)^2 - 2M\gamma(1+M\gamma) - M^2\gamma^2 \geq \delta - 4M\gamma - 3M^2\gamma^2 \geq \frac{\delta}{4}.$$

This implies $\mathfrak{F} \subset \sigma_{++}(B) \cup r(B)$.

If $\infty \in \mathfrak{F} \subset \sigma_{++}(A) \cup r(A)$ we choose two closed subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\overline{\mathbb{C}}$ with $\infty \notin \mathfrak{F}_1$, $\infty \in \mathfrak{F}_2$, $0 \notin \mathfrak{F}_2$ and $\mathfrak{F} = \mathfrak{F}_1 \cup \mathfrak{F}_2$. It follows from above that there exists a constant $\gamma_1 \in (0,1)$ such that for all closed linear relations B with $\hat{\delta}(A,B) < \gamma_1$ we have $\mathfrak{F}_1 \cap \sigma_{ap}(B) \subset \sigma_{++}(B)$. Moreover, the set $\{\lambda^{-1} : \lambda \in \mathfrak{F}_2 \setminus \{\infty\}\} \cup \{0\}$ is a subset of $\sigma_{++}(A^{-1}) \cup r(A^{-1})$, cf. Lemma 3.7, and by the first part of the proof there is a $\gamma_2 \in (0,1)$ with

$$\left\{\lambda^{-1} : \lambda \in \mathfrak{F}_2 \setminus \{\infty\}\right\} \cup \{0\} \subset \sigma_{++}(B^{-1}) \cup r(B^{-1})$$

for all closed linear relations B with $\hat{\delta}(A^{-1}, B^{-1}) < \gamma_2$. Another application of Lemma 3.7 gives $\mathfrak{F}_2 \subset \sigma_{++}(B) \cup \tilde{r}(B)$ and, as $\hat{\delta}(A,B) = \hat{\delta}(A^{-1}, B^{-1})$, we have $\mathfrak{F} \subset \sigma_{++}(B) \cup \tilde{r}(B)$ for all closed linear relations B with $\hat{\delta}(A,B) < \min\{\gamma_1, \gamma_2\}$. Thus, the first inclusion in (4.4) is proved.

The proof of the inclusion $\mathfrak{F} \subset \sigma_{--}(B) \cup \tilde{r}(B)$ is completely analogous and will therefore be omitted. □

In order to prove a stability result for spectral points of type π_+ and type π_- under perturbations small in the gap metric we prove the following statement first.

Proposition 4.7 *Let $\mathcal{L}$ and $\mathcal{M}$ be subspaces of some Hilbert space $\mathcal{G}$ with $\hat{\delta}(\mathcal{L}, \mathcal{M}) < 1$ and let $P_{\mathcal{L}}$ be the orthogonal projection in $\mathcal{G} \oplus \mathcal{G}$ onto $\mathcal{L}$. Let $\mathcal{M}_1$ be a subspace of $\mathcal{M}$ with* $\operatorname{codim}_{\mathcal{M}} \mathcal{M}_1 < \infty$. *Then the subspace $\mathcal{L}_1 = P_{\mathcal{L}}\mathcal{M}_1$ of $\mathcal{L}$ satisfies* $\operatorname{codim}_{\mathcal{L}} \mathcal{L}_1 = \operatorname{codim}_{\mathcal{M}} \mathcal{M}_1$ *and*

$$\hat{\delta}(\mathcal{L}_1, \mathcal{M}_1) \leq \hat{\delta}(\mathcal{L}, \mathcal{M}).$$

Proof. We denote by $P_{\mathcal{M}}$ the orthogonal projection on $\mathcal{M}$. For $z \in \mathcal{M}$ we have

$$\|P_{\mathcal{L}}z\| \geq \|z\| - \|(I - P_{\mathcal{L}})z\| \geq (1 - \|P_{\mathcal{M}} - P_{\mathcal{L}}\|)\|z\|.$$

The gap between $\mathcal{L}$ and $\mathcal{M}$ is smaller than one, hence the restriction $P_{\mathcal{L}}|\mathcal{M}$ of the projection $P_{\mathcal{L}}$ to the subspace $\mathcal{M}$, considered as a mapping from $\mathcal{M}$ into $\mathcal{L}$, is a bounded, injective operator with a closed range. Moreover, an element of $\mathcal{L}$ orthogonal to the range of $P_{\mathcal{L}}|\mathcal{M}$ belongs to $\mathcal{M}^\perp$, therefore, by $\mathcal{M}^\perp \cap \mathcal{L} = \{0\}$, the operator $P_{\mathcal{L}} : \mathcal{M} \to \mathcal{L}$ is an isomorphism.

