Analyticity of Semigroups Related to a Class of Block Operator Matrices

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Abstract. We derive various properties, e.g. analyticity of the associated semigroup and existence of a Riesz basis consisting of eigenfunctions, of the operator matrix $\mathcal{A} = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}$. Here the entries A_0 and D are unbounded operators. Such operator matrices are associated with second order problems of the form $\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0$ which are used as models for small motions of some hydrodynamical systems.

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1. Introduction

The aim of this paper is the study of second order equations of the form

$$\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0. \tag{1.1}$$

Here the operator A_0 is a possibly unbounded positive operator on a Hilbert space H and is assumed to be boundedly invertible. It is assumed that the operator D is the dominating operator, i.e. D is an unbounded operator such that, for some $\theta > 1$, the operator $A_0^{-\theta/2} D A_0^{-\theta/2}$ is bounded and non-negative in H. Problems of such type arise in hydrodynamics, see [19] and references therein. Moreover, we mention that, e.g., the second order equation (1.1) is obtained in the study of sytems of the form

$$\begin{aligned} -\frac{\partial v}{\partial t} &= \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial r^2}, \\ v &= \frac{\partial^4 u}{\partial r^4}. \end{aligned}$$

The second order equation (1.1) is equivalent to the standard first-order equation $\dot{x}(t) = \mathcal{A}x(t)$, where $\mathcal{A} : \mathrm{dom}\mathcal{A} \subset \mathrm{dom}(A_0^{1/2}) \times H \to \mathrm{dom}(A_0^{1/2}) \times H$, is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix}, \tag{1.2}$$

$$\mathrm{dom}\mathcal{A} = \left\{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) \in \mathrm{dom}(A_0^{1/2}) \times \mathrm{dom}(A_0^{1/2}) \mid A_0 z + Dw \in H \right\}.$$

In this paper we focus on two properties of the operator $\mathcal{A}$: Analyticity of the generated semigroup and the Riesz basis property in the phase space $H_{1/2} \times H$. In a first step, we show under the assumption that $A_0^{-\theta}D$ is boundedly invertible in $H_{1/2}$, where $H_{1/2} = \mathrm{dom}(A_0^{1/2})$ is equipped with the norm $\|x\|_{1/2} = \|A_0^{1/2}x\|_H$, that $\mathcal{A}$ generates a C_0-semigroup of contractions in $H_{1/2} \times H$ and thus the spectrum of $\mathcal{A}$ is located in the closed left half plane. Moreover, under the additional assumption that A_0 has a compact resolvent, we show that the essential spectrum of $\mathcal{A}$ consits of the point zero only. In this case we are able to show that $\mathcal{A}$ is the generator of an analytic semigroup and we develop some conditions for the existence of a Riesz basis in the phase space $H_{1/2} \times H$ consisting of eigenvectors of $\mathcal{A}$.

The block operator matrix in (1.2) has been studied in the literature for more than 20 years. Interest in this particular model is motivated by various problems such as stabilization, see for example [6, 26, 27, 29], solvability of the Riccati equations [11], minimum-phase property [16] and compensator problems with partial observations [12].

The case $\theta \leq 1$ is very well studied. We mention here only a few results. In this case, it is well-known that $\mathcal{A}$ generates a C_0-semigroup of contractions in $H_{1/2} \times H$. This goes back to [3] and [25], see also [4, 8]. Analyticity has been studied in [3, 4, 8, 9, 13, 14]. The Riesz basis property has been shown in [21] in the situation where A_0^{-1} is a compact operator, D is of the form $D = \alpha A_0 + B$, for some $\alpha \geq 0$ with a symmetric operator B and $-1/\alpha \notin \sigma_p(\mathcal{A})$, if $\alpha \neq 0$ (and with some additional assumptions in the case $\alpha = 0$). Similar results were obtained [9, Appendix A] in a more special situation. All these assumptions guarantee that the essential spectrum of $\mathcal{A}$ consists at most of one point.

We proceed as follows. In Section 2 we provide some useful results on the spectrum of operators in Krein spaces. In particular, we recall the notion of spectral points of positive and negative type and of type π_+ and type π_-. One main tool of this paper is to show that certain spectral points of $\mathcal{A}$ are of positive/negative type or of type π_+/π_-. In Section 3 we give the precise definition of the operator $\mathcal{A}$ and prove some of its properties. In particular we show that $\mathcal{A}$ generates a C_0-semigroup of contractions and, if A_0^{-1} is compact, that zero is the only point in the essential spectrum. The main results of this paper are contained in Section 4 where we always assume that that A_0^{-1} is a compact operator and that $A_0^{-\theta}D$ is boundedly invertible in $H_{1/2}$. We show that $\mathcal{A}$ generates an analytic strongly continuous semigroup. Further, we show that ∞ is a spectral point of negative type and that every real spectral point is of type π_+. As a consequence we obtain that $\mathcal{A}$ is definitizable and that the non-real spectrum of $\mathcal{A}$ consists of at most finitely many points belonging to the point spectrum of $\mathcal{A}$. Under additional weak conditions there exists a Riesz basis of $H_{1/2} \times H$ consisting of eigenvalues of $\mathcal{A}$.

