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Spectrum and Analyticity of Semigroups arising in Elasticity Theory and Hydromechanics

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Abstract Cauchy problems for a second order linear differential operator equation

$$\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0$$

in a Hilbert space H are studied. Equations of this kind arise for example in elasticity and hydrodynamics. It is assumed that A_0 is a uniformly positive operator and that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded accretive operator in H. The location of the spectrum of the corresponding semigroup generator is described and sufficient conditions for analyticity are given.

Keywords Block operator matrices · analytic semigroups · spectrum · second order equations · accretive operators

Mathematics Subject Classification (2000) 47A10 · 34G10 · 47D06

1 Introduction

The small transverse oscillations of a horizontal pipe of length 1, carrying steady-state fluid of ideal incompressible fluid are described by the equation [20]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2} + C\frac{\partial^3 u}{\partial r^2 \partial t}\right] + K\frac{\partial^2 u}{\partial t \partial x} = 0, \qquad r \in (0,1), t > 0. \tag{1}$$

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Let $u(r,t)$ denote the transverse oscillations at time t and position r, and E, C, K are positive physical constants. The fourth term in the left hand side of (1) is called the gyroscopic term.

The existence and behaviour of solutions u depend also on boundary and initial conditions. In the example above we are interested in a solution having finite energy, i.e. solutions such that $\|u(\cdot,t)\|^2 + \|u''(\cdot,t)\|^2 < \infty$ for all $t > 0$ where $\|\cdot\|$ denotes the usual norm in the Hilbert space $L^2(0,1)$. Identifying the function $u(\cdot,t)$ with an element $z(t) \in L^2(0,1)$ by $z(t)(r) = u(r,t)$ we obtain from the partial differential equation above a second order equation in $L^2(0,1)$ of the form

$$\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) = 0, \tag{2}$$

where $A_0 = E\frac{\partial^4}{\partial r^4}$, $D = \frac{\partial^2}{\partial r^2} C \frac{\partial^2}{\partial r^2} + K\frac{\partial}{\partial r}$ acting in $L^2(0,1)$ with appropriate domains encoding the boundary conditions under consideration. We will come back to this example in Section 5.

We mention that problems of the form (2) with an positive, boundedly invertible operator A_0 and a bounded accretive operator D arise in many problems in hydrodynamics, we mention here only [18, Chapters 6.4 and 6.5].

The aim of this paper is the study of second order equations of the form (2). Here the stiffness operator A_0 is a possibly unbounded positive operator on a Hilbert space H and is assumed to be boundedly invertible, and D, the damping operator, is an unbounded operator satisfying that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded accretive operator in H.

This second order equation is equivalent to the standard first-order equation $\dot{x}(t) = \mathcal{A}x(t)$ where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{D}(A_0^{1/2}) \times H \to \mathcal{D}(A_0^{1/2}) \times H$ is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] \in \mathcal{D}(A_0^{1/2}) \times \mathcal{D}(A_0^{1/2}) \mid A_0 z + Dw \in H \right\}.$$

This operator matrix has been studied in the literature for more than 20 years, see for example [1–7, 10–14, 16, 17, 19, 22, 23], and the references therein. However, most of the papers require the damping operator $A_0^{-1/2} D A_0^{-1/2}$ to be self-adjoint.

Papers dealing with accretive damping are [10, 11, 20]. In [20] an instability index formula is developed. Exponential stability of the semigroup generated by $\mathcal{A}$ is studied in [11] and in [10] sufficient conditions for analyticity of the semigroup generated by $\mathcal{A}$ are given. The first main result of [10] shows that the semigroup is analytic, if D, considered as an operator in H, is maximal sectorial satisfying some restrictions on the semiangle (cf. [10, Theorem 3]) and the domains of the Friedrichs extension of D and its adjoint are subsets of $\mathcal{D}(A_0^{-1/2})$. The second main result of [10] proves analyticity of the semigroup if there exist constants $\rho_1, \rho_2 > 0$ with $\rho_1 A_0^\theta \le \operatorname{Re} D \le \rho_2 A_0^\theta$ for some $\theta \in [1/2, 1]$.

In this paper we focus on two properties of the operator $\mathcal{A}$: Location of the spectrum of the operator $\mathcal{A}$ and analyticity of the generated semigroup.

For self-adjoint dampings the location of the spectrum is well-understood, but not studied in detail for accretive damping. Our aim of this paper is to extend the results of [16] to this more general situation. We use various upper and lower bounds of the quantity $\operatorname{Re}\langle Dz, z\rangle$ divided by the norm of z, where the norm of z is taken in different spaces. This enables us to give new results and pictures for the location of the spectrum of $\mathcal{A}$.