We define a linear bounded operator $D : \mathcal{L} \to \mathcal{L}^\perp$ by

$$D = (I - P_\mathcal{L})(P_\mathcal{L}|\mathcal{M})^{-1}.$$

For $z \in \mathcal{M}$ we set $x := P_\mathcal{L}|\mathcal{M}\, z$. Then we have

$$z = (I - P_\mathcal{L})(P_\mathcal{L}|\mathcal{M})^{-1}x + P_\mathcal{L}|\mathcal{M}(P_\mathcal{L}|\mathcal{M})^{-1}x = Dx + x$$

and, as $P_\mathcal{L}|\mathcal{M}$ is an isomorphism,

$$\mathcal{M} = \left\{ \begin{pmatrix} x \\ Dx \end{pmatrix} : x \in \mathcal{L} \right\} \quad \text{and } \mathcal{M}^\perp = \left\{ \begin{pmatrix} -D^*w \\ w \end{pmatrix} : w \in \mathcal{L}^\perp \right\}$$

with respect to the decomposition $\mathcal{G} = \mathcal{L} \oplus \mathcal{L}^\perp$. Moreover, see e.g. [30] or [33],

$$P_\mathcal{M} = \begin{bmatrix} (I + D^*D)^{-1} & D^*(I + DD^*)^{-1} \\ D(I + D^*D)^{-1} & DD^*(I + DD^*)^{-1} \end{bmatrix}.$$

Then, by $D^*(I+DD^*)^{-1} = (I+D^*D)^{-1}D^*$ and $D(I+D^*D)^{-1} = (I+DD^*)^{-1}D$ we obtain

$$(P_\mathcal{M} - P_\mathcal{L})^2 = \begin{bmatrix} D^*D(I + D^*D)^{-1} & 0 \\ 0 & DD^*(I + DD^*)^{-1} \end{bmatrix}.$$

Making use of the functional calculi of the bounded selfadjoint operators D^*D and DD^* in $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively, we find that

$$\hat{\delta}(\mathcal{L}, \mathcal{M}) = \|P_\mathcal{L} - P_\mathcal{M}\| = \frac{\|D\|}{\sqrt{1 + \|D\|^2}} \tag{4.5}$$

holds. Let

$$\mathcal{L}_1 = P_\mathcal{L}\mathcal{M}_1.$$

As $P_\mathcal{L}|\mathcal{M}$ is an isomorphism, $\operatorname{codim}_\mathcal{L} \mathcal{L}_1 = \operatorname{codim}_\mathcal{M} \mathcal{M}_1 < \infty$ follows. We choose a finite dimensional subspace $\mathcal{L}_1'$ of $\mathcal{L}$ such that

$$\mathcal{G} = \mathcal{L}_1 \oplus \mathcal{L}_1' \oplus \mathcal{L}^\perp. \tag{4.6}$$

We denote by D_1, $D_1 : \mathcal{L}_1 \to \mathcal{L}^\perp$, the restriction of D to $\mathcal{L}_1$, $D_1 = D|\mathcal{L}_1$. Then, with respect to the decomposition (4.6), we have

$$\mathcal{M}_1 = \left\{ \begin{pmatrix} x \\ 0 \\ D_1x \end{pmatrix} : x \in \mathcal{L}_1 \right\} \quad \text{and } \mathcal{M}_1^\perp = \left\{ \begin{pmatrix} -D_1^*w \\ u \\ w \end{pmatrix} : w \in \mathcal{L}^\perp, u \in \mathcal{L}_1' \right\}$$

and the orthogonal projection $P_{\mathcal{M}_1}$ on $\mathcal{M}_1$ satisfies

$$P_{\mathcal{M}_1} = \begin{bmatrix} (I + D_1^* D_1)^{-1} & 0 & D_1^*(I + D_1 D_1^*)^{-1} \\ 0 & 0 & 0 \\ D_1(I + D_1^* D_1)^{-1} & 0 & D_1 D_1^*(I + D_1 D_1^*)^{-1} \end{bmatrix}.$$

