Performance funnels and tracking control*

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Abstract: Tracking of an absolutely continuous reference signal (assumed bounded with essentially bounded derivative) is considered in the context of a class of nonlinear, single-input, single-output, dynamical systems modelled by functional differential equations satisfying certain structural hypotheses (which, interpreted in the particular case of linear systems, translate into assumptions – ubiquitous in the adaptive control literature – of (i) relative degree one, (ii) positive high-frequency gain and (iii) stable zero dynamics). The control objective is evolution of the tracking error within a prespecified funnel, thereby guaranteeing prescribed transient performance and prescribed asymptotic tracking accuracy. This objective is achieved by a so-called funnel controller, which takes the form of linear error feedback with time-varying gain. The gain is generated by a nonlinear feedback law in which the reciprocal of the distance of the instantaneous tracking error to the funnel boundary plays a central role. In common with many established high-gain adaptive control methodologies, the overall feedback structure exploits an intrinsic high-gain property of the system, but differs from these approaches in two fundamental respects: the funnel control gain is not dynamically generated and is not necessarily monotone. The main distinguishing feature of the present paper vis à vis previous contributions on funnel control is twofold: (a) a larger system class can be accommodated – in particular, nonlinearities of a general nature can be tolerated in the input channel; (b) a wider choice of formulations of prescribed transient behaviour (including, for example, practical (M,μ)-stability wherein, for prescribed parameter values $M>1$, $\mu>0$ and $\lambda>0$, the tracking error $e(\cdot)$ is required to satisfy $|e(t)| < \max\{Me^{-\mu t}|e(0)|\,,\,\lambda\}$ for all $t\geq 0$) is encompassed.
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1 Introduction

Feedback stabilization or tracking for nonlinear systems is investigated in many textbooks, see, for example, [8, 9, 14, 16, 13]. Restricting attention to single-input, single-output systems of relative degree one (the latter means, loosely speaking, that the input u appears explicitly in the expression for the first derivative of the output y), many authors study systems in the following Byrnes-Isidori normal form (or variants thereof):

$$\dot y(t) = a(y(t),z(t)) + b(y(t),z(t))\; u(t), \quad \dot z(t) = c(y(t),z(t)), \quad (y(0),z(0)) = (\xi,\zeta). \qquad (1.1)$$

Under suitable assumptions on the continuous functions $a,b\colon \mathbb{R}\times\mathbb{R}^{n-1}\to\mathbb{R}$ and $c\colon \mathbb{R}\times\mathbb{R}^{n-1}\to\mathbb{R}^{n-1}$, the objective is a dynamic control law of the form

$$u(t) = k\big(y(t),\eta(t)\big)\; y(t), \qquad \dot\eta(t) = p(y(t),z(t)), \qquad (1.2)$$

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where $k\colon \mathbb{R}\times\mathbb{R}^\ell \to \mathbb{R}$ and $p\colon \mathbb{R}\times\mathbb{R}^\ell \to \mathbb{R}^\ell$ are continuous, which ensures (semi) global (practical) stabilization of the closed-loop system; see, to name but two, [8, pp. 143,174,189] and [9, p. 79]. Standard assumptions are: (i) the continuous function b is bounded away from zero (the *relative-degree-one* assumption); (ii) *stable zero dynamics*, that is in particular, 0 is a globally asymptotically stable equilibrium of the system $\dot z = c(0,z)$.

Let $C(X,Y)$ denote the space of continuous functions $X\to Y$, with the conventions $C(X,\mathbb{R})\equiv C(X)$ and, for $[a,b]\subset\mathbb{R}$, $C([a,b])\equiv C[a,b]$. Define $\mathbb{R}_+ := [0,\infty)$. Assuming that the subsystem $\dot z = c(y,z)$ generates a controlled semi-flow ϕ in the sense that, if we temporarily regard y as an independent (continuous) input, then, for each $(z^0, y(\cdot))\in\mathbb{R}^{n-1}\times C(\mathbb{R}_+)$, the initial-value problem $\dot z = c(y,z)$, $z(0)=\zeta$, has unique solution $z:\mathbb{R}_+\to\mathbb{R}^{n-1}$ given by $z(t) := \phi(t;\zeta,y)$. Thus, with the equation $\dot z = c(y,z)$, we may associate a family of operators $T_\zeta\colon C(\mathbb{R}_+)\to C(\mathbb{R}_+)$, parameterized by the initial data ζ, given by $\big(T_\zeta y\big)(t) := \phi(t;\zeta,y(\cdot))$. Introducing $T\colon C(\mathbb{R}_+)\to C(\mathbb{R}_+,\mathbb{R}^2)$ defined by $\big(Ty\big)(t) := \big(y(t),(T_\zeta)(t)\big)$, the initial-value problem (1.1) may be reformulated (in terms of the input and output variables) as

$$\dot y(t) = a\big((Ty)(t)\big) + b\big((Ty)(t)\big)\,u(t), \qquad y(0)=\xi.$$

The above reformulation of (1.1) as an initial-value problem for a functional differential equation may be regarded as a prototype for the *system class* considered in the present paper (and made precise in Section 2 below) which consists of systems of the form

$$\dot y(t) = a\big(d_1(t),(Ty)(t)\big) + b\big(d_2(t),(Ty)(t)\big)\,g\big(u(t)+d_3(t)\big) \tag{1.3}$$

wherein a, b and g are continuous functions, d_1, d_2 and d_3 are disturbances, and T is a causal operator. This class affords considerably more generality than that of system (1.1). Firstly, in (1.1) the variable u occurs affine linearly on the right hand side and thus (with the assumption that the function b is bounded away from zero) this system has relative degree one: by contrast, the allowable input nonlinearities g in (1.3) (to be made precise in Section 2) are such the system does not necessarily have a well-defined relative degree (for definition of the latter see [8, p. 137] or, more generally, [11]): for example, g may be supported only on a set of finite measure. Secondly, (1.1) is finite dimensional whilst the system class of the present paper encompasses – via the generality of the operators T allowable in (1.3) – infinite-dimensionality (e.g. delays, both point and distributed) and hysteretic effects (e.g. backlash, Prandtl and Preisach hysteresis). We elaborate on such examples in Appendix A. In essence, the system class of the present paper consists of systems satisfying considerably weaker counterparts of the assumptions of relative-degree-one and stable zero dynamics alluded to above, viz. we assume only that (i) T has a bounded-input, bounded-output property and (ii) for each fixed pair (d,w), $\limsup_{v\to\infty} b(d,w)g(v) = +\infty$ and $\liminf_{v\to\infty} b(d,w)g(-v) = -\infty$.

Many approaches in the literature, both adaptive or non-adaptive, are concerned with asymptotic behaviour of solutions of the feedback system. In contrast, the present paper is concerned with both asymptotic and transient performance of the feedback system. In particular, the *control objective* is to ensure that the tracking error, i.e. the difference between output and reference signal, evolves within a prespecified funnel, which is “shaped” to ensure the requisite transients and asymptotics. With reference to Figure 1, we remark that the funnel radius is not permitted to shrink to zero at infinity: it may, however, approach an arbitrarily small prescribed value $\lambda > 0$, thereby ensuring tracking with prescribed asymptotic accuracy λ. The main contribution of the paper is to establish that the above objective is achieved by the application of a variant of a so-called ‘funnel controller’, introduced in [5]. Novel features of funnel

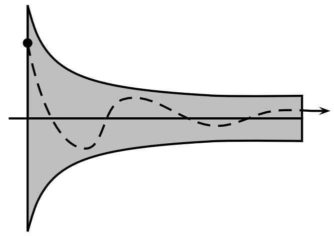

Figure 1: Performance funnel $\mathcal{F}_\varphi$

control include the following.

– The control law is a simple time-varying error feedback of the form $u(t) = -k(t)\, e(t)$, where $e = y - r$ denotes the tracking error between the output y and a given reference signal r and the gain function k is generated by a feedback of the form $k(t) = f\big(t, e(t), |e(0)|\big)$. The intuition underpinning this control structure is exploiting an inherent high-gain property of the system class in order to maintain the error evolution within the funnel by ensuring that, if the error approaches the funnel boundary, then the gain takes values sufficiently large to preclude contact with the boundary. Whilst the control exploits an inherent high-gain property, it is not a high-gain controller in the usual sense: in particular, and in contrast to high-gain adaptive control methodologies, k is not monotone and decreases as the error recedes from the funnel boundary.

– The gain k is adapted but $u(t) = -k(t)\, e(t)$ is not an adaptive controller in the usual sense: in particular, and in contrast to (1.2), it is non-dynamic.

– The approach does not invoke any identification mechanism or internal model principle.

Funnel control has been applied to temperature control in chemical reactor models [7], even in the presence of input constraints, and to speed control of electric drives [6], the latter has been tested successfully in the laboratory.

The main distinguishing feature of the present paper vis à vis previous contributions on funnel control (see, e.g. [5]) is twofold: (a) a larger system class can be accommodated – in particular, nonlinearities g of a general nature can be tolerated in the input channel; (b) a wider choice of formulations of prescribed transient behaviour is encompassed, including, for example, a variant of (M, μ)-stability (see [3, Section 5.5]).

The paper is organized as follows. In Section 2, we make precise the underlying system class and the control objective is formulated in Section 3: illustrative examples are provided in Appendix A. The main result is presented in Section 4, wherein the closed-loop system gives rise to an initial-value problem for a functional differential equation: the nature of this problem is such that it falls outside the scope of existence theories in the literature known to the authors. For this reason, an existence theory – of sufficient generality to include the closed-loop initial-value problem – is developed in Appendix B. The paper concludes with a numerical simulation.

2 System class

We consider single-input, single-output systems described by a functional differential equation of the form (1.3) and having the general structure depicted in Figure 2, wherein T is a causal operator and d_1, d_2 and d_3 are extraneous disturbances. For example, as already shown in the Introduction, the initial-value problem (1.1) may be reformulated (in terms of the input and output variables) as an initial-value problem for a disturbance-free system of the form depicted in Figure 2 with $g = \mathrm{id}$ (the identity map on $\mathbb{R}$). Other examples can be found in Appendix A.

