On Spanning Tree Congestion
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Abstract. We prove that every connected graph \( G \) of order \( n \) has a spanning tree \( T \) such that for every edge \( e \) of \( T \) the edge-cut defined in \( G \) by the vertex sets of the two components of \( T - e \) contains at most \( n^{3/2} \) many edges which solves a problem posed by Ostrovskii (Minimal congestion trees, Discrete Math. 285 (2004), 219-226.)

Keywords. Congestion; edge cut; cutwidth; Gomory-Hu tree; layout; embedding

1 Introduction

Let \( G = (V, E_G) \) be a connected graph and let \( T = (V, E_T) \) be a tree on the same set of vertices. For an edge \( e \in E_T \) of \( T \) we consider the congestion \( c(e, (G, T)) \) of \( e \) with respect to \( (G, T) \) as the number of edges \( uv \in E_G \) of \( G \) for which \( e \) lies on the path in \( T \) from \( u \) to \( v \), i.e. \( c(e, (G, T)) \) is the cardinality of the edge-cut defined in \( G \) by the vertex sets of the two components of \( T - e \). The maximum congestion \( \max\{c(e, (G, T)) \mid e \in E_T\} \) is denoted by \( c(G, T) \).

Following Ostrovskii [10] we consider the tree congestion of \( G \)
\[ t(G) = \min\{c(G, T) \mid T = (V, E_T) \text{ is a tree}\} \]
and spanning tree congestion of \( G \)
\[ s(G) = \min\{c(G, T) \mid T = (V, E_T) \text{ is a tree with } E_T \subseteq E_G\}. \]

In [10] he proves that \( t(G) \) always equals the maximum number of edge-disjoint paths connecting two vertices of \( G \) which is also a consequence of the existence of Gomory-Hu trees [5]. Furthermore, he studies the rate of growth of the maximum possible value of \( s(G) \) for graphs of order \( n \)
\[ \mu(n) = \max\{s(G) \mid G = (V, E), |V| = n\}. \]

He proves that \( s(G) < \left\lfloor \frac{n^2}{4} \right\rfloor \) for connected graphs \( G = (V, E) \) with \( n = |V| \geq 6 \) and for all odd \( k \in \mathbb{N} \) he constructs connected graphs \( G_k \) of order \( n_k = 3k^2 - 2k \) with \( s(G_k) \geq \frac{1}{4}k^3 \), i.e. \( s(G_k) = \Omega \left( \frac{n_k^3}{k^3} \right) \). As the main open problem he asks for more precise estimates on the rate of growth of \( \mu(n) \). In the present paper we prove that \( \mu(n) \leq n^{3/2} \). In view of the graphs \( G_k \) this determines the growth rate of \( \mu(n) \) quite accurately.
The reader should be aware that \( t(G) \) and \( s(G) \) are two special examples of the numerous graph embedding and layout problems which were considered in connection with applications to networking and circuit design. Restricting \( T \) to paths, \( t(G) \) corresponds exactly to the very well studied cutwidth [4]. Several other host graphs instead of trees such as cycles [3], grids [1] and binary trees [2] were considered. In [7] Hruska determines the exact values of \( t(G) \) and \( s(G) \) for several special graphs and we refer the reader to [7, 10] for further references.

2 Results

Before we proceed to our main result, we recall a great theorem due to Győri [6] and Lovász [8] concerning highly connected graphs.

**Theorem 1** (Győri [6], Lovász [8]) For \( k \in \mathbb{N} \) with \( k \geq 2 \) let \( G = (V,E) \) be a \( k \)-connected graph of order \( n \). If \( v_1, v_2, ..., v_k \in V \) are \( k \) distinct vertices of \( G \) and the integers \( n_1, n_2, ..., n_k \in \mathbb{N} \) are such that \( n_1 + n_2 + \cdots + n_k = n \), then there exists a partition \( V = V_1 \cup V_2 \cup \cdots \cup V_k \) such that \( v_i \) lies in \( V_i \), \( |V_i| = n_i \) and \( G[V_i] \) is connected for all \( 1 \leq i \leq k \).

With this tool at hand, we can proceed to our main result.

**Theorem 2** If \( G = (V,E_G) \) is a connected graph of order \( n \), then \( s(G) \leq n^3/2 \).

*Proof:* If \( G \) has a vertex of degree at least \( n - 2 \), then \( G \) has a spanning tree \( T \) which arises by subdividing at most one edge of a star. In this case \( c(G,T) \leq \max\{n-1,2(n-2)\} \leq n^3/2 \). Hence we may assume that \( G \) has no such vertex which implies that \( G \) has at most \( \frac{n(n-3)}{2} \) edges. Since for every tree \( T \), we have \( c(G,T) \leq |E_G| \) and for \( n \leq 9 \), we have \( \frac{n(n-3)}{2} \leq n^3/2 \), the result holds for \( n \leq 9 \). We may assume that \( n \geq 10 \) and prove the result by an inductive argument considering two cases.

**Case 1** \( G \) has a cutset of cardinality at most \( \sqrt{n} \).

Let \( Y \) be a cutset of minimum cardinality and let \( Z \) denote the vertex set of a smallest component of \( G[V \setminus Y] \). If \( X = V \setminus (Y \cup Z) \), then the subgraph \( G[X \cup Y] \) induced by \( X \cup Y \) is connected, \( x = |X| \geq z = |Z| \), \( y = |Y| \leq \sqrt{n} \), and there is no edge between \( X \) and \( Z \).