Similar as in (4.5) we have

$$\|P_{\mathcal{L}_1} - P_{\mathcal{M}_1}\| = \frac{\|D_1\|}{\sqrt{1 + \|D_1\|^2}}.$$

Together with (4.5) and the fact that the function $t \mapsto \frac{t}{\sqrt{1+t^2}}$, $t \geq 0$, is increasing, we conclude

$$\hat{\delta}(\mathcal{L}_1, \mathcal{M}_1) = \|P_{\mathcal{L}_1} - P_{\mathcal{M}_1}\| = \frac{\|D_1\|}{\sqrt{1 + \|D_1\|^2}} \leq \frac{\|D\|}{\sqrt{1 + \|D\|^2}} = \hat{\delta}(\mathcal{L}, \mathcal{M}).$$

This completes the proof of Proposition 4.7. □

With the help of Proposition 4.7 we are now able to prove a variant of Theorem 4.6 for spectral points of type π_+ and type π_-.

Theorem 4.8 *Let A be a closed linear relation in $\mathcal{H}$ and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$ ($\mathfrak{F} \subset \sigma_{\pi_-}(A) \cup \widetilde{r}(A)$). Then there exists a constant $\gamma \in (0,1)$ such that for all closed linear relations B with $\hat{\delta}(A, B) < \gamma$ we have*

$$\mathfrak{F} \subset \sigma_{\pi_+}(B) \cup \widetilde{r}(B) \quad (\textit{resp. } \mathfrak{F} \subset \sigma_{\pi_-}(B) \cup \widetilde{r}(B)). \tag{4.7}$$

Proof. We verify only the first inclusion in (4.7). Let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \sigma_{\pi_+}(A) \cup \widetilde{r}(A)$. In order to prove (4.7) we choose S as in Corollary 4.5. According to Theorem 4.6 there exists a constant $\gamma \in (0,1)$ such that for all closed linear relations S' with $\hat{\delta}(S, S') < \gamma$ we have

$$\mathfrak{F} \subset \sigma_{++}(S') \cup \widetilde{r}(S'). \tag{4.8}$$

Let B be a closed linear relation with $\hat{\delta}(A, B) < \gamma$ and let P_B be the orthogonal projection in $\mathcal{H} \oplus \mathcal{H}$ onto B. It follows from Proposition 4.7 that the closed linear relation $\widetilde{S} := P_B S \subset B$ satisfies

$$\operatorname{codim}_B \widetilde{S} = \operatorname{codim}_A S < \infty$$

and $\hat{\delta}(S, \widetilde{S}) \leq \hat{\delta}(A, B) < \gamma$, hence, by (4.8),

$$\mathfrak{F} \subset \sigma_{++}(\widetilde{S}) \cup \widetilde{r}(\widetilde{S}).$$

Then Corollary 4.5 implies $\mathfrak{F} \subset \sigma_{\pi_+}(B) \cup \widetilde{r}(B)$. $\square$

For a closed linear relation the intersection of the set of all spectral points of positive type with the set of all spectral points of negative type is void. This implies the following corollary.

Corollary 4.9 *Let A be a closed linear relation and let $\mathfrak{F} \subset \overline{\mathbb{C}}$ be a compact set with $\mathfrak{F} \subset \widetilde{r}(A)$. Then there exists a constant $\gamma \in (0,1)$ such that for all closed linear relations B with $\hat{\delta}(A,B) < \gamma$ we have*

$$\mathfrak{F} \subset \widetilde{r}(B).$$

4.3 Perturbations of fundamentally reducible relations

In this subsection we prove a result on small perturbations in the gap metric for fundamentally reducible closed linear relations in Krein spaces. Let

$$\mathcal{H} = \mathcal{H}_+ \, [\hat{+}] \, \mathcal{H}_-, \qquad \text{direct sum}, \tag{4.9}$$

be a fundamental decomposition of the Krein space $(\mathcal{H}, [\cdot,\cdot])$, see e.g. [10]. A relation A is said to be *fundamentally reducible* if A can be written as

$$A = A_+ \,\hat{+}\, A_-, \qquad \text{direct sum}, \tag{4.10}$$

where $A_+ := A \cap \mathcal{H}_+^2$ and $A_- := A \cap \mathcal{H}_-^2$ are closed linear relations in the Hilbert spaces $(\mathcal{H}_+, [\cdot,\cdot])$ and $(\mathcal{H}_-, -[\cdot,\cdot])$. Recall that according to Lemma 2.2 for a point λ of regular type of A_- the estimate