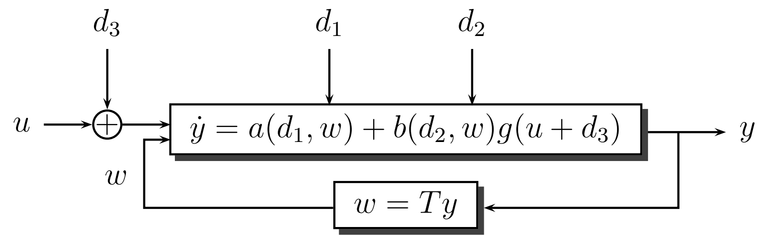

Figure 2: The open loop system

We proceed to a description of the general system class, first making precise the associated class of operators T. Throughout, $L^\infty(\mathbb{R}_+, \mathbb{R}^\ell)$ is the space of measurable, essentially bounded functions $\mathbb{R}_+ \to \mathbb{R}^\ell$, with norm given by $\|y\|_\infty := \operatorname{ess\,sup}_{t\in\mathbb{R}_+} \|y(t)\|$; the space of measurable, locally essentially bounded functions $\mathbb{R}_+ \to \mathbb{R}^\ell$ is denoted by $L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^\ell)$; $W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^\ell)$ is the space of absolutely continuous functions $y\colon \mathbb{R}_+ \to \mathbb{R}^\ell$ with $y, \dot y \in L^\infty(\mathbb{R}_+, \mathbb{R}^\ell)$.

Definition 2.1 (Operator class $\mathcal{T}_{\mathbf{h}}^{\mathbf{q}}$) *For $h, t \in \mathbb{R}_+$, $w \in C[-h, t]$, $\tau > t$ and $\delta > 0$, define*

$$\mathcal{C}(w; h, t, \tau, \delta) := \big\{ v \in C[-h, \tau] \;\big|\; v|_{[-h,t]} = w,\; |v(s) - w(t)| \le \delta \;\; \forall\, s \in [t, \tau] \big\},$$

that is, the space of all continuous extensions v of $w \in C[-h, t]$ to the interval $[-h, \tau]$ with the property that $|v(s) - w(t)| \le \delta$ for all $s \in [t, \tau]$.
An operator T is said to be of class $\mathcal{T}_h^q$, for some $q \in \mathbb{N}$, if, and only if, the following hold.

(i) $T\colon C[-h, \infty) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+, \mathbb{R}^q)$. (ii) *T is a causal operator.*

(iii) *For each $t \ge 0$ and each $w \in C[-h, t]$, there exist $\tau > t$, $\delta > 0$ and $c_0 > 0$ such that*

$$\text{ess-sup}_{s\in[t,\tau]} \|(Ty)(s) - (Tz)(s)\| \;\le\; c_0 \max_{s\in[t,\tau]} |y(s) - z(s)| \quad \forall\, y, z \in \mathcal{C}(w; h, t, \tau, \delta)\,.$$

(iv) *For every $c_1 > 0$ there exists $c_2 > 0$ such that, for all $y \in C[-h, \infty)$,*

$$\sup_{t\in[-h,\infty)} |y(t)| \le c_1 \quad \Longrightarrow \quad \|(Ty)(t)\| \le c_2 \quad \textit{for a.a. } t \ge 0\,.$$

Remark 2.2 Property (iii) is a technical assumption of local Lipschitz type which is used in establishing well-posedness of the closed-loop system. To interpret (iii) correctly, we need to give meaning to Ty, for a function $y \in C(I)$ on a bounded interval I of the form $[-h, \rho)$ or $[-h, \rho]$, where $0 < \rho < \infty$. This we do by showing that T "localizes", in a natural way, to an operator $\tilde{T}\colon C(I) \to L^\infty_{\rm loc}(J,\mathbb{R}^q)$, where $J := I \setminus [-h,0)$. Let $y \in C(I)$. For each $\sigma \in J$, define $y_\sigma \in C[-h,\infty)$ by

$$y_\sigma(t) := \begin{cases} y(t), & t \in [-h,\sigma], \\ y(\sigma), & t > \sigma\,. \end{cases}$$

By causality, we may define $\tilde{T}y \in L^\infty_{\rm loc}(J,\mathbb{R}^q)$ by the property $\tilde{T}y|_{[0,\sigma]} = Ty_\sigma|_{[0,\sigma]}$ for all $\sigma \in J$. Henceforth, we will not distinguish notationally an operator T and its "localisation" $\tilde{T}$: the correct interpretation being clear from context.
Property (iv) is a bounded-input, bounded-output assumption on the operator T. This assumption is a weak counterpart of the "stable zero dynamics" assumption ubiquitous in the context of high-gain control of linear systems.

Definition 2.3 **(System class $\Sigma_h^{p,q}$)** *Let $p,q \in \mathbb{N}$ and $h \geq 0$. The functional differential equation*

$$\dot{y}(t) \;=\; a\big(d_1(t),(Ty)(t)\big) \;+\; b\big(d_2(t),(Ty)(t)\big)\; g\big(u(t)+d_3(t)\big), \tag{2.1}$$

defines a system of class $\Sigma_h^{p,q}$, written $(a,b,g,T,d_1,d_2,d_3) \in \Sigma_h^{p,q}$, if, and only if, the following hold.

(i) *$a\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is continuous.*

(ii) *$b\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is continuous and sign definite, that is,*

$$|b(d,s)| > 0 \qquad \forall\ (d,s) \in \mathbb{R}^p \times \mathbb{R}^q\,. \tag{2.2}$$

(iii) *$g : \mathbb{R} \to \mathbb{R}$ is continuous with*

$$\text{(a)} \quad \limsup_{v\to\infty} b_1 g(v) = +\infty, \qquad \text{(b)} \quad \liminf_{v\to\infty} b_1 g(-v) = -\infty\,, \tag{2.3}$$

where $b_1 :=$ sgn(b), the polarity of the sign-definite function b.

(iv) *$T \in \mathcal{T}_h^q$.* (v) *$d_1, d_2 \in L^\infty(\mathbb{R}_+,\mathbb{R}^p)$, $d_3 \in W^{1,\infty}(\mathbb{R}_+)$.*

Remark 2.4 Some remarks on the nature of the input nonlinearity are warranted. The function $g \in C(\mathbb{R},\mathbb{R})$ can be interpreted in two distinct ways.
(i) The function g may form part of the overall control structure in the sense that it is a synthesizable element which may be designed to compensate for lack of knowledge of the sgn(b) of the input connection function b. In this context, the role of g is akin to that of a so-called "Nussbaum function" in adaptive control, see [15]. For example, the choice $g : u \mapsto u\cos u$ ensures that properties (2.3) hold.
(ii) Alternatively, g may be regarded as an uncertain intrinsic component of the plant, in which case, assumption (2.3) places some restrictions on the manner in which the functions g and b interact. In this context, note that g may influence/reverse the polarity of a control input u in a somewhat arbitrary manner. Note also that (assuming $d_3 = 0$ for simplicity) a control input u is nullified on the zero set $g^{-1}(0) \subset \mathbb{R}$ of g and the measure of this set may be infinite in the sense that the function g may be supported only on a set of finite Lebesgue measure: a simple example of such a function is a continuous unbounded odd function $g : \mathbb{R} \to \mathbb{R}$ with $g(u) := \sum_{n=1}^\infty \text{sgn}(b)\, g_n(u)$ for all $u \in \mathbb{R}_+$ (and so $g(u) = -g(-u)$ for all

$u < 0$), where, for each $n \in \mathbb{N}$, $g_n : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz function supported on $I_n := [n, n+2^{-n}]$ in which case, the measure of the set $\mathbb{R}\backslash g^{-1}(0)$ is bounded from above by the quantity $2\sum_{n=1}^{\infty} |I_n| = 2\sum_{n=1}^{\infty} 2^{-n} = 2$.
In more extreme cases, the function g may reverse the intended polarity of the control input: moreover, the "bad" set on which the polarity is reversed may be large in comparison with the "good" set on which polarity is maintained. In particular, assume (2.3) holds and let g^+ and g^- denote the positive and negative parts of $\operatorname{sgn}(b)g$, in which case $\operatorname{sgn}(b)g = g^+ - g^-$; the Lebesgue measure of the support of g^- (the "bad" polarity-reversing set) may be infinite, whilst the support of g^+ (the "good" set) may have only finite measure. Since, in this second context, knowledge of g is not available to the controller, it is perhaps counter-intuitive that the approximate tracking objective, as described in the next section, is achievable in the presence of input nonlinearities of such generality.

3 The control objective and performance funnel

Let $(a, b, g, T, d_1, d_2, d_3) \in \Sigma_h^{p,q}$ and consider the initial-value problem

$$\dot{y}(t) = a\big(d_1(t), (Ty)(t)\big) + b\big(d_2(t), (Ty)(t)\big)\, g\big(u(t) + d_3(t)\big), \quad y|_{[-h,0]} = y^0 \in C[-h, 0]\,.$$

The control objective is to design a simple tracking error feedback controller of the form $u(t) = -k(t)e(t)$, with gain $k(t)$ also generated by feedback of the error $e(t) = y(t) - r(t)$, so that, for all initial functions $y^0 \in C[-h, 0]$ and all reference signals $r \in W^{1,\infty}(\mathbb{R}_+)$, every solution of the closed-loop initial-value problem is bounded and approximate tracking with prescribed asymptotic accuracy and transient behaviour is achieved in the sense that the tracking error satisfies an *a priori* bound and asymptotically approaches a prescribed (arbitrarily small) neighbourhood of zero. The prescription of asymptotic and transient behaviour is formulated, in a manner to be made precise, via the following class Ψ_λ of functions $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$.

