On Spanning Tree Congestion

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Abstract. We prove that every connected graph G of order n has a spanning tree T such that for every edge e of T the edge-cut defined in G by the vertex sets of the two components of $T - e$ contains at most $n^{\frac{3}{2}}$ many edges which solves a problem posed by Ostrovskii (Minimal congestion trees, *Discrete Math.* **285** (2004), 219-226.)

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1 Introduction

Let $G = (V, E_G)$ be a connected graph and let $T = (V, E_T)$ be a tree on the same set of vertices. For an edge $e \in E_T$ of T we consider *the congestion* $c(e, (G, T))$ *of* e *with respect to* (G, T) as the number of edges $uv \in E_G$ of G for which e lies on the path in T from u to v, i.e. $c(e, (G, T))$ is the cardinality of the edge-cut defined in G by the vertex sets of the two components of $T - e$. The maximum congestion $\max\{c(e, (G, T)) \mid e \in E_T\}$ is denoted by $c(G, T)$.

Following Ostrovskii [10] we consider the *tree congestion* of G

$$t(G) = \min\{c(G, T) \mid T = (V, E_T) \text{ is a tree}\}$$

and *spanning tree congestion* of G

$$s(G) = \min\{c(G, T) \mid T = (V, E_T) \text{ is a tree with } E_T \subseteq E_G\}.$$

In [10] he proves that $t(G)$ always equals the maximum number of edge-disjoint paths connecting two vertices of G which is also a consequence of the existence of Gomory-Hu trees [5]. Furthermore, he studies the rate of growth of the maximum possible value of $s(G)$ for graphs of order n

$$\mu(n) = \max\{s(G) \mid G = (V, E), |V| = n\}.$$

He proves that $s(G) < \left\lfloor \frac{n^2}{4} \right\rfloor$ for connected graphs $G = (V, E)$ with $n = |V| \geq 6$ and for all odd $k \in \mathbb{N}$ he constructs connected graphs G_k of order $n_k = 3k^2 - 2k$ with $s(G_k) \geq \frac{1}{4}k^3$, i.e. $s(G_k) = \Omega\left(n_k^{\frac{3}{2}}\right)$. As the main open problem he asks for more precise estimates on the rate of growth of $\mu(n)$. In the present paper we prove that $\mu(n) \leq n^{\frac{3}{2}}$. In view of the graphs G_k this determines the growth rate of $\mu(n)$ quite accurately.

The reader should be aware that $t(G)$ and $s(G)$ are two special examples of the numerous graph embedding and layout problems which were considered in connection with applications to networking and circuit design. Restricting T to paths, $t(G)$ corresponds exactly to the very well studied cutwidth [4]. Several other *host graphs* instead of trees such as cycles [3], grids [1] and binary trees [2] were considered. In [7] Hruska determines the exact values of $t(G)$ and $s(G)$ for several special graphs and we refer the reader to [7, 10] for further references.

2 Results

Before we proceed to our main result, we recall a great theorem due to Győri [6] and Lovász [8] concerning highly connected graphs.

Theorem 1 (Győri [6], Lovász [8]) *For $k \in \mathbb{N}$ with $k \geq 2$ let $G = (V, E)$ be a k-connected graph of order n. If $v_1, v_2, ..., v_k \in V$ are k distinct vertices of G and the integers $n_1, n_2, ..., n_k \in \mathbb{N}$ are such that $n_1 + n_2 + ... + n_k = n$, then there exists a partition $V = V_1 \cup V_2 \cup ... \cup V_k$ such that v_i lies in V_i, $|V_i| = n_i$ and $G[V_i]$ is connected for all $1 \leq i \leq k$.*

With this tool at hand, we can proceed to our main result.

Theorem 2 *If $G = (V, E_G)$ is a connected graph of order n, then $s(G) \leq n^{\frac{3}{2}}$.*

Proof: If G has a vertex of degree at least $n-2$, then G has a spanning tree T which arises by subdividing at most one edge of a star. In this case $c(G,T) \leq \max\{n-1, 2(n-2)\} \leq n^{\frac{3}{2}}$. Hence we may assume that G has no such vertex which implies that G has at most $\frac{n(n-3)}{2}$ edges. Since for every tree T, we have $c(G,T) \leq |E_G|$ and for $n \leq 9$, we have $\frac{n(n-3)}{2} \leq n^{\frac{3}{2}}$, the result holds for $n \leq 9$. We may assume that $n \geq 10$ and prove the result by an inductive argument considering two cases.

Case 1 G has a cutset of cardinality at most $\sqrt{n}$.

Let Y be a cutset of minimum cardinality and let Z denote the vertex set of a smallest component of $G[V \setminus Y]$. If $X = V \setminus (Y \cup Z)$, then the subgraph $G[X \cup Y]$ induced by $X \cup Y$ is connected, $x = |X| \geq z = |Z|$, $y = |Y| \leq \sqrt{n}$, and there is no edge between X and Z.