$$\|\tilde{y}^- - \lambda y^-\| \geq k_{\lambda,-} \|y^-\| \tag{4.11}$$

holds for some $k_{\lambda,-} > 0$ and all $\binom{y^-}{\tilde{y}^-} \in A_-$. Analogously, for a point λ of regular type of A_+ the estimate

$$\|\tilde{y}^+ - \lambda y^+\| \geq k_{\lambda,+} \|y^+\| \tag{4.12}$$

holds for some $k_{\lambda,+} > 0$ and all $\binom{y^+}{\tilde{y}^+} \in A_+$.

The following theorem can be viewed as a generalization of [28, Theorem 4.2].

Theorem 4.10 *Let A be a fundamentally reducible closed linear relation in $\mathcal{H}$ as in* (4.10) *and let B be a closed linear relation in $\mathcal{H}$. Then the following holds.*

(i) *If for some* $\lambda \in r(A_-)$, $k_{\lambda,-} > 0$ *as in* (4.11) *and* $\gamma > 0$

$$\hat{\delta}(A-\lambda, B-\lambda) < \gamma \quad \textit{and} \quad \gamma^2\left(1+\frac{1}{k_{\lambda,-}^2}\right) < \frac{1}{4}$$

hold, then

$$\lambda \in \sigma_{++}(B) \cup r(B).$$

(ii) *If for some* $\lambda \in r(A_+)$, $k_{\lambda,+} > 0$ *as in* (4.12) *and* $\gamma > 0$

$$\hat{\delta}(A-\lambda, B-\lambda) < \gamma \quad \textit{and} \quad \gamma^2\left(1+\frac{1}{k_{\lambda,+}^2}\right) < \frac{1}{4}$$

hold, then

$$\lambda \in \sigma_{--}(B) \cup r(B).$$

Proof. We prove only assertion (i). The proof of (ii) is analogous. Suppose that $\lambda \in \sigma_{ap}(B)$ and let $\binom{x_n^+ + x_n^-}{\tilde{x}_n^+ + \tilde{x}_n^-} \in B$, $n = 1, 2 \ldots$, $x_n^+, \tilde{x}_n^+ \in \mathcal{H}_+$, $x_n^-, \tilde{x}_n^- \in \mathcal{H}_-$, be a sequence with

$$\|x_n^+\|^2 + \|x_n^-\|^2 = 1 \quad \text{and} \quad \lim_{n\to\infty} \|\tilde{x}_n^+ - \lambda x_n^+\|^2 + \|\tilde{x}_n^- - \lambda x_n^-\|^2 = 0. \tag{4.13}$$

We have

$$\operatorname{dist}\left(\begin{pmatrix} x_n^+ + x_n^- \\ \tilde{x}_n^+ + \tilde{x}_n^- - \lambda x_n^+ - \lambda x_n^- \end{pmatrix}, A-\lambda\right) < \gamma \left\| \begin{pmatrix} x_n^+ + x_n^- \\ \tilde{x}_n^+ + \tilde{x}_n^- - \lambda x_n^+ - \lambda x_n^- \end{pmatrix} \right\|.$$

Hence, there exist $\binom{y_n^+}{\tilde{y}_n^+} \in A_+$ and $\binom{y_n^-}{\tilde{y}_n^-} \in A_-$ with

$$\begin{aligned} \|x_n^+ - y_n^+\|^2 + \|\tilde{x}_n^+ - \lambda x_n^+ - (\tilde{y}_n^+ - \lambda y_n^+)\|^2 \\ + \|x_n^- - y_n^-\|^2 + \|\tilde{x}_n^- - \lambda x_n^- - (\tilde{y}_n^- - \lambda y_n^-)\|^2 \\ < \gamma^2 \left\| \begin{pmatrix} x_n^+ + x_n^- \\ \tilde{x}_n^+ + \tilde{x}_n^- - \lambda x_n^+ - \lambda x_n^- \end{pmatrix} \right\|^2. \end{aligned}$$