Definition 3.1 (Function class Ψ) *A continuous function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class Ψ if, and only if, the following hold:*
(i) $\psi(t, \zeta) > 0\ \forall\, t > 0\ \forall\, \zeta \geq 0$; (ii) $\psi(\cdot, \zeta) \in W^{1,\infty}(\mathbb{R}_+)\ \forall\, \zeta \geq 0$; (iii) $\psi(0, \zeta) < \zeta^{-1}\ \forall\, \zeta > 0$.

Let $\psi \in \Psi$, $y^0 \in C[-h, 0]$ and $r \in W^{1,\infty}(\mathbb{R}_+)$ be arbitrary, and write $e^0 := y^0(0) - r(0)$. Then the performance objective of prescribed asymptotic and transient behaviour of the tracking error e is now specified in a predetermined manner through choice of the function ψ and captured by the requirement that

$$\psi(t, |e^0|)|e(t)| < 1 \quad \forall\, t \in \mathbb{R}_+\,. \tag{3.1}$$

In addition, if, for some prescribed $\lambda > 0$ arbitrarily small, ψ satisfies

$$\limsup_{t\to\infty} [\psi(t, \zeta)]^{-1} \leq \frac{1}{\lambda} \quad \forall\, \zeta \geq 0\,, \tag{3.2}$$

then asymptotic tracking accuracy $\limsup_{t\to\infty} |e(t)| \leq \lambda$ is achieved. With reference to Figure 1 and writing $\varphi(\cdot) = \psi(\cdot, |e^0|) \in W^{1,\infty}(\mathbb{R}_+)$, we see that (3.1) may, in turn, be identified as the requirement that the tracking error should evolve within a performance funnel

$$\mathcal{F}_\varphi := \text{graph}\ \Big(t \mapsto \big\{z \in \mathbb{R} \big|\ \ \varphi(t)|z| < 1\big\}\Big),$$

i.e. the graph of a set-valued map defined on $\mathbb{R}_+$, the value of which, at $t \in \mathbb{R}_+$, is the interval $(-1/\varphi(t)\,,\, 1/\varphi(t))$. Note that the boundary of $\mathcal{F}_\varphi$ is determined by the reciprocal of φ.

Example 3.2

(A) Fix $\lambda > 0$ and choose $\varphi \in W^{1,\infty}(\mathbb{R}_+)$ such that $\varphi(0) = 0$, $\varphi(t) \in (0, 1/\lambda)$ for all $t > 0$ and $\lim_{t\to\infty} \varphi(t) = 1/\lambda$. Define $\psi \in \Psi$ by the property that $\psi(\cdot, \zeta) = \varphi(\cdot)$ for all $\zeta \in \mathbb{R}_+$. Then satisfaction of (3.1) (equivalently, error evolution within the funnel $\mathcal{F}_\varphi$) implies that

$$|e(t)| < 1/\varphi(t) \qquad \forall\, t > 0.$$

For example, the choice

$$t \mapsto \varphi(t) = \frac{\min\{t/\tau, 1\}}{\lambda}, \qquad \text{with} \;\; \tau, \lambda > 0,$$

ensures that the modulus of the error decays at rate $\tau\lambda/t$ in the "initial (transient) phase" $(0, \tau]$, and, since (3.2) holds, is bounded by λ in the "terminal phase" $[\tau, \infty)$.

(B) In this second example, and in contrast with Example (A) above, the function ψ has non-trivial dependence on its second argument: for $M > 1$, $\mu > 0$ and $\lambda > 0$, define $\psi \in \Psi$ by

$$\psi(t, \zeta) := 1/\max\{Me^{-\mu t}\zeta\,,\, \lambda\}, \qquad \forall\, t, \zeta \in \mathbb{R}_+.$$

In doing so, we adopt the objective of "practical (M, μ)-stability" of the tracking error in the sense that, for every $y^0 \in C[-h, 0]$ and $r \in W^{1,\infty}(\mathbb{R}_+)$, the tracking error $e = y - r$ (with $e^0 = e(0)$) is required to satisfy

$$|e(t)| < \max\big\{Me^{-\mu t}|e^0|\,,\, \lambda\big\} \qquad \forall\, t \geq 0.$$

For example, if $\lambda M|e^0| > 1$ and (3.1) holds, then, defining $\tau := \ln(\lambda M|e^0|)/\mu$, the tracking error decays at prescribed exponential rate in the "initial (transient) phase" $[0, \tau]$, and is bounded by λ in the "terminal phase" $[\tau, \infty)$.

4 Main result: funnel output feedback

Loosely speaking, funnel control exploits an inherent benign high-gain property of the system by designing – with appropriate choice of $\psi \in \Psi$ – a proportional error feedback $u(t) = -k(t)\, e(t)$ in such a way that $k(t)$ becomes large if $|e(t)|$ approaches the performance funnel boundary (equivalently, if $\psi(t, |e(0)|)|e(t)|$ approaches the value 1), thereby precluding contact with the funnel boundary. We emphasize that the gain is non-monotone and decreases as the error recedes from the funnel boundary. The essence of the proof of the main result lies in showing that the closed-loop system is well-posed in the sense that u and k are bounded functions and the error evolves strictly within the performance funnel.

For $\psi \in \Psi$, the "funnel controller" can be expressed in its simplest form as

$$u(t) = -k(t)\, e(t), \qquad k(t) = \frac{\psi(t, |e(0)|)}{1 - \psi(t, |e(0)|)|e(t)|}\,. \tag{4.1}$$

This form is a special case of a more general structure

$$u(t) = -k(t)\, e(t), \qquad k(t) = \alpha\big(\psi(t, |e(0)|)|e(t)|\big)\psi(t, |e(0)|)\,, \tag{4.2}$$

wherein α is any continuously differentiable unbounded injection $[0,1) \to \mathbb{R}_+$ with $\alpha(0) > 0$. Note that $\alpha(s) \to \infty$ as $s \uparrow 1$ and the choice $\alpha : s \mapsto 1/(1-s)$ yields (4.1). Let $p, q \in \mathbb{N}$, $h \geq 0$ and consider control (4.2) applied to the system $(a, b, g, T, d_1, d_2, d_3) \in \Sigma_h^{p,q}$. In view of the potential for "blow up" in the gain generation in (4.2), some care must be exercised in formulating the closed-loop system. Introducing

$$\Omega := \{(t, z, \zeta) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \mid \ \psi(t, \zeta)|z| < 1\}, \tag{4.3}$$

we define

$$f : \Omega \to \mathbb{R}, \quad (t, z, \zeta) \to f(t, z, \zeta) := \alpha\big(\psi(t, \zeta)|z|\big)\psi(t, \zeta)\,, \tag{4.4}$$

in which case, the control (4.1) can be interpreted, in explicit feedback form, as

$$u(t) = -f(t, e(t), |e(0)|)\ e(t). \tag{4.5}$$

Let $y^0 \in C[-h, 0]$ and $r \in W^{1,\infty}(\mathbb{R}_+)$ be arbitrary and write $e^0 := y^0(0) - r(0)$. The closed-loop initial-value problem now takes the form

$$\left.\begin{array}{l} \dot{y}(t) = a\big(d_1(t), (Ty)(t)\big) + b\big(d_2(t), (Ty)(t)\big)\ g\big(d_3(t) - f(t, y(t) - r(t), |e^0|)e(t)\big), \\ y|_{[-h,0]} = y^0 \in C[-h, 0]. \end{array}\right\} \tag{4.6}$$

Setting $\varphi(\cdot) := \psi(\cdot, |e^0|)$ (with associated performance funnel $\mathcal{F}_\varphi$), (4.6) may, in turn, be rewritten as

$$\dot{y}(t) = F(t, y(t), (Ty)(t)), \quad y|_{[-h,0]} = y^0 \in C[-h, 0], \tag{4.7}$$

where

$$F \colon \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}, \quad \mathcal{D} := \{(t, v) \in \mathbb{R}_+ \times \mathbb{R} \mid \ (t, v - r(t)) \in \mathcal{F}_\varphi\}, \tag{4.8}$$

is a Carathéodory function (see App. B for the definition) given by

$$F(t, v, w) := a(d_1(t), w) + b(d_2(t), w)g\big(d_3(t) - f(t, v - r(t), |e^0|)(v - r(t))\big). \tag{4.9}$$

By a *solution* of (4.7) we mean a function $y \in C[-h, \omega)$, $0 < \omega \leq \infty$, such that $y|_{[-h,0]} = y^0$, $y|_{[0,\omega)}$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$ and $\dot{y}(t) = F(t, y(t), (\tilde{T}y)(t))$ for almost all $t \in [0, \omega)$. A solution is said to be *maximal* if it has no proper right extension that is also a solution. A solution defined on $[-h, \infty)$ is said to be *global*.

In Appendix B, we develop an existence theory of sufficient generality to encompass the closed-loop initial-value problem (4.7): this theory is a variant of that in [5] – the distinguishing feature of the present paper resides in the nature of the domain of the function F in (4.8) which places the initial-value problem (4.7) outside the scope of the existence theory in [5].

Now we are in a position to state the main result.

Theorem 4.1 *Let $\psi \in \Psi$ specify the prescribed transient behaviour Let $\alpha : [0,1) \to \mathbb{R}_+$ be a continuously differentiable unbounded injection with $\alpha(0) > 0$, and let $r \in W^{1,\infty}(\mathbb{R}_+)$ and $y^0 \in C[-h, 0]$ be arbitrary. Then the "funnel controller" (4.2) applied to any system $(a, b, g, T, d_1, d_2, d_3) \in \Sigma_h^{p,q}$, with $p, q \in \mathbb{N}$, $h \geq 0$ and initial data $y^0 \in C[-h, 0]$, is such that the resulting closed-loop initial-value problem has a solution and every solution can be extended to a global solution. Every global solution y has the properties:*

(a) *the functions y and u (given by (4.5) with $e := y - r$) are bounded;*

(b) *there exists* $\varepsilon \in (0,1)$ *such that*

$$|e(t)| \;\le\; \frac{1-\varepsilon}{\psi(t,|e(0)|)} \qquad \forall\, t > 0\,.$$

Remark 4.2 Assertion (b) of Theorem 4.1 is its essence. Writing $\varphi(\cdot) := \psi(\cdot, |e(0)|)$, it asserts that the tracking error evolves within the performance funnel $\mathcal{F}_\varphi$ as depicted in Figure 1; moreover, the error evolution is strictly bounded away from the funnel boundary, thereby ensuring that the gain function k and the control function u in (4.1) are bounded.