2. Spectrum of operators in Krein spaces

We briefly recall that a complex linear space $\mathcal{H}$ with a hermitian nondegenerate sesquilinear form $[\cdot\,,\cdot]$ is called a *Krein space* if there exists a so called *fundamental decomposition*

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \tag{2.1}$$

with subspaces $\mathcal{H}_\pm$ being orthogonal to each other with respect to $[\cdot\,,\cdot]$ such that $(\mathcal{H}_\pm, \pm[\cdot\,,\cdot])$ are Hilbert spaces. In the following all topological notions are understood with respect to some Hilbert space norm $\|\cdot\|$ on $\mathcal{H}$ such that $[\cdot\,,\cdot]$ is $\|\cdot\|$-continuous. Any two such norms are equivalent. For the basic theory of Krein space and operators acting therein we refer to [7] and [1].

Let A be a closed operator in $\mathcal{H}$. We define the extended spectrum $\sigma_e(A)$ of A by $\sigma_e(A) := \sigma(A)$ if A is bounded and $\sigma_e(A) := \sigma(A) \cup \{\infty\}$ if A is unbounded. The resolvent set of A is denoted by $\rho(A)$. The operator A is called *Fredholm* if the dimension of the kernel of A and the codimension of the range of A are finite. The set

$$\sigma_{ess}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not Fredholm}\}$$

is called the *essential spectrum* of A. We say that $\lambda \in \mathbb{C}$ belongs to the *approximate point spectrum of* A, denoted by $\sigma_{ap}(A)$, if there exists a sequence $(x_n) \subset \mathrm{dom}(A)$ with $\|x_n\| = 1$, $n = 1, 2, \ldots$, such that

$$\|x_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|Ax_n - \lambda x_n\| = 0.$$

Obviously, the continuous and the point spectrum of a closed operator are subsets of the approximate point spectrum. Moreover, we have the following, cf. [10, §IV 1.10].

Remark 2.1. The boundary points of $\sigma(A)$ in $\mathbb{C}$ belong to $\sigma_{ap}(A)$.

Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$, i.e., A coincides with its adjoint A^+ with respect to the indefinite inner product $[\cdot,\cdot]$. Then all real spectral points of A belong to $\sigma_{ap}(A)$ (see e.g. Corollary VI.6.2 in [7]).

The indefiniteness of the scalar product on $\mathcal{H}$ leads to the definition of several subsets of the spectrum of an operator. The following definition was given in [20], [24] and [2].

Definition 2.2. For a self-adjoint operator A in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of *positive* (*negative*) *type of* A if $\lambda_0 \in \sigma_{ap}(A)$

and for every sequence $(x_n) \subset \operatorname{dom}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0 I)x_n\| \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0\big).$$

The point ∞ is said to be of *positive* (*negative*) *type of* A if A is unbounded and for every sequence $(x_n) \subset \operatorname{dom}(A)$ with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0\big).$$

We denote the set of all points of $\sigma_e(A)$ of positive (negative) type by $\sigma_{++}(A)$ (resp. $\sigma_{--}(A)$).

The sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ are contained in $\overline{\mathbb{R}}$. Indeed, for $\lambda \in \sigma_{++}(A) \setminus \{\infty\}$ and (x_n) as in the first part of Definition 2.2 we have $-(\operatorname{Im}\lambda)[x_n, x_n] = \operatorname{Im}[(A - \lambda)x_n, x_n] \to 0$ for $n \to \infty$ which implies $\operatorname{Im}\lambda = 0$. Here $\overline{\mathbb{R}}$ denotes the set $\mathbb{R} \cup \{\infty\}$ and $\overline{\mathbb{C}}$ the set $\mathbb{C} \cup \{\infty\}$, each equipped with the usual topology.

In a similar way as above we define subsets $\sigma_{\pi_+}(A)$ and $\sigma_{\pi_-}(A)$ of $\sigma_e(A)$ containing $\sigma_{++}(A)$ and $\sigma_{--}(A)$, respectively (cf. Definition 5 in [2]). They will play an important role in the following.

Definition 2.3. For a self-adjoint operator A in $\mathcal{H}$ a point $\lambda_0 \in \sigma(A)$ is called a spectral point of *type* π_+ (*type* π_-) *of* A if $\lambda_0 \in \sigma_{ap}(A)$ and if there exists a linear submanifold $\mathcal{H}_0 \subset \mathcal{H}$ with $\operatorname{codim}\mathcal{H}_0 < \infty$ such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \operatorname{dom}(A)$ with $\|x_n\| = 1$ and $\|(A - \lambda_0 I)x_n\| \to 0$ as $n \to \infty$ we have

$$\liminf_{n\to\infty} [x_n, x_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [x_n, x_n] < 0\big).$$

The point ∞ is said to be of *type* π_+ (*type* π_-) *of* A if A is unbounded and if there exists a linear submanifold $\mathcal{H}_0 \subset \mathcal{H}$ with $\operatorname{codim}\mathcal{H}_0 < \infty$ such that for every sequence $(x_n) \subset \mathcal{H}_0 \cap \operatorname{dom}(A)$ with $\lim_{n\to\infty} \|x_n\| = 0$ and $\|Ax_n\| = 1$ we have

$$\liminf_{n\to\infty} [Ax_n, Ax_n] > 0 \quad \big(\textit{resp.}\ \limsup_{n\to\infty} [Ax_n, Ax_n] < 0\big).$$

We denote the set of all points of $\sigma_e(A)$ of type π_+ (type π_-) of A by $\sigma_{\pi_+}(A)$ (resp. $\sigma_{\pi_-}(A)$).

For example, if $A - \lambda_0$ is a Fredholm operator, then $\lambda_0 \in \sigma_{\pi_+}(A)$ (and $\lambda_0 \in \sigma_{\pi_-}(A)$).