In particular,

$$\|x_n^- - y_n^-\|^2 + \|\tilde{x}_n^- - \lambda x_n^- - (\tilde{y}_n^- - \lambda y_n^-)\|^2 < \gamma^2 \left\| \begin{pmatrix} x_n^+ + x_n^- \\ \tilde{x}_n^+ + \tilde{x}_n^- - \lambda x_n^+ - \lambda x_n^- \end{pmatrix} \right\|^2$$

and together with (4.13) we have

$$\limsup_{n\to\infty} \left(\|x_n^- - y_n^-\|^2 + \|\tilde{y}_n^- - \lambda y_n^-\|^2\right) < \gamma^2. \tag{4.14}$$

With (4.13), (4.14) and $\|\tilde{y}_n^- - \lambda y_n^-\| \geq k_{\lambda,-}\|y_n^-\|$ we obtain

$$\begin{aligned}
\liminf_{n\to\infty} & \left[x_n^+ + x_n^-, x_n^+ + x_n^-\right] = \\
&= \liminf_{n\to\infty}(\|x_n^+\|^2 - \|x_n^-\|^2) = \liminf_{n\to\infty}(1 - 2\|x_n^-\|^2) \\
&= 1 - 2\limsup_{n\to\infty}\|x_n^- - y_n^- + y_n^-\|^2 \\
&\geq 1 - 2\limsup_{n\to\infty}\left(2\|x_n^- - y_n^-\|^2 + 2\|y_n^-\|^2\right) \\
&\geq 1 - 4\limsup_{n\to\infty}\left(1 + \frac{1}{k_{\lambda,-}^2}\right)\left(\|x_n^- - y_n^-\|^2 + \|\tilde{y}_n^- - \lambda y_n^-\|^2\right) \\
&\geq 1 - 4\gamma^2\left(1 + \frac{1}{k_{\lambda,-}^2}\right) > 0.
\end{aligned}$$

This implies $\lambda \in \sigma_{++}(B)$. $\square$

5 Spectral points of type π for selfadjoint operators and relations in Krein spaces

In this section we study the properties of spectral points of type π_+ and type π_- for selfadjoint relations in Krein spaces. In particular it will turn out in Theorem 5.3 below that selfadjoint relations are locally definitizable in the sense of [23,24] (or even definitizable, cf. [13,27]) over subintervals of $\overline{\mathbb{R}}$ which consist of spectral points of type π_+, type π_- and regular points only.

Recall first that the *adjoint* A^+ of a linear relation A in the Krein space $\mathcal{H}$ is defined as

$$A^+ = \left\{ \begin{pmatrix} y \\ \tilde{y} \end{pmatrix} : [\tilde{x}, y] = [x, \tilde{y}] \text{ for all } \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} \in A \right\}.$$

It is clear that A^+ is a closed linear relation in $\mathcal{H}$ and that this definition generalizes the usual definition of the adjoint of a densely defined operator. The relation A is said to be *selfadjoint* if $A = A^+$ holds. We mention that every real point in the spectrum of a selfadjoint relation A belongs to $\widetilde{\sigma}_{ap}(A)$ and that $\sigma_{++}(A) \cup \sigma_{--}(A) \subset \overline{\mathbb{R}}$ holds.

For the operator case Theorem 5.1 below coincides with [4, Theorem 18] and [9, Theorem 4.1]. The idea of the proof is the same as part (ii) of the proof of [28, Theorem 5.1].

Theorem 5.1 *Let A be a selfadjoint relation in $\mathcal{H}$ with $\rho(A) \neq \emptyset$ and let Δ be a compact subset of $\overline{\mathbb{R}}$ such that*

$$\Delta \cap \widetilde{\sigma}(A) \subset \sigma_{\pi_+}(A) \quad (\textit{resp. } \Delta \cap \widetilde{\sigma}(A) \subset \sigma_{\pi_-}(A))$$

holds. Assume, in addition, that each point of Δ is an accumulation point of $\rho(A)$, i.e. $\Delta \subset \overline{\rho(A)}$. Then there exists an open neighbourhood $\mathfrak{U}$ in $\overline{\mathbb{C}}$ of Δ such that the following holds.

(i) $\mathfrak{U} \setminus \overline{\mathbb{R}} \subset \widetilde{\rho}(A)$.