Proof of Theorem 4.1:
Write $e^0 := e(0) = y^0(0) - r(0)$ and $\varphi(\cdot) := \psi(\cdot, |e^0|) \in W^{1,\infty}(\mathbb{R}_+)$ (by Definition 3.1(ii)). We have seen that the closed-loop initial-value problem (4.6) may be expressed in the form (4.7). Invoking Theorem 7.1 of Appendix B, we may conclude that (4.7) has a solution and every solution can be extended to a maximal solution; moreover (noting that F is locally essentially bounded), if $y : [-h, \omega) \to \mathbb{R}$ is a maximal solution, then the closure of $\text{graph}\big(y|_{[0,\omega)}\big)$ is not a compact subset of $\mathcal{D}$.

Let $y : [-h, \omega) \to \mathbb{R}$, $0 < \omega \le \infty$, be a maximal solution. Then $e := y - r$ is bounded with $\varphi(t)|e(t)| < 1$, for all $t \in [0, \omega)$. Since r is bounded, it follows that $y = e + r$ is bounded and so, by property (iv) of $T \in \mathcal{T}_h^q$, the function $w := Ty$ is also bounded. Define $c : [0, \omega) \to \mathbb{R}$ by $c(t) := a(d_1(t), w(t)) - \dot r(t)$. By continuity of a, boundedness of w, and essential boundedness of $\dot r$ and d_1, it follows that $c \in L^\infty[0, \omega)$. By (4.7) and (4.9), we have

$$\dot e(t) = c(t) + b\big(d_2(t), w(t)\big)\; g\big(u(t) + d_3(t)\big) \quad \text{for a.a. } t \in [0, \omega). \tag{4.10}$$

Let $\beta : [0,1) \to \mathbb{R}_+$ be the continuously differentiable bijection given by $\beta(s) := s\alpha(s)$; we record that $\beta'(s) = \alpha(s) + s\alpha'(s) \ge \alpha(0) > 0$ for all $s \in \mathbb{R}_+$. Write

$$\kappa(t) := \beta(\varphi(t)|e(t)|) \quad \forall\, t \in [0, \omega),$$

in which case, in view of (4.2), we have

$$t \in [0, \omega), \;\; e(t) \ne 0 \qquad \Longrightarrow \qquad u(t) = -\kappa(t)\,\text{sgn}(e(t))\,.$$

Seeking a contradiction, suppose that the function κ is unbounded on $[0, \omega)$. Then there exists a strictly-increasing sequence (t_n) in $[0, \omega)$, with $t_n \uparrow \omega$ as $n \to \infty$, such that the sequence $\big(\kappa(t_n)\big)$ is a strictly-increasing unbounded sequence in $\mathbb{R}_+$ and $\big(\varphi(t_n)|e(t_n)|\big)$ is a sequence in $(0,1)$ with $\varphi(t_n)|e(t_n)| \to 1$ as $n \to \infty$. Since φ is bounded with $\varphi(t) > 0$ for all $t \in (0, \omega)$ and passing to a subsequence if necessary, we may infer the existence of $c_0 \in \{-1, 1\}$ such that $c_0 e(t_n) < 0$ for all $n \in \mathbb{N}$. By property (2.2) of the continuous function b, together with boundedness of w and essential boundedness of d_2, there exists $b_0 > 0$ such that

$$|b(d_2(t), w(t))| \ge b_0 \quad \text{for a.a. } t \in [0, \omega). \tag{4.11}$$

By properties (2.3) of g, there exist strictly-increasing unbounded sequences (u_n) and (v_n) in $\mathbb{R}_+$ such that

$$b_1\, g(u_n) \to \infty \;\; \text{and} \;\; -\, b_1\, g(-v_n) \to \infty \;\; \text{as } n \to \infty\,. \tag{4.12}$$

(Recall that $b_1 = \operatorname{sgn}(b)$, the polarity of the sign-definite function b.) Define the sequence (s_n) and the continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ by

$$s_n := \begin{cases} u_n, & \text{if } c_0 = +1 \\ -v_n, & \text{if } c_0 = -1 \end{cases}, \quad \gamma(s) := b_0 b_1 c_0\, g(s) \quad \forall\, s \in \mathbb{R}.$$

Clearly,

$$c_0 = +1 \implies \gamma(s_n) = b_0 b_1 g(u_n) \quad \text{and} \quad c_0 = -1 \implies \gamma(s_n) = -b_0 b_1 g(-v_n),$$

and so, invoking (4.12) and recalling that $b_0 > 0$, we may infer that $\gamma(s_n) \to \infty$ as $n \to \infty$. Passing to a subsequence if necessary, we may assume that $(\gamma(s_n))$ is a strictly-increasing sequence in $\mathbb{R}_+$. Now define the sequence $(\kappa_n) := (c_0 s_n)$. Observe that (κ_n) is a strictly-increasing unbounded sequence in $\mathbb{R}_+$ and so, extracting a subsequence (which we do not relabel), we may assume that $\kappa_n \geq 1 + \kappa(0) + \|d_3\|_\infty$ for all $n \in \mathbb{N}$. Again passing to a subsequence of (t_n) if necessary, we may also assume that $\kappa(t_n) \geq \kappa_{n+1}$ for all $n \in \mathbb{N}$. Define the sequence (t_n^*) by

$$t_n^* := \begin{cases} \sup T_n, & T_n \neq \emptyset, \\ 0, & T_n = \emptyset, \end{cases} \qquad \text{where} \quad T_n := \big\{ t \in [0, t_n] \,\big|\, \ e(t) = 0 \big\}.$$

Observe that $\kappa(t_n^*) + c_0 d_3(t_n^*) \leq \kappa(0) + \|d_3\|_\infty < \kappa_n$ for all $n \in \mathbb{N}$ and so the following are well defined for each $n \in \mathbb{N}$

$$\begin{aligned} \tau_n &:= \inf \big\{ t \in [t_n^*, t_n] \,\big|\, \ \kappa(t) + c_0 d_3(t) = \kappa_{n+1} \big\} \\ \sigma_n &:= \sup \big\{ t \in [t_n^*, \tau_n] \,\big|\, \ \gamma(c_0 \kappa(t) + d_3(t)) = \gamma(s_n) \big\} < \tau_n\,, \end{aligned}$$

wherein the strict inequality $\sigma_n < \tau_n$ holds because

$$\gamma(c_0 \kappa(\tau_n) + d_3(\tau_n)) = \gamma(c_0 \kappa_{n+1}) = \gamma(s_{n+1}) > \gamma(s_n).$$

Suppose that $\kappa(\sigma_n) + c_0 d_3(\sigma_n) \geq \kappa(\tau_n) + c_0 d_3(\tau_n)$ for some $n \in \mathbb{N}$. Then

$$\kappa(t_n^*) + c_0 d_3(t_n^*) < \kappa_{n+1} = \kappa(\tau_n) + c_0 d_3(\tau_n) \leq \kappa(\sigma_n) + c_0 d_3(\sigma_n)$$

and so, by continuity, there exists $s \in (t_n^*, \sigma_n]$ such that $\kappa(s) + c_0 d_3(s) = \kappa_{n+1}$, whence the contradiction:

$$\tau_n = \inf\{ t \in [t_n^*, t_n] |\ \kappa(t) + c_0 d_3(t) = \kappa_{n+1} \} \leq s \leq \sigma_n < \tau_n.$$

Therefore, $\kappa(\sigma_n) + c_0 d_3(\sigma_n) < \kappa(\tau_n) + c_0 d_3(\tau_n)$ for all $n \in \mathbb{N}$. Since $d_3 \in W^{1,\infty}(\mathbb{R}_+)$, there exists $c_1 > 0$ such that

$$\kappa(\sigma_n) < \kappa(\tau_n) + c_0(d_3(\tau_n) - d_3(\sigma)) \leq \kappa(\tau_n) + (\tau_n - \sigma_n) c_1 \quad \forall\, n \in \mathbb{N}\,. \tag{4.13}$$

By definition of σ_n, we have

$$\gamma(c_0 \kappa(t) + d_3(t)) > \gamma(s_n) \quad \forall\, t \in (\sigma_n, \tau_n], \quad \forall\, n \in \mathbb{N}\,. \tag{4.14}$$

We also record that

$$-|e(t)| = c_0 e(t) < 0 \quad \forall\, t \in [\sigma_n, \tau_n], \quad \forall\, n \in \mathbb{N}. \tag{4.15}$$

We may now conclude that

$$-c_0 b(d_2(t), w(t))g(u(t) + d_3(t)) = -|b(d_2(t), w(t))|\gamma(c_0\kappa(t) + d_3(t))/b_0$$
$$\leq -\gamma(c_0\kappa(t) + d_3(t)) \leq -\gamma(s_n) \quad \forall\, t \in [\sigma_n, \tau_n], \quad \forall\, n \in \mathbb{N}\,. \tag{4.16}$$

Observe that, for almost all $t \in [\sigma_n, \tau_n]$ and for all $n \in \mathbb{N}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)|e(t)|\big) = -c_0\big(\dot\varphi(t)e(t) + \varphi(t)\dot e(t)\big) = -c_0\big(\dot\varphi(t)e(t) + h(t) + b(d_2(t), w(t))g(u(t) + d_3(t))\big).$$

By boundedness of e, together with essential boundedness of $\dot\varphi$ and h, and invoking (4.16), we may conclude that, for some constant $c_2 > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)|e(t)|\big) \leq c_2 - \gamma(s_n) \quad \text{for a.a. } t \in [\sigma_n, \tau_n] \;\; \forall\, n \in \mathbb{N}. \tag{4.17}$$

Fix $n \in \mathbb{N}$ sufficiently large so that $\alpha(0)\big(c_2 - \gamma(s_n)\big) < -c_1$, in which case we have

$$\dot\kappa(t) = \beta'(\varphi(t)|e(t)|)\,\frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(t)|e(t)|\big) \leq \alpha(0)\big(c_2 - \gamma(s_n)\big) < -c_1 \quad \text{for a.a. } t \in [\sigma_n, \tau_n]$$

whence $\kappa(\tau_n) - \kappa(\sigma_n) < -(\tau_n - \sigma_n)c_1$, which contradicts (4.13). This proves boundedness of κ.