We will collect some properties of spectral point of positive/negative type and of type π_+/π_- in the following propositions.

Proposition 2.4. *Let λ_0 be a point of $\sigma_{\pi_+}(A)$ ($\sigma_{\pi_-}(A)$, respectively). Then there exists an open neighbourhood $\mathcal{U}$ in $\overline{\mathbb{C}}$ of λ_0 such that the following holds.*

(i) *We have*

$$\mathcal{U} \setminus \overline{\mathbb{R}} \subset \sigma_p(A).$$

Moreover, if $\lambda_0 \in \mathbb{C}$ is non-real then the operator $A - \lambda_0$ has a closed range and $\dim\ker(A - \lambda_0) < \infty$.

(ii) $\mathcal{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_+}(A)$ *(resp.* $\mathcal{U} \cap \sigma_{ap}(A) \subset \sigma_{\pi_-}(A)$*).*
(iii) *If* $\lambda_0 = \infty$ *then* $\infty \in \sigma_{++}(A)$*. If* $\infty \in \sigma_{\pi_-}(A)$ *then* $\infty \in \sigma_{--}(A)$*.*
(iv *Assume, in addition, that* $\lambda_0 \in \mathbb{R}$ *and that there is* $[a, b] \subset \mathcal{U}$*,* $\lambda_0 \in [a, b]$*, such that each point of* $[a, b]$ *is an accumulation point of* $\rho(A)$*. Then there exists an open neighbourhood* $\mathcal{V}$ *in* $\mathbb{C}$ *of* $[a, b]$ *such that* $\mathcal{V} \setminus \mathbb{R} \subset \rho(A)$ *and either* $\mathcal{V} \cap \sigma(A) \cap \mathbb{R} \subset \sigma_{++}(A)$ *or there exists a finite number of points* $\lambda_1, \ldots, \lambda_n \in \sigma_{\pi_+}(A) \cap \sigma_p(A)$ *such that*

$$(\mathcal{V} \cap \sigma(A) \cap \mathbb{R}) \setminus \{\lambda_1, \ldots, \lambda_n\} \subset \sigma_{++}(A).$$

Moreover, in this case there exist numbers $m \geq 1$ *and* $M > 0$ *such that*

$$\|(A - \lambda)^{-1}\| \leq \frac{M}{|\operatorname{Im} \lambda|^m} \quad \text{for all } \lambda \in \mathcal{V} \setminus \overline{\mathbb{R}}.$$

For a proof of Proposition 2.4 we refer to [2] and [5]. Recall that a Krein space $(\mathcal{H}, [\cdot\,, \cdot])$ is called a *Pontryagin space* if one of the spaces $\mathcal{H}_+, \mathcal{H}_-$ in (2.1) is finite dimensional. Moreover, we will call a Krein space $(\mathcal{H}, [\cdot\,, \cdot])$ an *anti Hilbert space* if $(\mathcal{H}, -[\cdot\,, \cdot])$ is a Hilbert space.

Theorem 2.5. *Let* A *be a self-adjoint operator in* $(\mathcal{H}, [\cdot\,, \cdot])$*. Let* A *satisfy*

$$\sigma_e(A) = \sigma_{++}(A) \quad (\text{resp. } \sigma_e(A) = \sigma_{--}(A)). \tag{2.2}$$

Then $(\mathcal{H}, [\cdot\,, \cdot])$ *is a Hilbert space (anti-Hilbert space, respectively).*

If A *satisfies instead of* (2.2) *the following condition*

$$\sigma_e(A) = \sigma_{++}(A) \cup \sigma_{--}(A),$$

then A *is similar to a self-adjoint operator in a Hilbert space.*

The proof of Theorem 2.5 will be given together with Theorem 2.6 below.

Observe, that in the case of an unbounded operator A condition (2.2) implies also $\infty \in \sigma_{++}(A)$.

Recall that a self-adjoint operator A in a Krein space $(\mathcal{H}, [\cdot\,, \cdot])$ is called *definitizable* if $\rho(A) \neq \emptyset$ and if there exists a rational function $p \neq 0$ having poles only in $\rho(A)$ such that $[p(A)x, x] \geq 0$ for all $x \in \mathcal{H}$. Then the spectrum of A is real or its non-real part consists of a finite number of points, cf. [22, 23]. Moreover, A has a spectral function $E(\cdot)$ defined on the ring generated by all connected subsets of $\overline{\mathbb{R}}$ whose endpoints do not belong to some finite set which is contained in $\{t \in \mathbb{R} : p(t) = 0\} \cup \{\infty\}$ (see [22, 23]).

We will use the following therorem in Section 4.

Theorem 2.6. *Let* A *be a self-adjoint operator in* $(\mathcal{H}, [\cdot\,, \cdot])$ *with* $\rho(A) \neq \emptyset$ *satisfying*

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \text{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A) \quad (\text{resp. } \sigma_e(A) = \sigma_{\pi_-}(A)). \tag{2.3}$$

Then $(\mathcal{H}, [\cdot\,, \cdot])$ *is a Pontryagin space and the space* $\mathcal{H}_-$ *(resp.* $\mathcal{H}_+$*) in the fundamental decomposition* (2.1) *is of finite dimension. Moreover, the set* $\sigma(A) \setminus \mathbb{R}$ *consists of at most finitely many eigenvalues with finite dimensional root subspaces, i.e.*

$$\sigma(A) \setminus \mathbb{R} \subset \sigma_p(A) \setminus \sigma_{ess}(A)$$

If A with $\rho(A) \neq \emptyset$ satisfies instead of (2.3) *the following condition*

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \textit{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A) \cup \sigma_{\pi_-}(A), \tag{2.4}$$

then the non-real spectrum of A consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{ess}(A)$. Moreover, the operator A is definitizable.