(ii) *Either* $\mathfrak{U} \cap \sigma(A) \cap \overline{\mathbb{R}} \subset \sigma_{++}(A)$ *(resp.* $\mathfrak{U} \cap \sigma(A) \cap \overline{\mathbb{R}} \subset \sigma_{--}(A)$*) or there exists a finite number of points* $\lambda_1, \ldots, \lambda_n$ *in* $\sigma_{\pi_+}(A)$ *(resp.* $\sigma_{\pi_-}(A)$*) such that*

$$(\mathfrak{U} \cap \widetilde{\sigma}(A) \cap \mathbb{R}) \setminus \{\lambda_1, \ldots, \lambda_n\} \subset \sigma_{++}(A)$$
$$\Big(\textit{resp. } (\mathfrak{U} \cap \widetilde{\sigma}(A) \cap \mathbb{R}) \setminus \{\lambda_1, \ldots, \lambda_n\} \subset \sigma_{--}(A)\Big). \tag{5.1}$$

Proof. We prove the statements only for $\Delta \cap \widetilde{\sigma}(A) \subset \sigma_{\pi_+}(A)$. Assume first that $\infty \notin \Delta$. As a consequence of Proposition 3.8 there is a bounded open neighbourhood $\mathfrak{U}$ in $\mathbb{C}$ of Δ such that $\mathfrak{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_+}(A)$. If a nonreal $\lambda \in \mathfrak{U} \cap \sigma(A)$ does not belong to $\sigma_{ap}(A)$ then $\overline{\lambda} \in \sigma_{ap}(A)$. Suppose the assertion of the theorem is not true. Then, cf. Lemma 2.2, there exists a sequence $(\mu_n) \subset \sigma_{ap}(A) \cap \mathfrak{U}$, $(\mu_n) \subset \sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$, such that $\mu_n \neq \mu_m$, $\mu_n \neq \overline{\mu_m}$ for $n \neq m$ and (μ_n) converges to some $\mu_0 \in \Delta$. By Lemma 3.2, $\mu_0 \in \sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$. By Theorem 3.10, for every $n \in \mathbb{N}$, we have $\mu_n \in \sigma_p(A)$ and there exists an eigenvector x_n of A corresponding to μ_n which is nonpositive, $[x_n, x_n] \leq 0$. We have $[x_n, x_m] = 0$ for $n \neq m$.

Let $\mathcal{L}_0$ be the linear span of the elements $\binom{x_n}{\mu_n x_n} \in \mathcal{H} \times \mathcal{H}$, $n \in \mathbb{N}$. Then $\mathcal{L} := \overline{\mathcal{L}_0}$ is a nonpositive subspace of $\mathcal{H} \times \mathcal{H}$. Let $A_{\mathcal{L}} := A \cap \mathcal{L}$. There are two possibilities.

a) $A_{\mathcal{L}} - \mu_0$ has closed range with $\dim\ker(A_{\mathcal{L}} - \mu_0) < \infty$. As all μ_n, $n \in \mathbb{N}$, are eigenvalues of $A_{\mathcal{L}}$ there exists a neighbourhood in $\mathbb{C}$ of μ_0 which consists only of eigenvalues of A, cf. [12, Theorem 2.4]. This contradicts the fact that μ_0 is an accumulation point of $\rho(A)$.

b) It is not true that $A_{\mathcal{L}} - \mu_0$ has a closed range with $\dim\ker(A_{\mathcal{L}} - \mu_0) < \infty$. Then for any $\epsilon > 0$ and an arbitrary subspace $\mathcal{N}$ of $A_{\mathcal{L}}$ with $\operatorname{codim}_{A_{\mathcal{L}}} \mathcal{N} < \infty$ there exists an $\binom{f}{\widetilde{f}} \in \mathcal{N}$ such that $\|f\| = 1$ and $\|\widetilde{f} - \mu_0 f\| < \epsilon$. The same construction as in the proof of Theorem 3.9 shows that there exists a sequence $\binom{f_n}{\widetilde{f}_n} \in A_{\mathcal{L}}$, $n = 1, 2, \ldots$, with $\|f_n\| = 1$ and $\|\widetilde{f}_n - \mu_0 f_n\| \to 0$ as $n \to \infty$ such that (f_n) converges weakly to zero. Since $\binom{f_n}{\widetilde{f}_n} \in \mathcal{L}$ we have $[f_n, f_n] \leq 0$, $n \in \mathbb{N}$. By Theorem 3.9 this contradicts $\mu_0 \in \sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$.