By boundedness of $t \mapsto \kappa(t) = \beta(\varphi(t)|e(t)|)$, we may conclude that $\sup_{t\in[0,\omega)} \varphi(t)|e(t)| < 1$, equivalently, there exists $\varepsilon \in (0,1)$ such that $\varphi(t)|e(t)| \leq 1 - \varepsilon$ for all $t \in [0, \omega)$.

It remains only to show that the solution $y : [-h, \omega) \to \mathbb{R}$ is global. Seeking a contradiction, suppose $\omega < \infty$. Then $\mathcal{K} := \{(t, y) \in \mathcal{D}|\;\; t \in [0, \omega],\; \varphi(t)|y - r(t)| \leq 1 - \varepsilon\}$ is a compact subset of $\mathcal{D}$ with the property $(t, y(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, which contradicts the fact that the closure of $\mathrm{graph}\big(y|_{[0,\omega)}\big)$ is not a compact subset of $\mathcal{D}$. Therefore, $\omega = \infty$. □

5 Illustrative simulation

Consider the system shown in Figure 3 consisting of a linear, single-input, single-output system (c, A, b) with state space $\mathbb{R}^n$, disturbance $d \in L^\infty(\mathbb{R}_+)$, a nonlinearity g in the input channel, and a feedback loop containing a hysteretic nonlinearity H :

$$\dot x(t) = Ax(t) + b\big(d(t) + (H(cx))(t) + g(u(t))\big), \quad x(0) = x^0 \in \mathbb{R}^n\,, \quad y(t) = cx(t). \tag{5.1}$$

As discussed in Section 6.2 of Appendix B, under the assumptions that the linear system

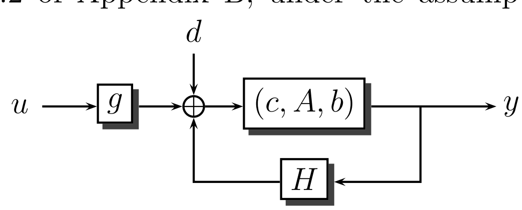

Figure 3: Linear system with hysteretic feedback loop and input nonlinearity

(c, A, b) has positive high-frequency gain $cb > 0$ and is minimum-phase, there exists a similarity transformation S that takes the triple into the form $(\hat{c}, \hat{A}, \hat{b})$, with

$$\hat{c} = cS^{-1} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \hat{A} = SAS^{-1} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \quad \hat{b} = Sb = \begin{pmatrix} cb \\ 0 \end{pmatrix},$$

and, in view of the minimum-phase assumption, $A_4 \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Hurwitz matrix. Writing $\begin{pmatrix} y^0 \\ z^0 \end{pmatrix} := Sx^0$, defining the operator T_1 by

$$(T_1 y)(t) := A_1 y(t) + A_2 \int_0^t (\exp A_4(t-s)) A_3 y(s) \mathrm{d}s$$

and writing $T_2 := cb\, H$, we see that (5.1) can be reformulated as

$$\dot{y}(t) = ((T_1 + T_2)(y)) + d_1(t) + cb\, g(u(t)), \quad y(0) = y^0, \quad d_1(t) := A_2\big(\exp A_4 t\big) z^0 + cb\, d(t). \tag{5.2}$$

Since A_2 is Hurwitz, it is readily verified that T_1 is in the operator class $\mathcal{T}_0^1$ and $d_1 \in L^\infty(\mathbb{R}_+)$. If we assume that the hysteresis operator H is also of class $\mathcal{T}_0^1$ (as discussed in Section 6.3 of Appendix A, many commonly encountered hysteretic components – including backlash and, more generally, Preisach operators – are of class $\mathcal{T}_0^1$), then $T := T_1 + T_2$ is of class $\mathcal{T}_0^1$. Defining $a\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tilde{b}\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $a(d, w) := d + w$ and $\tilde{b}(d, w) = cb$, (5.2) may be expressed as

$$\dot{y}(t) = a(d_1(t), (Ty)(t)) + \tilde{b}(0, (Ty)(t)) g(u(t)), \quad y(0) = y^0,$$

which is an initial-value problem for the system $(a, \tilde{b}, g, T, d_1, 0, 0)$ of class $\Sigma_0^{1,1}$. For purposes of illustration, as reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$ and disturbance $d \in W^{1,\infty}(\mathbb{R}_+)$, we take $r = \zeta_1$ and $d = \zeta_3$, where ζ_1 and ζ_3 are the first and third components of the (chaotic) solution of the following initial-value problem for the Lorenz system

$$\left.\begin{aligned} \dot{\zeta}_1(t) &= \zeta_2(t) - \zeta_1(t), & \zeta_1(0) &= 1, \\ \dot{\zeta}_2(t) &= c_0\zeta_1(t) - c_1\zeta_2(t) - \zeta_1(t)\zeta_3(t), & \zeta_2(0) &= 0, \\ \dot{\zeta}_3(t) &= \zeta_1(t)\zeta_2(t) - c_2\zeta_3(t), & \zeta_3(0) &= 3. \end{aligned}\right\} \tag{5.3}$$

with parameter values $c_0 = 28/10$, $c_1 = 1/10$ and $c_2 = 8/30$. It is well known that the unique global solution of (5.3) is bounded with bounded derivative, see for example [18].

Let (c, A, b) be given by

$$c = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

let g be given by $g(u) := (1 + u)|u|$, and let $H = \mathcal{B}_{\sigma,\xi}$ be the backlash hysteresis operator of Section 7.3 Appendix B, with parameter values $\sigma = 1/2$ and $\xi = 0$. We adopt the objective of "practical (M, μ)-stability", as described in Example 3.2 (B), with parameter values $\lambda = 0.02$, $\mu = 0.2$ and $M = 2$, and the simple control structure given by (4.1). For initial data $x^0 = 0$, Figure 4 depicts the evolution of the output y and reference signal r; Figure 5 depicts the error evolution within the funnel; Figures 6 and 7 show the control signal u and the gain k.

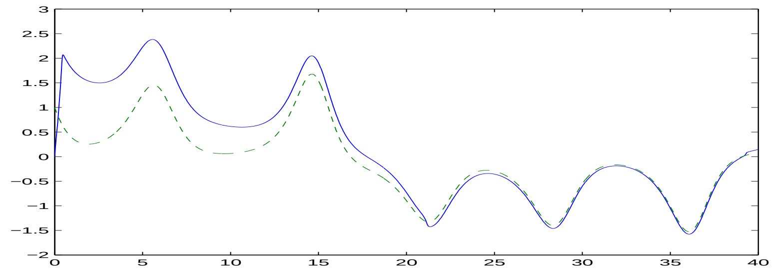

Figure 4: The output y (solid line) and reference r (dashed line)

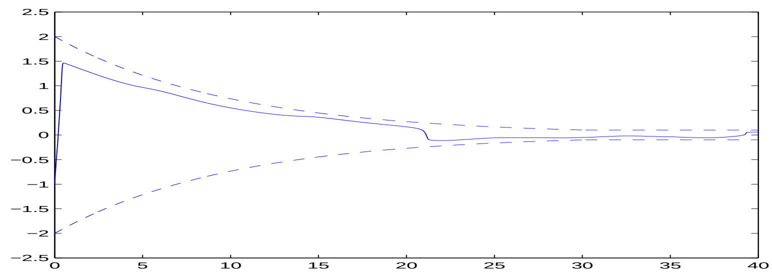

Figure 5: Error evolution within funnel

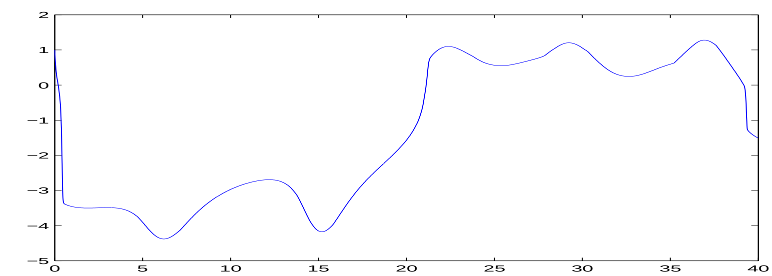

Figure 6: The control u

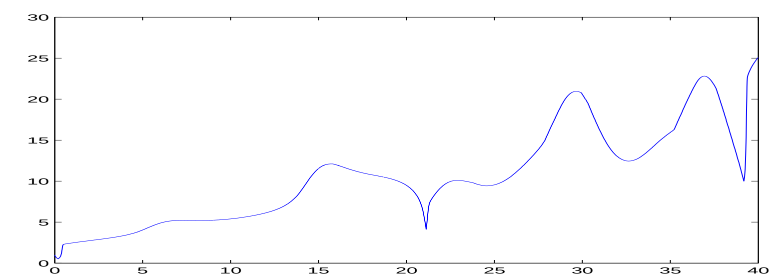

Figure 7: The gain function k

6 Appendix A: Examples of the system class $\Sigma_{\mathbf{h}}^{\mathbf{p},\mathbf{q}}$ and the operator class $\mathcal{T}_{\mathbf{h}}^{\mathbf{q}}$