Proof of the Theorems 2.5 *and* 2.6. Let A be a self-adjoint operator in the Krein space $(\mathcal{H}, [\cdot\,,\cdot])$ with $\rho(A) \neq \emptyset$ satisfying (2.4). The resolvent set of a self-adjoint operator in a Krein space is symmetric with respect to the real axis (cf. [7]), hence there are points from $\rho(A)$ in the upper and in the lower half-plane. This and $\sigma_{ess}(A) \subset \mathbb{R}$ imply that $\sigma(A) \setminus \mathbb{R}$ consists only of isolated eigenvalues with finite algebraic multiplicity (see §5.6 in [18]). In particular, each point in $\overline{\mathbb{R}}$ is an accumulation point of $\rho(A)$ and Proposition 2.4 implies that the spectrum of A cannot accumulate to a real point. Moreover, from (2.4) and Proposition 2.4 (iii) we conclude $\infty \in \sigma_{++}(A) \cup \sigma_{--}(A) \cup \rho(A)$. Therefore the non-real spectrum of A is bounded and consists of at most finitely many points which belong to $\sigma_p(A) \setminus \sigma_{ess}(A)$.

Relation (2.4), Theorem 23 in [2] and Theorem 4.7 in [17] imply that A is a definitizable operator and the second part of Theorem 2.6 is proved. In order to show the first part of Theorem 2.6 we assume without loss of generality

$$\sigma_{ess}(A) \subset \mathbb{R} \quad \text{and} \quad \sigma_e(A) = \sigma_{\pi_+}(A). \tag{2.5}$$

It remains to show that $(\mathcal{H}, [\cdot\,,\cdot])$ is a Pontryagin space. As $\infty \in \sigma_{++}(A) \cup \rho(A)$, there are at most only finitely many points in $\sigma_e(A)$ which belong to $\sigma_{\pi_+}(A) \setminus \sigma_{++}(A)$, cf. Proposition 2.4. Then, with Proposition 25, Theorem 26 of [2] and the properties of the spectral function of a definitizable operator (see [22, 23]), the assertions of Theorem 2.6 hold true.

Theorem 2.5 is now a consequence of Theorem 2.6: Assume without loss of generality

$$\sigma_e(A) = \sigma_{++}(A).$$

Then Theorem 2.6 implies that $(\mathcal{H}, [\cdot\,,\cdot])$ is a Pontryagin space and the space $\mathcal{H}_-$ in the fundamental decomposition (2.1) is of finite dimension. If $\mathcal{H}_- \neq 0$, then there exists at least one non-positive eigenvector of A (see §12 in [15]) for some eigenvalue λ_0. This implies $\lambda_0 \notin \sigma_{++}(A)$, a contradiction. Hence $\mathcal{H}_- = 0$ and $(\mathcal{H}, [\cdot\,,\cdot])$ is a Hilbert space. The second part of Theorem 2.5 follows from the fact that the sets $\sigma_{++}(A)$ and $\sigma_{--}(A)$ have a positive distance, hence, with the help of the spectral function of the definitizable operator A, $\mathcal{H}$ can be decomposed into the orthogonal sum of a A-invariant Hilbert and a A-invariant anti-Hilbert space. □

3. Framework and first results

Throughout this paper we make the following assumptions.

(I) The operator $A_0 : \mathrm{dom}A_0 \subset H \to H$ is a self-adjoint, positive definite linear operator on a Hilbert space H such that zero is in the resolvent set of A_0, i.e., A_0 is uniformly positive. Here $\mathrm{dom}A_0$ denotes the domain of A_0. Since A_0 is self-adjoint and positive definite, A_0^α is well-defined for $\alpha \geq 0$. A scale of Hilbert spaces H_α is defined as follows: For $\alpha \geq 0$, we define

$$H_\alpha = \mathrm{dom}(A_0^\alpha)$$

equipped with the norm induced by the inner product

$$\langle x, y\rangle_{H_\alpha} = \langle A_0^\alpha x, A_0^\alpha y\rangle_H, \qquad x, y \in H_\alpha$$

and $H_{-\alpha} = H_\alpha^*$. Here the duality is taken with respect to the pivot space H, that is, equivalently $H_{-\alpha}$ is the completion of H with respect to the norm

$$\|z\|_{H_{-\alpha}} = \|A_0^{-\alpha} z\|_H.$$

Thus A_0 extends (restricts) to $A_0 : H_\alpha \to H_{\alpha-1}$ for $\alpha \in \mathbb{R}$. We use the same notation A_0 to denote this extension (restriction).

We denote the inner product on H by $\langle\cdot,\cdot\rangle_H$ or $\langle\cdot,\cdot\rangle$, and the duality pairing on $H_{-\alpha} \times H_\alpha$ by $\langle\cdot,\cdot\rangle_{H_{-\alpha}\times H_\alpha}$. Note that for $(z', z) \in H \times H_\alpha$, $\alpha > 0$, we have

$$\langle z', z\rangle_{H_{-\alpha}\times H_\alpha} = \langle z', z\rangle_H.$$

(II) For some $\theta > 1$ the operator $D : H_\theta \to H_{-\theta}$ is a bounded operator such that $A_0^{-\theta/2} D A_0^{-\theta/2}$ is a bounded non-negative self-adjoint operator in H, that is,

$$\langle Dz, z\rangle_{H_{-\frac{\theta}{2}}\times H_{\frac{\theta}{2}}} \geq 0, \qquad z \in H_{\frac{\theta}{2}}.$$

In particular, we have that $A_0^{-\theta} D$ is a bounded self-adjoint operator in $H_{\frac{\theta}{2}}$.