If $\infty \in \Delta$ we choose two closed subsets Δ_1 and Δ_2 of $\overline{\mathbb{R}}$ with $\infty \notin \Delta_1$, $\infty \in \Delta_2$, $0 \notin \Delta_2$ and $\Delta = \Delta_1 \cup \Delta_2$. The relation A^{-1} is selfadjoint and each point of the set $\{\lambda^{-1} : \lambda \in \Delta_2 \setminus \{\infty\}\} \cup \{0\}$ is an accumulation point of $\rho(A^{-1})$. Moreover, each point of that set belongs to $\sigma_{\pi_+}(A^{-1}) \cup \rho(A^{-1})$, cf. Lemma 3.7. By the first part of this proof and Lemma 3.7 the assertion of Theorem 5.1 follows. □

Next we recall the notion of *locally definitizable* selfadjoint relations, see, e.g. [24]. For this let Ω be some domain in $\overline{\mathbb{C}}$ symmetric with respect to the real axis such that $\Omega \cap \overline{\mathbb{R}} \neq \emptyset$ and the intersections of Ω with the upper and lower open half-planes are simply connected.

Definition 5.2 *Let A be a selfadjoint relation in the Krein space $\mathcal{H}$ such that $\sigma(A) \cap (\Omega \backslash \overline{\mathbb{R}})$ consists of isolated points which are poles of the resolvent of A, and no point of $\Omega \cap \overline{\mathbb{R}}$ is an accumulation point of the non-real spectrum of A in Ω. The relation A is said to be* definitizable over Ω, *if the following holds.*

(i) *Every point $\mu \in \Omega \cap \overline{\mathbb{R}}$ has an open connected neighbourhood I_μ in $\overline{\mathbb{R}}$ such that each component of $I_\mu \backslash \{\mu\}$ belongs either to $\sigma_{++}(A) \cup \widetilde{\rho}(A)$ or to $\sigma_{--}(A) \cup \widetilde{\rho}(A)$.*

(ii) *For every finite union Δ of open connected subsets of $\overline{\mathbb{R}}$, $\overline{\Delta} \subset \Omega \cap \overline{\mathbb{R}}$, there exists $m \geq 1$, $M > 0$ and an open neighbourhood $\mathfrak{U}$ of $\overline{\Delta}$ in $\overline{\mathbb{C}}$ such that*

$$\|(A - \lambda)^{-1}\| \leq M \frac{(1 + |\lambda|)^{2m-2}}{|\mathrm{Im}\, \lambda|^m}$$

holds for all $\lambda \in \mathfrak{U} \backslash \overline{\mathbb{R}}$.

By [24, Theorem 4.7] a selfadjoint relation A in $\mathcal{H}$ is definitizable over $\overline{\mathbb{C}}$ if and only if A is *definitizable*, that is, the resolvent set of A is nonempty and there exists a rational function $\mathfrak{r} \neq 0$ with poles only in $\rho(A)$ such that $\mathfrak{r}(A) \in \mathcal{L}(\mathcal{H})$ is a nonnegative operator in $\mathcal{K}$, that is

$$[\mathfrak{r}(A)x, x] \geq 0$$

holds for all $x \in \mathcal{H}$ (see [27] and [13, §4 and §5]).

Theorem 5.3 *Let A be a selfadjoint relation in $\mathcal{H}$ and let Δ be a closed connected subset of $\overline{\mathbb{R}}$ such that*

$$\Delta \cap \widetilde{\sigma}(A) \subset \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A)$$

holds. Assume that each point of Δ is an accumulation point of $\rho(A)$, i.e. $\Delta \subset \overline{\rho(A)}$. Then there exists a domain $\Omega \subset \overline{\mathbb{C}}$ symmetric with respect to real line with $\Omega \cap \mathbb{C}^+$ and $\Omega \cap \mathbb{C}^-$ simply connected such that $\Delta \subset \Omega$ and A is definitizable over Ω.