6.1 Finite-dimensional nonlinear prototype

Consider again the initial-value problem for the nonlinear prototype system (1.1). In the Introduction, we have seen that, if the equation $\dot{z} = c(y,z)$ is assumed to generate a controlled semiflow ϕ, then with this equation we may associate a family of operators $T_\zeta \colon C(\mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R}^{n-1})$, parameterized by the initial data $\zeta \in \mathbb{R}^{n-1}$, given by $\big(T_\zeta y\big)(t) := \phi(t;\zeta, y(\cdot))$. Introducing $T \colon C(\mathbb{R}_+) \to C(\mathbb{R}_+, \mathbb{R} \times \mathbb{R}^{n-1})$ defined by $\big(Ty\big)(t) := \big(y(t), (T_\zeta)(t)\big)$, the initial-value problem (1.1) may be reformulated (in terms of the input and output variables) as

$$\dot{y}(t) = a\big((Ty)(t)\big) + b\big((Ty)(t)\big)u(t), \quad y(0) = \xi\,. \tag{6.1}$$

If, in addition, we assume that the system $\dot{z} = c(y,z)$ is input-to-state stable (ISS) (see, [17]), then it is readily verified that the operator T is of class $\mathcal{T}_0^n$. Assuming that the function b is positive-valued and bounded away from zero and introducing the functions $\tilde{a} \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\tilde{b} \colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ (these are simply convenient artifacts) given by

$$\tilde{a}(d,w) := d + a(w), \qquad \tilde{b}(d,w) := d + b(w),$$

we see that (6.1) is equivalent to

$$\dot{y}(t) = \tilde{a}\big(0, (Ty)(t)\big) + \tilde{b}\big(0, (Ty)(t)\big)u(t), \quad y(0) = \xi\,,$$

which is an initial-value problem for the system $(\tilde{a}, \tilde{b}, \mathrm{id}, T, 0, 0, 0)$ of class $\Sigma_0^{1,n}$. Therefore, under the assumptions that the system $\dot{z} = c(y,z)$ is ISS and b is positive-valued and bounded away from zero, Theorem 4.1 implies that, for all $(\zeta, \xi) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and all $r \in W^{1,\infty}(\mathbb{R}_+)$, the control (4.2) applied to (1.1) ensures attainment of the tracking objectives.

6.2 Linear (retarded) systems with input nonlinearities

Let $h > 0$, let A be an $n \times n$-matrix with entries in $BV[0,h]$ (the space of real-valued functions of bounded variation on $[a,b] \subset \mathbb{R}$) and let $b, c^T \in \mathbb{R}^n$. Consider the linear retarded system with nonlinearity g in the input channel

$$\dot{x} = \mathrm{d}A * x + bg(u)\,, \qquad x|_{[-h,0]} = x^0 - \in C([-h,0], \mathbb{R}^n), \qquad y = cx\,, \tag{6.2}$$

where $(\mathrm{d}A * x)(t) := \int_0^h \mathrm{d}A(\tau)x(t-\tau)$ for all $t \in \mathbb{R}_+$, satisfying

- *minimum-phase* condition, i.e.,

$$\det \begin{pmatrix} sI - \hat{A}(s) & -b \\ c & 0 \end{pmatrix} \neq 0 \quad \forall\, s \in \mathbb{C}\,, \mathrm{Re}(s) > 0, \quad \text{where } \hat{A}(s) := \int_0^h \exp(-s\tau)\mathrm{d}A(\tau)$$

- *positive high-frequency gain* condition, i.e., $cb > 0$
- $\limsup_{v\to\infty} g(v) = +\infty, \quad \liminf_{v\to\infty} g(-v) = -\infty\,.$

It is well-known that, under these assumptions, there exists a similarity transformation which takes system (6.2) into the form

$$\dot{y} = \mathrm{d}A_1 * y + \mathrm{d}A_2 * z + cb\, g(u)\,, \quad y|_{[-h,0]} = y^0\,, \tag{6.3a}$$

$$\dot{z} = \mathrm{d}A_3 * y + \mathrm{d}A_4 * z\,, \quad z|_{[-h,0]} = z^0\,, \tag{6.3b}$$

where, by the minimum-phase condition, A_4 has the property that

$$\det(sI - \hat{A}_{22}(s)) \neq 0 \quad \forall\, s \in \mathbb{C}\,, \mathrm{Re}(s) > 0, \tag{6.4}$$

see [4] for details. For given $z^0 \in C([-h,0], \mathbb{R}^{n-1})$ and given $\xi \in C[-h,\infty)$, let $z(\cdot; z_0, \xi)$ denote the unique solution of the initial-value problem

$$\dot{z} = \mathrm{d}A_4 * z + \mathrm{d}A_3 * \xi\,, \qquad z|_{[-h,0]} = z^0\,.$$

Defining the operator T and function d_1 by

$$T(\xi) := \mathrm{d}A_1 * \xi + \mathrm{d}A_2 * z(\,\cdot\,; 0, \xi)\,, \quad d_1 := \mathrm{d}A_2 * z(\,\cdot\,; z_0, 0)\,,$$

equation (6.3a) can be expressed as

$$\dot{y} = d_1 + T(y) + cb\, g(u)\,, \qquad y^0 = cx^0\,. \tag{6.5}$$

By the standard theory of retarded functional differential equations (see [2, Corollary 6.1, p. 215]), (6.4) implies that the zero solution of the retarded equation $\dot{z} = \mathrm{d}A_4 * z$ is exponentially stable, so that there exists $K > 0$ such that, for all $z^0 \in C([-h,0], \mathbb{R}^{n-1})$ and all $\xi \in C[-h,\infty)$,

$$\sup_{t\in[0,\infty)} |z(t; z_0, \xi)| \leq K\big(\sup_{t\in[-h,0]} |z^0(t)| + \sup_{t\in[-h,\infty)} |\xi(t)|\big)\,.$$

We conclude that d is bounded and that $T \in \mathcal{T}_h^1$. Finally, defining $a\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\tilde{b}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ (as in the previous example, these are simply artifacts) by $a(d,w) := d + w$ and $\tilde{b}(d,w) = cb$, we see that (6.5) is equivalent to

$$\dot{y}(t) = a\big(d_1(t), (Ty)(t)\big) + \tilde{b}\big(0, (Ty)(t)\big)u(t), \quad y^0 = cx^0\,,$$

which is an initial-value problem for the system $(a, \tilde{b}, g, T, d_1, 0, 0)$ of class $\Sigma_h^{1,(n-1)}$.
The above example is readily modified to include the non-retarded case $h = 0$. In this case, we simply replace $\mathrm{d}A * x$ by Ax ($A \in \mathbb{R}^{n\times n}$), and $\mathrm{d}A_1 * y$, $\mathrm{d}A_2 * z$, $\mathrm{d}A_3 * y$, $\mathrm{d}A_4 * z$ by $A_1, \ldots, A_4$ at the appropriate places. In this way, we see that the class of linear, single-input, single-output, minimum-phase, relative-degree-one systems (c, A, b), with $cb > 0$ and with nonlinearity g in the input channel, is subsumed by our system class $\Sigma_0^{1,1}$.

6.3 Systems with hysteresis

An operator $T : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is a *hysteresis operator* if it is causal and rate independent. Here *rate independence* means that $T(y \circ \zeta) = (Ty) \circ \zeta$ for every $y \in C(\mathbb{R}_+)$ and every time transformation ζ, where $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a *time transformation* if it is continuous, non-decreasing and surjective. The so-called Preisach operators are among the most general and most important hysteresis operators: in particular, they can model complex hysteresis effects such as nested loops in input-output characteristics.

A basic building block for these operators is the *backlash* operator. A discussion of the *backlash* operator (also called *play* operator) can be found in a number of references, see for example [1], [10] and [12]. Let $\sigma \in \mathbb{R}_+$ and introduce the function $b_\sigma : \mathbb{R}^2 \to \mathbb{R}$ given by

$$b_\sigma(v_1, v_2) := \max\bigl\{v_1 - \sigma\,,\, \min\{v_1 + \sigma, v_2\}\bigr\}.$$

Let $C_{\mathrm{pm}}(\mathbb{R}_+)$ denote the space of continuous piecewise monotone functions defined on $\mathbb{R}_+$. For all $\sigma \in \mathbb{R}_+$ and $\xi \in \mathbb{R}$, define the operator $\mathcal{B}_{\sigma,\,\xi} : C_{\mathrm{pm}}(\mathbb{R}_+) \to C(\mathbb{R}_+)$ by

$$\mathcal{B}_{\sigma,\,\xi}(y)(t) = \begin{cases} b_\sigma(y(0), \xi) & \text{for } t = 0\,, \\ b_\sigma(y(t), (\mathcal{B}_{\sigma,\,\xi}(u))(t_i)) & \text{for } t_i < t \le t_{i+1},\ i = 0, 1, 2, \dots\,, \end{cases}$$

where $0 = t_0 < t_1 < t_2 < \dots$, $\lim_{n\to\infty} t_n = \infty$ and u is monotone on each interval $[t_i, t_{i+1}]$. We remark that ξ plays the role of an "initial state". It is not difficult to show that the definition is independent of the choice of the partition (t_i). Figure 8 illustrates how $\mathcal{B}_{\sigma,\,\xi}$ acts. It is well-

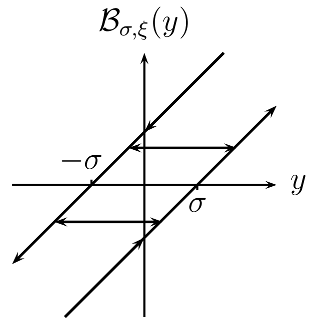

Figure 8: Backlash hysteresis

known that $\mathcal{B}_{\sigma,\,\xi}$ extends to a Lipschitz continuous operator on $C(\mathbb{R}_+)$ (with Lipschitz constant $L = 1$), the so-called backlash operator, which we shall denote by the same symbol $\mathcal{B}_{\sigma,\,\xi}$. It is well-known that $\mathcal{B}_{\sigma,\,\xi}$ is a hysteresis operator.