(III) The operator $A_0^{-\theta} D$ admits a bounded extension to an operator in H. We use the same notation $A_0^{-\theta} D$ to denote this extension. Note, that (by complex interpolation), the operator $A_0^{-\theta} D$ admits a bounded extension to an operator in $H_{\frac{1}{2}}$, which we also denote by $A_0^{-\theta} D$.

Example 3.1. Let $A_0 : \mathrm{dom}A_0 \subset H \to H$ be a uniformly positive operator on a Hilbert space H. Choose

$$D := A_0^\theta \quad \text{for some } \theta > 1.$$

Then, obviously, we have $A_0^{-\theta/2} D A_0^{-\theta/2} = A_0^{-\theta} D = I$ and (I)-(III) are valid.

The equation (1.1) is equivalent to the following standard first-order equation

$$\dot{x}(t) \quad = \quad \mathcal{A}x(t),$$

where $\mathcal{A} : \mathrm{dom}\mathcal{A} \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$, is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix},$$

$$\mathrm{dom}\mathcal{A} = \left\{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + Dw \in H \right\}.$$

Proposition 3.2. *The operator $\mathcal{A}$ is a closed operator in $H_{\frac{1}{2}} \times H$.*

Proof. Let $\binom{z_n}{w_n} \subset \operatorname{dom}\mathcal{A}$ be a sequence and let $x, z \in H_{\frac{1}{2}}$, $y, w \in H$ such that $z_n \to z$ in $H_{\frac{1}{2}}$ and $w_n \to w$ in H as $n \to \infty$ and

$$\mathcal{A}\binom{z_n}{w_n} \to \binom{x}{y} \quad \text{in } H_{\frac{1}{2}} \times H \quad \text{as } n \to \infty. \tag{3.1}$$

Hence, $w_n \to x$ in $H_{\frac{1}{2}}$ and $w = x$, $x \in H_{\frac{1}{2}}$. As $A_0^{-\theta} D$ is bounded in $H_{\frac{\theta}{2}}$,

$$A_0^{-\theta} D w_n \to A_0^{-\theta} D x \quad \text{in } H_{\frac{1}{2}} \quad \text{as } n \to \infty. \tag{3.2}$$

Moreover, by (3.1),

$$-A_0 z_n - D w_n \to y \text{ in } H \quad \text{and } -A_0^{-\theta+1} z_n - A_0^{-\theta} D w_n \to A_0^{-\theta} y \text{ in } H \text{ as } n \to \infty.$$

From (3.2) we conclude

$$-A_0^{-\theta+1} z_n \to A_0^{-\theta} D x + A_0^{-\theta} y \quad \text{in } H \quad \text{as } n \to \infty.$$

Furthermore, $-A_0^{-\theta+1} z_n \to -A_0^{-\theta+1} z$ in H as $n \to \infty$ and we have

$$-A_0^{-\theta+1} z = A_0^{-\theta} D x + A_0^{-\theta} y,$$

hence $-A_0 z = Dx + y$. This shows $A_0 z + Dx = -y \in H$ and $\mathcal{A}$ is a closed operator. □

For $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in H_{\frac{1}{2}} \times H$ we define an indefinite inner product on $H_{\frac{1}{2}} \times H$ by

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] := \left\langle J \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\rangle = \langle x_1, x_2 \rangle_{H_{\frac{1}{2}}} - \langle y_1, y_2 \rangle, \tag{3.3}$$

where

$$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \tag{3.4}$$

is a self-adjoint operator in the Hilbert space $H_{\frac{1}{2}} \times H$. Then $(H_{\frac{1}{2}} \times H, [\cdot\,,\cdot])$ is a Krein space and we obtain the following theorem.

Theorem 3.3. *Assume, in addition to our standard assumptions (I)-(III), that the operator $A_0^{-\theta} D$ is boundedly invertible in $H_{\frac{1}{2}}$. Then $\mathcal{A}$ is a self-adjoint operator in the Krein space $(H_{\frac{1}{2}} \times H, [\cdot\,,\cdot])$, its spectrum is contained in the closed left half plane and lies symmetric with respect to the real line. The operator $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions on $H_{\frac{1}{2}} \times H$.*

Proof. Observe, that the operator $A_0^{-\theta} D$ is a uniformly positive operator in the Hilbert space $H_{\frac{1}{2}}$. The operator $A_0^{-\theta+1} + A_0^{-\theta}$ is a bounded non-negative operator in the Hilbert space $H_{\frac{1}{2}}$, therefore, for $\lambda > 0$, the operator

$$V(\lambda) := A_0^{-\theta+1} + \lambda A_0^{-\theta} D + \lambda^2 A_0^{-\theta} \tag{3.5}$$

is a bounded, uniformly positive operator in the Hilbert space $H_{\frac{1}{2}}$. In particular, $V(\lambda)$ is boundedly invertible in $H_{\frac{1}{2}}$ for all $\lambda > 0$. Let $x \in H_{\frac{1}{2}}$ and $y \in H$. We set

$$z := -V(1)^{-1}\left(A_0^{-\theta} y + A_0^{-\theta} D x + A_0^{-\theta} x\right) \quad \text{and} \quad w := x + z.$$