Proof. Assume first $\infty \notin \Delta$. Then Δ is a closed bounded interval, $\Delta = [a, b]$. By Theorem 5.1 we find real numbers a_0, a_0', b_0, b_0' with $a_0 < a_0' < a < b < b_0' < b_0$ such that either $[a_0, a) \subset \sigma_{++}(A) \cup \rho(A)$ or $[a_0, a) \subset \sigma_{--}(A) \cup \rho(A)$ and such that either $(b, b_0] \subset \sigma_{++}(A) \cup \rho(A)$ or $(b, b_0] \subset \sigma_{--}(A) \cup \rho(A)$. Moreover, we choose a_0 and b_0 in such a way that no point of $[a_0, b_0]$ is an accumulation point of the non-real spectrum of A, see Theorem 5.1. Then [24, Theorem 3.18] implies the existence of a (local) spectral function of A on

(a_0, a_0') and on (b_0', b_0). Therefore, the Krein space $\mathcal{H}$ can be written as the direct orthogonal sum

$$\mathcal{H} = E((a_0, a_0') \cup (b_0', b_0))\mathcal{H}\,[\hat{+}]\,\Big(I - E((a_0, a_0') \cup (b_0', b_0))\Big)\mathcal{H}$$

and with respect to this decomposition the selfadjoint relation A becomes the direct orthogonal sum of the selfadjoint relations

$$A_1 := A \cap \Big(E((a_0, a_0') \cup (b_0', b_0))\mathcal{H}\Big)^2$$

and

$$A_2 := A \cap \Big(\Big(I - E((a_0, a_0') \cup (b_0', b_0))\Big)\mathcal{H}\Big)^2,$$

$A = A_1\,[\hat{+}]\,A_2$, where the spectrum of A_2 is a subset of $[a_0, a_0'] \cup [b_0', b_0]$ and the spectrum of A_2 belongs to $\mathbb{C} \setminus \{(a_0, a_0') \cup (b_0', b_0)\}$. Then, with Theorem 5.1, the interval (a_0', b_0') is a spectral set for the operator A_2, hence the Riesz-Dunford projection $E_{(a_0', b_0')}$ corresponding to (a_0', b_0') and A_2 is defined. Now, by [4, Theorem 23], there exists a domain Ω in $\mathbb{C}$ with the properties stated in Theorem 5.3 such that $A_2 E_{(a_0', b_0')}$ is definitizable over Ω. Thus, A is definitizable over Ω.

If $\infty \in \Delta$ we choose two closed connected subsets Δ_1 and Δ_2 of $\overline{\mathbb{R}}$ with $\infty \notin \Delta_1$, $\infty \in \Delta_2$, $0 \notin \Delta_2$ and $\Delta = \Delta_1 \cup \Delta_2$. Then by the first part of this proof and by Lemma 3.7 there exist domains Ω_1 and Ω_2, $\Delta_1 \subset \Omega_1$, $\Delta_2 \subset \Omega_2$, with the properties stated in the Theorem 5.3 such that A is definitizable over Ω_1 and A^{-1} is definitizable over $\{\lambda^{-1} : \lambda \in \Omega_2 \setminus \{\infty\}\} \cup \{0\}$. Then it follows that A is definitizable over $\Omega = \Omega_1 \cup \Omega_2$. □

Theorem 5.3 together with Corollary 4.3 now implies a result on compact perturbations which is well-known, see [8]. We mention that it was proved for bounded operators in [28] and in [23] for unbounded operators under some additional assumptions.

Theorem 5.4 *Let A be a selfadjoint relation in $\mathcal{H}$ which is definitizable over some domain $\Omega \subset \overline{\mathbb{C}}$ and let $\Omega \setminus \overline{\mathbb{R}} \subset \rho(A)$.*[3] *Assume that $\Delta = \Omega \cap \overline{\mathbb{R}}$ is an open connected set such that*

$$\Delta \cap \tilde{\sigma}(A) \subset \sigma_{\pi_+}(A) \quad (\Delta \cap \tilde{\sigma}(A) \subset \sigma_{\pi_-}(A))$$

holds. If B is a selfadjoint relation in $\mathcal{H}$, $\rho(B) \cap \Omega \neq \emptyset$ and $(A-\mu)^{-1} - (B-\mu)^{-1}$ is a compact operator for some, and hence for all, $\mu \in \rho(A) \cap \rho(B)$, then B is also definitizable over Ω and

$$\Delta \cap \tilde{\sigma}(B) \subset \sigma_{\pi_+}(B) \quad (\textit{resp. } \Delta \cap \tilde{\sigma}(B) \subset \sigma_{\pi_-}(B)).$$

[3] We remark that in the formulation of Theorem 29 in [4] the assumption $\Omega \setminus \overline{\mathbb{R}} \subset \rho(A)$ is missing.

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