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be a compactly supported and globally Lipschitz function with Lipschitz constant 1. Let μ be a signed Borel measure on $\mathbb{R}_+$ such that $|\mu|(K) < \infty$ for all compact sets $K \subset \mathbb{R}_+$, where $|\mu|$ denotes the total variation of μ. Denoting Lebesgue measure on $\mathbb{R}$ by μ_L, let $w : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ be a locally $(\mu_L \otimes \mu)$-integrable function and let $w_0 \in \mathbb{R}$. The operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ defined by

$$(\mathcal{P}_\xi(y))(t) = \int_0^\infty \int_0^{(\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)} w(s,\sigma)\mu_L(\mathrm{d}s)\mu(\mathrm{d}\sigma) + w_0 \quad \forall\, y \in C(\mathbb{R}_+)\,,\ \forall\, t \in \mathbb{R}_+\,, \tag{6.6}$$

is called a *Preisach* operator, cf. [1, p. 55]. It is well-known that $\mathcal{P}_\xi$ is a hysteresis operator (this follows from the fact that $\mathcal{B}_{\sigma,\,\xi(\sigma)}$ is a hysteresis operator for every $\sigma \ge 0$). Under the assumption that the measure μ is finite and w is essentially bounded, the operator $\mathcal{P}_\xi$ is Lipschitz continuous with Lipschitz constant $L = |\mu|(\mathbb{R}_+)\|w\|_\infty$ (see [12]) in the sense that

$$\sup_{t\in\mathbb{R}_+} |\mathcal{P}_\xi(y_1)(t) - \mathcal{P}_\xi(y_2)(t)| \le L \sup_{t\in\mathbb{R}_+} |y_1(t) - y_2(t)| \quad \forall\, y_1, y_2 \in C(\mathbb{R}_+).$$

This property ensures that the Preisach operator belongs to our operator class $\mathcal{T}_0^1$.

Setting $w(\cdot,\cdot) = 1$ and $w_0 = 0$ in (6.6), yields the *Prandtl* operator $\mathcal{P}_\xi : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ given by

$$\mathcal{P}_\xi(y)(t) = \int_0^\infty (\mathcal{B}_{\sigma,\,\xi(\sigma)}(y))(t)\mu(\mathrm{d}\sigma) \quad \forall\, y \in C(\mathbb{R}_+)\,, \ \ \forall\, t \in \mathbb{R}_+\,. \tag{6.7}$$

For $\xi \equiv 0$ and μ given by $\mu(E) = \int_E \chi_{[0,5]}(\sigma)\mathrm{d}\sigma$ (where $\chi_{[0,5]}$ denotes the indicator function of the interval $[0,5]$), the Prandtl operator is of class $\mathcal{T}_0^1$ and is illustrated in Figure 9. These

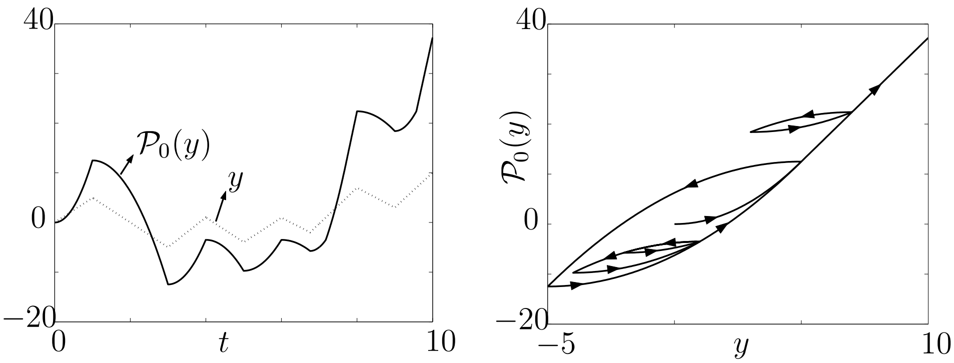

Figure 9: Example of Prandtl hysteresis

examples serve to illustrate that systems (1.3) incorporating rather general hysteresis operators T fall within the scope of our theory.

7 Appendix B: Existence theory

Let $\mathcal{D}$ be a domain in $\mathbb{R}_+ \times \mathbb{R}$ (that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}$). Let $q \in \mathbb{N}$ and assume that $F : \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}$ is a Carathéodory function[1]. Let $T \in \mathcal{T}_h^q$ and $t_0 \in \mathbb{R}_+$. Consider the initial-value problem

$$\dot{y}(t) = F(t, y(t), (Ty)(t)), \quad y|_{[-h,t_0]} = y^0 \in C[-h, t_0], \quad (t_0, y^0(t_0)) \in \mathcal{D}. \tag{7.1}$$

A *solution* of (7.1) is a function $y \in C(I)$ on an interval of the form $I = [-h, \rho]$, $t_0 < \rho < \infty$, or $[-h, \omega)$, $t_0 < \omega \le \infty$, such that $y|_{[-h,t_0]} = y^0$, $y|_J$ is locally absolutely continuous, with $(t, y(t)) \in \mathcal{D}$ for all $t \in J$ and $\dot{y}(t) = F(t, y(t), (Ty)(t))$ for almost all $t \in J$, where $J := I \setminus [-h, t_0]$. A solution is *maximal* if it has no proper right extension that is also a solution.

Theorem 7.1 *For all initial data* $(t_0, y^0) \in \mathbb{R}_+ \times C[-h, t_0]$ *with* $(t_0, y^0(t_0)) \in \mathcal{D}$,

(i) *the initial-value problem* (7.1) *has a solution,*

(ii) *every solution can be extended to a maximal solution* $y \in C[-h, \omega)$,

(iii) *if* F *is locally essentially bounded and* $y \in C[-h, \omega)$ *is a maximal solution, then the closure of* $\mathrm{graph}\big(y|_{[t_0,\omega)}\big)$ *is not a compact subset of* $\mathcal{D}$.

[1] Let $\mathcal{D}$ be a *domain* in $\mathbb{R}_+ \times \mathbb{R}$ (that is, a non-empty, connected, relatively open subset of $\mathbb{R}_+ \times \mathbb{R}$). A function $F\colon \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}$, is deemed to be a *Carathéodory function* if, for every "rectangle" $[a,b] \times [c,d] \subset \mathcal{D}$ and every compact set $K \subset \mathbb{R}^q$, the following hold: (i) $F(t,\cdot,\cdot) : [c,d] \times K \to \mathbb{R}$ is continuous for all $t \in [a,b]$; (ii) $F(\cdot, x, w) : [a,b] \to \mathbb{R}$ is measurable for each fixed $(x,w) \in [c,d] \times K$; (iii) there exists an integrable function $\gamma : [a,b] \to \mathbb{R}_+$ such that $|F(t,x,w)| \le \gamma(t)$ for almost all $t \in \mathbb{R}_+$ and all $(x,w) \in [c,d] \times K$.

Proof. By Property (iii) of the class $\mathcal{T}_h^q$, there exist $\tau > t_0$, $\delta > 0$ and $c_0 > 0$ such that

$$\text{ess-sup}_{s\in[t,\tau]}\|(Ty)(s) - (Tz)(s)\| \;\le\; c_0 \max_{s\in[t,\tau]}|y(s) - z(s)| \qquad \forall\, y, z \in \mathcal{C}(y^0; h, t, \tau, \delta)\,.$$

We may assume that $\delta \in (0,1)$ and $\tau - t_0 > 0$ are sufficiently small so that

$$\mathcal{D}_0 := [t_0, \tau] \times [y^0(t_0) - \delta\,,\, y^0(t_0) + \delta] \subset \mathcal{D}.$$

By Property (iv) of $\mathcal{T}_h^q$, there exists $c_2 > 0$ such that

$$\forall\, y \in C[-h,\infty) \;\&\; \text{a.a. } t \in [t_0,\tau] : \quad \sup_{t\in[-h,\infty)} |y(t)| < c_1 := \delta + \|y^0\|_\infty \implies \|(Ty)(t)\| < c_2\,.$$

Since F is a Carathéodory function, there exists integrable $\gamma : [t_0,\tau] \to \mathbb{R}_+$ such that

$$|F(t,\xi,\zeta)| \le \gamma(t) \quad \forall\, (t,\xi,\zeta) \in \mathcal{D}_0 \times \{\zeta \in \mathbb{R}^q |\; \|\zeta\| < c_2\} \tag{7.2}$$

Define $\Gamma \in C[-h,\tau]$ by

$$\Gamma(t) := \begin{cases} 0, & t \in [-h, t_0) \\ \int_{t_0}^t \gamma(s)\,\mathrm{d}s, & t \in [t_0,\tau]. \end{cases}$$

Since Γ is continuous and non-decreasing with $\Gamma(t_0) = 0$, there exists $\rho \in (t_0,\tau)$ such that $\Gamma(\rho) \in [0,\delta)$. We will establish the existence of a solution of the initial-value problem (7.1) on the interval $[-h,\rho]$. This we do by constructing a sequence (y_n) in $C[-h,\rho]$ with a subsequence converging to a solution $y \in C[-h,\rho]$ of (7.1). Let $n \in \mathbb{N}$ be arbitrary and define

$$\rho_m := t_0 + m\Delta_n \quad \text{for} \quad m = 0,...,n, \qquad \Delta_n := (\rho - t_0)/n\,.$$

For each $m \in \{1,...,n\}$ let $P(m)$ be the statement

$$P(m): \quad \begin{cases} \text{there exists } y_m \in C[-h,\rho_m] \text{ such that} \\ |y_m(t)| < c_1 \;\; \forall\, t \in [-h,\rho_m], \quad |y_m(t) - y^0(t_0)| < \delta \;\; \forall\, t \in [t_0,\rho_m] \\ y_m(t) = y^0(t) \;\; \forall\, t \in [-h,t_0], \quad y_m(t) = y^0(t_0) \;\; \forall\, t \in (t_0,\rho_1) \\ y_m(t) = y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_m(s), (Ty_m(s)))\,\mathrm{d}s \;\; \forall\, t \in [\rho_1,\rho_m]. \end{cases}$$