Then $z, w \in H_{\frac{1}{2}}$ and we have

$$V(1)z = A_0^{-\theta+1}z + A_0^{-\theta}Dz + A_0^{-\theta}z = -\left(A_0^{-\theta}y + A_0^{-\theta}Dx + A_0^{-\theta}x\right),$$

hence

$$A_0 z + Dz + z = -y - Dx - x.$$

This implies

$$A_0 z + Dw = A_0 z + Dx + Dz = -y - x - z = -y - w \in H,$$

that is, $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in \mathrm{dom}\mathcal{A}$. Moreover, we have

$$-z + w = x \quad \text{and} \quad -A_0 z - Dw - w = y$$

and, thus, $\mathcal{A} - 1$ is surjective. Assume that 1 is an eigenvalue of $\mathcal{A}$ with an corresponding eigenvector $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)$. Then $z = w$ follows and $A_0 z + Dz + z = 0$. Then,

$$A_0^{-\theta+1}z + A_0^{-\theta}Dz + A_0^{-\theta}z = V(1)z = 0.$$

This gives $z = w = 0$, a contradiction. From Proposition 3.2 we conclude

$$1 \in \rho(\mathcal{A}).$$

It is easy to see that $J\mathcal{A}$, where J is defined as in (3.4), is a symmetric operator in the Hilbert space $H_{\frac{1}{2}} \times H$. Then $J(\mathcal{A}-1)$ is a symmetric and boundedly invertible operator in the Hilbert space $H_{\frac{1}{2}} \times H$. Thus $J(\mathcal{A}-1)$ and $J\mathcal{A}$ are self-adjoint operators in the Hilbert space $H_{\frac{1}{2}} \times H$ and $\mathcal{A}$ is a self-adjoint operator in the Krein space $(H_{\frac{1}{2}} \times H, [\cdot\,,\cdot])$ with

$$\mathcal{A}^* = J\mathcal{A}J, \qquad \text{with } \mathrm{dom}(\mathcal{A}^*) = J\mathrm{dom}(\mathcal{A}).$$

For $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in \mathrm{dom}\mathcal{A}$

$$\mathrm{Re}\left\langle \mathcal{A}\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle = \mathrm{Re}\left\langle \begin{pmatrix} w \\ -A_0 z - Dw \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle = -\langle Dw, w\rangle_{H_{-\frac{\theta}{2}} \times H_{\frac{\theta}{2}}} \le 0$$

and, by a similar argument,

$$\mathrm{Re}\left\langle \mathcal{A}^*\begin{pmatrix} z \\ w \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \right\rangle \le 0$$

for all $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in \mathrm{dom}\mathcal{A}^*$. Hence, $\mathcal{A}$ is the generator of a strongly continuous semigroup of contractions on the state space $H_{\frac{1}{2}} \times H$. □

The next example shows that, in general, $\mathcal{A}$ is not a boundedly invertible operator.

Example 3.4. Let $A_0 : \mathrm{dom}A_0 \subset H \to H$ be a uniformly positive operator on a Hilbert space H. Choose $D := A_0^2$. Then (cf. Example 3.1) (I)-(III) are valid. Assume $y_0 \in H_{\frac{1}{2}} \setminus H_1$. For all $x \in H_{\frac{1}{2}}$ we have

$$-A_0 x - A_0^2 y_0 \in H_{-\frac{3}{2}} \setminus H_{-1},$$

that is $-A_0x - A_0^2 y_0 \neq 0$. Hence

$$\mathcal{A}\begin{pmatrix} x \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ -A_0x - A_0^2 y_0 \end{pmatrix} \neq \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

for all $x \in H_{\frac{1}{2}}$ and $\mathcal{A}$ is not a boundedly invertible operator.

If the operator A_0 has a compact resolvent, then we have the following description of the essential spectrum of $\mathcal{A}$.

Proposition 3.5. *Assume, in addition to our standard assumptions (I)-(III), that A_0^{-1} is a compact operator in H and the operator $A_0^{-\theta}D$ is boundedly invertible in $H_{\frac{1}{2}}$. Then*

$$\sigma_{ess}(\mathcal{A}) = \{0\}. \tag{3.6}$$

Proof. The operator $V(1)$ (see (3.5)) is a bounded, uniformly positive operator in the Hilbert space $H_{\frac{1}{2}}$. It is easily seen, that

$$(\mathcal{A} - 1)^{-1} = \begin{bmatrix} V(1)^{-1}A^{-\theta+1} - I & -V(1)^{-1}A_0^{-\theta} \\ V(1)^{-1}A_0^{-\theta+1} & -V(1)^{-1}A_0^{-\theta} \end{bmatrix}. \tag{3.7}$$

The left upper corner of the matrix of the right hand side of (3.7) is the difference between a compact operator and the identity, the other entries of this matrix are compact operators. Hence,

$$\sigma_{ess}\left((\mathcal{A} - 1)^{-1}\right) = \{0\} \cup \sigma_{ess}(-I) = \{0\} \cup \{-1\}$$

and $\sigma_{ess}(\mathcal{A}) = \{0\}$ follows. □

4. Analyticity and expansions in eigenfunctions

The following result is the main result of this paper.

Theorem 4.1. *Assume, in addition to our standard assumptions (I)-(III), that A_0^{-1} is a compact operator in H and the operator $A_0^{-\theta}D$ is boundedly invertible in $H_{\frac{1}{2}}$. Then $\mathcal{A}$ generates an analytic semigroup on $H_{1/2} \times H$.*

The proof of Theorem 4.1 will be given after Theorem 4.2 below.