Let $T(X \cup Y)$ be a spanning tree of the subgraph $G[X \cup Y]$ with

$$c(G[X \cup Y], T(X \cup Y)) \leq (x+y)^{\frac{3}{2}}$$

and let $T(Z)$ be a spanning tree of $G[Z]$ with

$$c(G[Z], T(Z)) \leq z^{\frac{3}{2}}.$$

Let $uv \in E_G$ with $u \in Y$ and $v \in Z$ and let

$$T = \left(V, E_{T(X \cup Y)} \cup \{uv\} \cup E_{T(Z)}\right).$$

Note that there are at most yz edges between $X \cup Y$ and Z. This implies that, if $e \in E_{T(X \cup Y)}$, then

$$c(e,(G,T)) \leq (x+y)^{\frac{3}{2}} + yz = (n-z)^{\frac{1}{2}} \cdot (n-z) + yz \leq \sqrt{n} \cdot (n-z) + \sqrt{n} \cdot z = n^{\frac{3}{2}},$$

if $e \in E_{T(Z)}$, then

$$c(e,(G,T)) \leq z^{\frac{3}{2}} + yz = z \cdot \left(\sqrt{z} + y\right) \leq \frac{1}{2} n \cdot \left(\sqrt{n} + \sqrt{n}\right) = n^{\frac{3}{2}}$$

and, finally, if $e = uv$, then $c(e,(G,T)) \leq yz < n^{\frac{3}{2}}$. Altogether, $c(G,T) \leq n^{\frac{3}{2}}$ which completes the proof in this case.

Case 2 G has no cutset of cardinality at most $\sqrt{n}$, i.e. G is $(\lfloor \sqrt{n} \rfloor + 1)$-connected.

Let u be a vertex of degree at least $d = \lfloor \sqrt{n} \rfloor + 1$ and let $v_1, v_2, \ldots, v_d$ be d neighbours of u. If $a, b \in \mathbb{N}_0$ with $0 \leq b \leq \lfloor \sqrt{n} \rfloor$ are such that $n = a \cdot (\lfloor \sqrt{n} \rfloor + 1) - b$, then

$$a = \frac{n}{\lfloor \sqrt{n} \rfloor + 1} + \frac{b}{\lfloor \sqrt{n} \rfloor + 1} < (\lfloor \sqrt{n} \rfloor + 1) + 1 = \lfloor \sqrt{n} \rfloor + 2,$$

i.e. $a \leq \sqrt{n} + 1$. This implies that, if $n = n_1 + n_2 + \ldots + n_d$ and $|n_i - n_j| \leq 1$ for $1 \leq i < j \leq d$, then $n_i \leq \sqrt{n} + 1$.

By Theorem 1, there is a partition $V = V_1 \cup V_2 \cup \ldots \cup V_d$ such that $v_i \in V_i$ and $G[V_i]$ is connected for $1 \leq i \leq d$. We may assume that $u \in V_1$. For $1 \leq i \leq d$ let T_i be an arbitrary spanning tree of $G[V_i]$ and let

$$T = (V, E_T) = \left(V, E_{T_1} \cup \bigcup_{i=2}^{d} \{uv_i\} \cup E_{T_i}\right).$$

Since for every edge $e \in E_T$ one component of $T - e = (V, E_T \setminus \{e\})$ has at most $\sqrt{n} + 1$ many vertices and $n \geq 10$, we obtain

$$c(G,T) \leq \max_{1 \leq x \leq \sqrt{n}+1} x(n-x) = \left(\sqrt{n} + 1\right)\left(n - \sqrt{n} - 1\right) < n^{\frac{3}{2}},$$

which completes the proof. $\square$

In view of the exact values of $s(G)$ and $t(G)$ for special graphs given in [7] and also as a possible strengthening of Theorem 1 one might be tempted to conjecture $\frac{s(G)}{t(G)} = O\left(n^{\frac{1}{2}}\right)$ for a connected G of order n. Nevertheless, considering random d-regular graphs it follows (cf. Theorem 6.4 in [9]) that there are d-regular graphs H_d of arbitrarily large order n_d with $s\left(H_d\right) > \frac{n_d - 1}{d-1}\left(\frac{d}{2} - (1 + o(1))\sqrt{d}\right)$. Since $t(H_d) \leq d$ for these graphs, we see that $\frac{s(G)}{t(G)}$ can be linear in n and our next result is best possible.

Proposition 3 *If $G = (V, E_G)$ be a connected graph of order n, then $s(G) \leq nt(G)$.*

Proof: We prove the result by induction on the order of G. For $n \leq 2$ the result is trivial. Hence let $n \geq 3$.

Let $V_1 \cup V_2$ be a partition of V such that $E(V_1, V_2) = \{uv \in E_G \mid u \in V_1, v \in V_2\}$ is a minimum edge cut of G, i.e. $|E(V_1, V_2)| \leq t(G)$. Since G is connected, the choice of $V_1 \cup V_2$ implies that $G_i = G[V_i]$ is connected for $i = 1, 2$. Let T_i be a spanning tree of G_i with $c(G_i, T_i) \leq |V_i|t(G_i)$. If $uv \in E(V_1, V_2)$ and $T = (V, E_{T_1} \cup E_{T_2} \cup \{uv\})$, then $c(G, T) \leq \max\{c(G_2, T_2), c(G_2, T_2)\} + |E(V_1, V_2)| \leq (n-1)t(G) + t(G) = t(G)$, which completes the proof. □

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