Let $m \in \{1,...,(n-1)\}$ and assume that $P(m)$ is a true statement. Then,

$$\big(s, y_m(s), (Ty_m)(s)\big) \in \mathcal{D}_0 \times \{\zeta \in \mathbb{R}^q |\; \|\zeta\| < c_2\} \quad \forall\, s \in [t_0,\rho_m]$$

and so, by (7.2),

$$\left|\int_{t_0}^{t-\Delta_n} F(s, y_m(s), (Ty_m)(s))\,\mathrm{d}s\right| \le \int_{t_0}^{t-\Delta_n} \gamma(s)\,\mathrm{d}s = \Gamma(t-\Delta_n) \le \Gamma(\rho_m) < \delta \;\; \forall\, t \in [\rho_m,\rho_{m+1}].$$

Now, define $y_{m+1} : [-h,\rho_{m+1}] \to \mathbb{R}$ by

$$y_{m+1}(t) := \begin{cases} y^0(t), & t \in [-h,t_0] \\ y^0(t_0), & t \in (t_0,\rho_1) \\ y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_m(s), (Ty_m)(s))\,\mathrm{d}s, & t \in [\rho_1,\rho_{m+1}]. \end{cases}$$

It immediately follows that $|y_{m+1}(t)| < c_1$ for all $t \in [-h, \rho_{m+1}]$ and $|y_{m+1}(t) - y^0(t_0)| < \delta$ for all $t \in [t_0, \rho_{m+1}]$. Clearly, y_{m+1} is continuous at all points $t \in [-h, \rho_{m+1}]$ with $t \neq \rho_1$. Moreover, since $\rho_1 - \Delta_n = \rho_0 = t_0$, we see that $y_{m+1}(\rho_1) = y^0(t_0)$, whence continuity at $t = \rho_1$. Therefore, $y_{m+1} \in C[-h, \rho_{m+1}]$. Finally, observing that $y_{m+1}(t) = y_m(t)$ for all $t \in [-h, \rho_m]$ and invoking causality of T, we may infer that

$$y_{m+1}(t) = y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_{m+1}(s), (Ty_{m+1})(s))\, \mathrm{d}s \quad \forall\, t \in [\rho_1, \rho_{m+1}].$$

We have now established the following

$$P(m) \text{ true for some } m \in \{1, ..., (n-1)\} \implies P(m+1) \text{ true.}$$

Defining $y_1 \in C[-h, \rho_1]$ by

$$y_1(t) := \begin{cases} y^0(t), & t \in [-h, \rho_0) \\ y^0(t_0), & t \in [\rho_0, \rho_1]. \end{cases}$$

we see that $P(1)$ is a true statement. Therefore, $P(m)$ is true for $m = 1, ..., n$. We may now conclude that, for each $n \in \mathbb{N}$, there exists $y_n \in C[-h, \rho]$ such that

$$y_n(t) = \begin{cases} y^0(t), & t \in [-h, t_0] \\ y^0(t_0), & t \in (t_0, \rho_1) \\ y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s, y_n(s), (Ty_n)(s))\, \mathrm{d}s, & t \in [\rho_1, \rho]. \end{cases}$$

Moreover, $\max_{t\in[-h,\rho]} |y_n(t)| < c_1$ for all $n \in \mathbb{N}$ and so (y_n) is a bounded sequence in the Banach space $C[-h, \rho]$ with norm $\|y\|_\infty = \max_{t\in[-h,\rho]} |y(t)|$. We proceed to prove that the bounded sequence (y_n) is also equicontinuous. Let $\varepsilon > 0$ be arbitrary. By uniform continuity of $\Gamma \in C[-h, \rho]$, there exists $\bar{\delta} > 0$ such that

$$t, s \in [-h, \rho] \text{ with } |t - s| < \bar{\delta} \implies |\Gamma(t) - \Gamma(s)| < \varepsilon.$$

Let $t, s \in [t_0, \rho]$ be such that $|t - s| < \bar{\delta}$. Without loss of generality, we may assume that $s \leq t$. Observe that

(a) $t, s \in [t_0, \rho_1) \implies |y_n(t) - y_n(s)| = 0,$

(b) $s \leq \rho_1 \leq t \implies t - \rho_1 < \bar{\delta}\ \ \&\ \ |y_n(t) - y_n(s)| = |y_n(t) - y^0(t_0)| \leq \Gamma(t - \Delta_n)$
$= |\Gamma(t - \rho_1 + t_0) - \Gamma(t_0)| < \varepsilon\,,$

(c) $t, s \in [\rho_1, \rho] \implies |y_n(t) - y_n(s)| \leq |\Gamma(t - \Delta_n) - \Gamma(s - \Delta_n)| < \varepsilon\,.$

Therefore, the sequence $(y_n|_{[t_0,\rho]})$ is equicontinuous. Since $y_n|_{[-h,t_0]} = y^0$ for all n, it follows that (y_n) is an equicontinuous sequence in $C[-h, \rho]$. By the Arzelà-Ascoli Theorem, it follows that (y_n) has a subsequence (which we do not relabel) converging to $y \in C[-h, \rho]$. Clearly, $y|_{[-h,t_0]} = y^0$. By Property (iii) of $\mathcal{T}_h^q$, $\lim_{n\to\infty}(Ty_n)(t) = (Ty)(t)$ for almost all $t \in [t_0, \rho]$ and so, by continuity of $(\xi, \zeta) \mapsto F(t, \xi, \zeta)$,

$$\lim_{n\to\infty} F(t, y_n(t), (Ty_n)(t)) = F(t, y(t), (Ty)(t)) \quad \text{for a.a. } t \in [t_0, \rho].$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{n\to\infty}\int_{t_0}^{t} F(s,y_n(s),(Ty_n)(s))\ \mathrm{d}s = \int_{t_0}^{t} F(s,y(s),(Ty)(s))\ \mathrm{d}s \quad \forall\ t\in[t_0,\rho]\,.$$

Noting that

$$\begin{aligned} y_n(t) &= y^0(t_0) + \int_{t_0}^{t-\Delta_n} F(s,y_n(s),(Ty_n)(s))\ \mathrm{d}s \\ &= y^0(t_0) + \left(\int_{t_0}^{t} - \int_{t-\Delta_n}^{t}\right) F(s,y_n(s),(Ty_n)(s))\ \mathrm{d}s \quad \forall\ t\in[t_0+\Delta_n,\rho], \end{aligned}$$

and since $\Delta_n \downarrow 0$ as $n\to\infty$, we may conclude that

$$y(t) = \begin{cases} y^0(t), & t\in[-h,t_0] \\ y^0(t_0) + \int_{t_0}^{t} F(s,y(s),(Ty)(s))\ \mathrm{d}s, & t\in(t_0,\rho]. \end{cases}$$

Therefore, $y\in C[-h,\rho]$ is a solution of the initial-value problem (7.1). This establishes Assertion (i) of the theorem.

Let $y\in C(I)$ be a solution of (7.1). Define

$$\mathcal{E} := \big\{(\omega,z)|\ \ \omega=\sup J,\ J\supset I,\ z\in C(J) \text{ is a solution of (7.1)},\ z|_I = y\big\},$$

and so, for $(\omega,z)\in\mathcal{E}$, either $z=y$ or z is a solution which extends y. On this non-empty set, define a partial order $\preceq$ by

$$(\omega_1,z_1)\preceq(\omega_2,z_2) \quad\Longleftrightarrow\quad \omega_1\le\omega_2\ \ \&\ \ z_1(t)=z_2(t)\ \ \forall\ t\in[-h,\omega_1).$$

Assertion (ii) follows if we can establish that $\mathcal{E}$ has a maximal element. This we do by an application of Zorn's Lemma, as follows. Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{E}$. Let $\omega^* := \sup\{\omega|\ \ (\omega,z)\in\mathcal{E}\}$ and define $z^*\in C[0,\omega^*)$ by the property that, for every $(\omega,z)\in\mathcal{O}$, $z^*|_{[-h,\omega)} = z$. Then (ω^*,z^*) is in $\mathcal{E}$ and is an upper bound for $\mathcal{O}$ (that is, $(\omega,z)\preccurlyeq(\omega^*,z^*)$ for all $(\omega,z)\in\mathcal{O}$). By Zorn's Lemma, it follows that $\mathcal{E}$ contains at least one maximal element.

Finally, we prove Assertion (iii). Assume that F is locally essentially bounded and let $y\in C[-h,\omega)$ be a maximal solution of (7.1). Seeking a contradiction, suppose that $G:=\mathrm{graph}\big(y|_{[t_0,\omega)}\big)$ has compact closure $\overline{G}$ in $\mathcal{D}$. Then, by boundedness of y, property (iv) of $\mathcal{T}_h^q$ and local essential boundedness of F, there exists $c_3>0$ such that $|\dot y(t)|\le c_3$ for almost all $t\in[t_0,\omega)$. We may now conclude that y is uniformly continuous on the bounded interval $[-h,\omega)$ and so extends to a function $\hat y\in C[-h,\omega]$ with $\mathrm{graph}\big(\hat y|_{[t_0,\omega]}\big)\subset\overline{G}\subset\mathcal{D}$. In particular, we have $(\omega,\hat y(\omega))\in\mathcal{D}$. An application of Assertion (i) (the the roles of t_0 and y^0 now being taken by ω and $\hat y$) yields the existence $z\in C[-h,\rho]$ with $\omega<\rho$ and $z|_{[-h,\omega]}=\hat y$ such that $\dot z(t) = F(t,z(t),(\tilde T z)(t))$ for almost all $t\in[\omega,\rho]$. Therefore, z is a solution of the initial-value problem (7.1) and is an extension of y. This contradicts maximality of the solution y. □

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