Theorem 4.2. *Assume, in addition to our standard assumptions (I)-(III), that A_0^{-1} is a compact operator in H and the operator $A_0^{-\theta}D$ is boundedly invertible in $H_{\frac{1}{2}}$. Then the operator $\mathcal{A}$ is definitizable and*

$$\infty \in \sigma_{--}(\mathcal{A}), \quad 0 \in \sigma_{++}(\mathcal{A}) \quad \textit{and} \quad \mathbb{R} \subset \sigma_{\pi_+}(\mathcal{A}) \cup \rho(\mathcal{A}). \tag{4.1}$$

Moreover, there exists a neighbourhood $\mathcal{U}$ of ∞ in $\overline{\mathbb{C}}$ and constants $M > 0$, $m \in \mathbb{N}$ and $\eta > 0$ such that

$$\mathcal{U} \setminus \overline{\mathbb{R}} \subset \rho(\mathcal{A}) \quad \textit{and} \quad \mathcal{U} \cap \mathbb{R} \subset \sigma_{--}(\mathcal{A}) \cup \rho(\mathcal{A}) \tag{4.2}$$

and

$$\|(\mathcal{A}-\lambda I)^{-1}\| \le \tfrac{M}{|\mathrm{Im}\,\lambda|} \quad \textit{for all } \lambda \in \mathcal{U}\setminus\overline{\mathbb{R}}, \tag{4.3}$$

$$\|(\mathcal{A}-\lambda I)^{-1}\| \le \tfrac{M}{|\mathrm{Im}\,\lambda|^m} \quad \textit{for all } \lambda \in \rho(A)\setminus\mathbb{R} \textit{ with } |\mathrm{Im}\lambda| \le \eta. \tag{4.4}$$

Further, the non-real spectrum of $\mathcal{A}$ consists of at most finitely many points.

Proof. In the first three steps we show that (4.1) holds.

Step 1. Let $\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right) \subset \mathrm{dom}\mathcal{A}$ be a sequence with $\|\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\|^2_{H_{\frac12}\times H} = \|x_n\|^2_{H_{\frac12}} + \|y_n\|^2 \to 0$ as $n\to\infty$ and $\|\mathcal{A}\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\|_{H_{\frac12}\times H} = 1$. Then

$$\|y_n\|^2_{H_{\frac12}} + \|A_0x_n + Dy_n\|^2 = 1, \tag{4.5}$$

We have $\|y_n\| \to 0$ as $n\to\infty$. Assume that there exists a subsequence of (y_n) which does not converge to zero in $H_{\frac12}$. Without loss of generality, let us assume that $\|y_n\|_{H_{\frac12}} \ge \gamma > 0$ for $n\in\mathbb{N}$. By (4.5), $A_0^{-\frac12}(A_0x_n + Dy_n)$ is bounded in H. As (x_n) converges to zero in $H_{\frac12}$, the sequence $\left(A_0^{-\frac12}Dy_n\right)$ is a bounded sequence in H, hence it has a subsequence $\left(A_0^{-\frac12}Dy_{n_k}\right)$ which converges weakly in H to some $y_0\in H$ for $k\to\infty$. Then $A_0^{-\theta}Dy_{n_k}$ converges in H to $A_0^{-\theta+\frac12}y_0$ as $k\to\infty$. As $A_0^{-\theta}D$ is a bounded operator in H, $(A_0^{-\theta}Dy_{n_k})$ converges to zero in H as $k\to\infty$ and we have

$$A_0^{-\theta+\frac12}y_0 = 0,$$

hence $y_0 = 0$. Therefore, $(A_0^{-\frac12}Dy_{n_k})$ converges weakly to zero in H. It follows from the compactness of the operator $A_0^{-\theta+\frac12}$ that

$$A_0^{-\theta+1}A_0^{-\frac12}Dy_{n_k} = A_0^{\frac12}A_0^{-\theta}Dy_{n_k} \to 0 \quad \text{in } H \quad \text{as } k\to\infty$$

and

$$A_0^{-\theta}Dy_{n_k} \to 0 \quad \text{in } H_{\frac12} \quad \text{as } k\to\infty.$$

By assumption, the operator $A_0^{-\theta}D$ is boundedly invertible in $H_{\frac12}$, that is

$$y_{n_k} \to 0 \quad \text{in } H_{\frac12} \quad \text{as } k\to\infty.$$

This is a contradiction, therefore we have

$$y_n \to 0 \quad \text{in } H_{\frac12} \quad \text{as } k\to\infty.$$

By (4.5)

$$\limsup_{n\to\infty}\left[\mathcal{A}\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right), \mathcal{A}\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right)\right] = \limsup_{n\to\infty}\left(\|y_n\|^2_{H_{\frac12}} - \|A_0x_n + Dy_n\|^2\right) = -1, \tag{4.6}$$

that is, $\infty\in\sigma_{--}(\mathcal{A})$.