Let \( T(X \cup Y) \) be a spanning tree of the subgraph \( G[X \cup Y] \) with

\[
c(G[X \cup Y], T(X \cup Y)) \leq (x + y)^{\frac{3}{2}}
\]

and let \( T(Z) \) be a spanning tree of \( G[Z] \) with

\[
c(G[Z], T(Z)) \leq z^3.
\]

Let \( uv \in E_G \) with \( u \in Y \) and \( v \in Z \) and let

\[
T = (V, E_T(X \cup Y) \cup \{uv\} \cup E_T(Z)).
\]
Note that there are at most $yz$ edges between $X \cup Y$ and $Z$. This implies that, if $e \in E_{T(X \cup Y)}$, then
\[
c(e, (G, T)) \leq (x+y)^2 + yz = (n-z)^2 \cdot (n-z) + yz \leq \sqrt{n} \cdot (n-z) + \sqrt{n} \cdot z = n^2,
\]
if $e \in E_{T(Z)}$, then
\[
c(e, (G, T)) \leq z^2 + yz = z \cdot (z + y) \leq \frac{1}{2} n \cdot (\sqrt{n} + \sqrt{n}) = n^2
\]
and, finally, if $e = uv$, then $c(e, (G, T)) \leq yz < n^2$. Altogether, $c(G, T) \leq n^2$ which completes the proof in this case.

**Case 2** $G$ has no cutset of cardinality at most $\sqrt{n}$, i.e. $G$ is $(\lceil \sqrt{n} \rceil + 1)$-connected.

Let $u$ be a vertex of degree at least $d = \lceil \sqrt{n} \rceil + 1$ and let $v_1, v_2, \ldots, v_d$ be $d$ neighbours of $u$. If $a, b \in \mathbb{N}_0$ with $0 \leq b \leq \lceil \sqrt{n} \rceil$ are such that $n = a \cdot (\lceil \sqrt{n} \rceil + 1) - b$, then
\[
a = \frac{n}{\lceil \sqrt{n} \rceil + 1} + \frac{b}{\lceil \sqrt{n} \rceil + 1} < (\lceil \sqrt{n} \rceil + 1) + 1 = \lceil \sqrt{n} \rceil + 2,
\]
i.e. $a \leq \sqrt{n} + 1$. This implies that, if $n = n_1 + n_2 + \ldots + n_d$ and $|n_i - n_j| \leq 1$ for $1 \leq i < j \leq d$, then $n_i \leq \sqrt{n} + 1$.

By Theorem 1, there is a partition $V = V_1 \cup V_2 \cup \ldots \cup V_d$ such that $v_i \in V_i$ and $G[V_i]$ is connected for $1 \leq i \leq d$. We may assume that $u \in V_1$. For $1 \leq i \leq d$ let $T_i$ be an arbitrary spanning tree of $G[V_i]$ and let
\[
T = (V, E_T) = \left(V, E_{T_1} \cup \bigcup_{i=2}^{d} \{uv_i \} \cup E_T \right).
\]
Since for every edge $e \in E_T$ one component of $T - e = (V, E_T \setminus \{e\})$ has at most $\sqrt{n} + 1$ many vertices and $n \geq 10$, we obtain
\[
c(G, T) \leq \max_{1 \leq x \leq \sqrt{n}+1} x(n-x) = (\sqrt{n}+1)(n-\sqrt{n}-1) < n^2,
\]
which completes the proof. $\square$

In view of the exact values of $s(G)$ and $t(G)$ for special graphs given in [7] and also as a possible strengthening of Theorem 1 one might be tempted to conjecture $\frac{s(G)}{t(G)} = O\left(n^{\frac{1}{2}}\right)$ for a connected $G$ of order $n$. Nevertheless, considering random $d$-regular graphs it follows (cf. Theorem 6.4 in [9]) that there are $d$-regular graphs $H_d$ of arbitrarily large order $n_d$ with $s(H_d) > \frac{d-1}{4d+1} \left(\frac{d}{2} - (1+o(1))\sqrt{d}\right)$. Since $t(H_d) \leq d$ for these graphs, we see that $\frac{s(G)}{t(G)}$ can be linear in $n$ and our next result is best possible.
**Proposition 3** If $G = (V, E_G)$ be a connected graph of order $n$, then $s(G) \leq nt(G)$.

**Proof:** We prove the result by induction on the order of $G$. For $n \leq 2$ the result is trivial. Hence let $n \geq 3$.

Let $V_1 \cup V_2$ be a partition of $V$ such that $E(V_1, V_2) = \{uv \in E_G \mid u \in V_1, v \in V_2\}$ is a minimum edge cut of $G$, i.e. $|E(V_1, V_2)| \leq t(G)$. Since $G$ is connected, the choice of $V_1 \cup V_2$ implies that $G_i = G[V_i]$ is connected for $i = 1, 2$. Let $T_i$ be a spanning tree of $G_i$ with $c(G_i, T_i) \leq |V_i|t(G_i)$. If $uv \in E(V_1, V_2)$ and $T = (V, E_{T_1} \cup E_{T_2} \cup \{uv\})$, then $c(G, T) \leq \max\{c(G_2, T_2), c(G_2, T_2)\} + |E(V_1, V_2)| \leq (n-1)t(G) + t(G) = t(G)$, which completes the proof. □

**Acknowledgement.** We thank Diego Scheide for suggesting some simplifications.

**References**