Step 2. Let $(\binom{x_n}{y_n}) \subset \text{dom}\mathcal{A}$ be a sequence with $\|\binom{x_n}{y_n}\|^2_{H_{\frac{1}{2}}\times H} = \|x_n\|^2_{H_{\frac{1}{2}}} + \|y_n\|^2 = 1$ and $\mathcal{A}\binom{x_n}{y_n} \to 0$ as $n \to \infty$. Then

$$\|y_n\|^2_{H_{\frac{1}{2}}} \to 0 \quad \text{and} \quad \|x_n\|^2_{H_{\frac{1}{2}}} \to 1 \quad \text{as } n \to \infty.$$

This gives

$$\liminf_{n\to\infty} [\binom{x_n}{y_n}, \binom{x_n}{y_n}] = \liminf_{n\to\infty} \left(\|x_n\|^2_{H_{\frac{1}{2}}} - \|y_n\|^2 \right) = 1,$$

that is, $0 \in \sigma_{++}(\mathcal{A})$.

Step 3. We now choose $\mu \in (-\infty, 0)$ and

$$G_\mu := \text{span}\,\{x \in H_{\frac{1}{2}} \mid A_0 x = \nu x, \nu \leq \mu^2\}.$$

Then G_μ is a finite dimensional subspace of $H_{\frac{1}{2}}$. For every sequence $\binom{x_n}{y_n}$ in $\text{dom}\mathcal{A} \cap (G_\mu \times G_\mu)^\perp$, where $\perp$ denotes the orthogonal complement in $H_{\frac{1}{2}} \times H$ with respect to the usual Hilbert space product in $H_{\frac{1}{2}} \times H$, with $\|\binom{x_n}{y_n}\|^2_{H_{\frac{1}{2}}\times H} = 1$ and $(\mathcal{A} - \mu I)\binom{x_n}{y_n} \to 0$ as $n \to \infty$ we have

$$\|y_n - \mu x_n\|_{H_{\frac{1}{2}}} \to 0 \quad \text{and} \quad \|A_0 x_n + D y_n + \mu y_n\| \to 0 \quad \text{as } n \to \infty.$$

This gives

$$\begin{aligned} \liminf_{n\to\infty} [\binom{x_n}{y_n}, \binom{x_n}{y_n}] &= \liminf_{n\to\infty} \left(\langle x_n, x_n \rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n \rangle \right) \\ &= \liminf_{n\to\infty} \left(\langle A_0 x_n, x_n \rangle - \mu^2 \langle x_n, x_n \rangle \right) > 0, \end{aligned}$$

where the last inequality follows from the fact that $x_n \in G_\mu^\perp$, $n \in \mathbb{N}$. Therefore $\mathbb{R} \subset \sigma_{\pi_+}(\mathcal{A})$ and (4.1) is valid.

Step 4. By Proposition 3.5 the essential spectrum of $\mathcal{A}$ is real. Moreover, for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $A - \lambda$ is a Fredholm operator and, therefore, $\mathbb{C} \setminus \mathbb{R} \subset \sigma_{\pi_+}(\mathcal{A}) \cup \rho(\mathcal{A})$, cf. [2]. Together with (4.1) we obtain

$$\sigma_e(\mathcal{A}) = \sigma_{\pi_+}(\mathcal{A}) \cup \sigma_{\pi_-}(\mathcal{A}).$$

Then, by Theorem 2.6, $\mathcal{A}$ is definitizable. From this and (4.1) the remaining assertions of Theorem 4.2 follow (see e.g. [23, Proposition II.2.1] or [2, Proposition 3 and Theorem 20]). □

Proof of Theorem 4.1. Since $\mathcal{A}$ is the generator of a strongly continuous semigroup, estimate (4.3) shows immediately that $\mathcal{A}$ generates an analytic semigroup. □

In the sequel we always assume the Hilbert space H to be separable. An at most countable set $\mathcal{M}$ of elements of a Hilbert space is said to be a *Riesz basis* if there exists an isomorphic mapping $\mathcal{M}$ onto an orthonormal basis, cf. [28, Lecture VI].

Theorem 4.3. *Assume, in addition to our standard assumptions (I)-(III), that A_0^{-1} is a compact operator in H and the operator $A_0^{-\theta}D$ is boundedly invertible in $H_{\frac{1}{2}}$. Then there exists a subspace of $H_{\frac{1}{2}} \times H$ of at most finite codimension which has a Riesz basis consisting of eigenvectors of $\mathcal{A}$ and there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors and finitely many associated vectors of $\mathcal{A}$. Moreover assume, in addition, that for all $\mu \in \sigma_{\pi_+}(\mathcal{A}) \setminus \sigma_{++}(\mathcal{A})$ there exists no non-zero $\binom{y}{\mu y} \in \ker(\mathcal{A} - \mu I)$ such that*

$$\langle y, y \rangle_{H_{\frac{1}{2}}} = \mu^2 \langle y, y \rangle. \tag{4.7}$$

Then $\mathcal{A}$ has no associated vectors, i.e. there are no Jordan chains of length greater than one, and there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors of $\mathcal{A}$.

Proof. The first part of Theorem 4.3 follows from the properties of the spectral function of $\mathcal{A}$ and Theorem 4.2. Relation (4.7) implies that there are no neutral eigenvectors. Hence the spectrum of $\mathcal{A}$ is real and there are no Jordan chains of $\mathcal{A}$ corresponding to the eigenvalue μ of length greater than one. Moreover,

$$\sigma(\mathcal{A}) \subset \sigma_{++}(\mathcal{A}) \cup \sigma_{--}(\mathcal{A}) \subset \mathbb{R}.$$

Then, by Theorem 2.5, the operator $\mathcal{A}$ is similar to a self-adjoint operator in a Hilbert space, hence there exists a Riesz basis of $H_{\frac{1}{2}} \times H$ consisting of eigenvectors of $\mathcal{A}$. □

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