We further develop conditions guaranteeing analyticity of the semigroup generated by $\mathcal{A}$. Contrary to [10], we always start with an operator D which acts between $\mathcal{D}(A_0^{1/2})$ and $\mathcal{D}(A_0^{-1/2})$, each equipped with the corresponding graph norm. Our setup has the advantage that the operator $\mathcal{A}$ is closed. Under the weak assumption that there exist constants $M_0, M_1 > 0$ and $\omega_0 > 0$ such that

$$\|A_0^{1/2}(D + (\omega_0 + z)E)^{-1}A_0^{1/2}x\| \le M_0\|x\|, \quad x \in H, \quad \operatorname{Re} z > 0, \tag{3}$$

and

$$\|(\omega_0 + z)(D + (\omega_0 + z)E)^{-1}x\| \le M_1\|x\|, \quad x \in H, \quad \operatorname{Re} z > 0 \tag{4}$$

we show that the operator $\mathcal{A}$ is the generator of an analytic C_0-semigroup. We are able to show that (3) and (4) are satisfied under the assumption that

$$\delta := \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{\|z\|^2_{H_{\frac{1}{2}}}} > 0.$$

Further, $\delta > 0$ implies that $\omega_0 I - \mathcal{A}$ is maximal-ω-accretive for some $\omega_0 > 0$, which is a stronger property than analyticity of the semigroup generated by $\mathcal{A}$. Note that this result improves the second main result of [10] for $\theta = 1$ in a slightly different setup.

We proceed as follows. In Section 2 we give the precise definition of the operator $\mathcal{A}$ and prove some of its properties. The main results of this paper are contained in Sections 3 and 4. In Section 3 sufficient conditions are given to guarantee that certain regions are contained in the resolvent set of $\mathcal{A}$. The main result of Section 4 is that $\mathcal{A}$ generates an analytic strongly continuous semigroup under suitable conditions on the damping operator D. Finally, in Section 5 the results are illustrated by an example: small transverse oscillations of a horizontal pipe of length one.

2 Framework and preliminary results

Throughout this paper we make the following assumptions.

(A1) The stiffness operator $A_0 : \mathcal{D}(A_0) \subset H \to H$ is a self-adjoint, positive definite linear operator on a Hilbert space H such that zero is in the resolvent set of A_0. A scale of Hilbert spaces H_α is defined as follows: For $\alpha \ge 0$, we define $H_\alpha = \mathcal{D}(A_0^\alpha)$ equipped with the norm $\|\cdot\|_{H_\alpha} := \|A_0^\alpha \cdot\|_H$ and

$H_{-\alpha} = H_\alpha^*$. Here the duality is taken with respect to the pivot space H, that is, equivalently $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{H_{-\alpha}} = \|A_0^{-\alpha} z\|_H$. Thus A_0 extends (restricts) to $A_0 : H_\alpha \to H_{\alpha-1}$ for $\alpha \in \mathbb{R}$. We use the same notation A_0 to denote this extension (restriction), but we will mention it explicitly if A_0 is considered as an operator acting between H_α and $H_{\alpha-1}$ for some $\alpha \in \mathbb{R}$.

We denote the inner product on H by $\langle \cdot, \cdot \rangle_H$ or $\langle \cdot, \cdot \rangle$, and the duality pairing on $H_{-\alpha} \times H_\alpha$ by $\langle \cdot, \cdot \rangle_{H_{-\alpha} \times H_\alpha}$. Note that for $(z', z) \in H \times H_\alpha$, $\alpha > 0$, we have

$$\langle z', z \rangle_{H_{-\alpha} \times H_\alpha} = \langle z', z \rangle_H.$$

In the following we will consider $\alpha = \frac{1}{2}$. There exists a constant a_0 such that

$$\|z\|_{H_{\frac{1}{2}}} = \|A_0^{\frac{1}{2}} z\| \geq a_0 \|z\|, \quad \text{for all } z \in H_{\frac{1}{2}}, \tag{5}$$

where a_0 can be chosen as $\|A_0^{-\frac{1}{2}}\|^{-1}$.

(A2) The damping operator $D : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is a bounded operator such that $A_0^{-1/2} D A_0^{-1/2}$ is a bounded accretive operator in H, that is,

$$\operatorname{Re} \langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \qquad z \in H_{\frac{1}{2}}.$$

The system (2) is equivalent to the following standard first-order equation

$$\dot{x}(t) = \mathcal{A} x(t) \tag{6}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset H_{\frac{1}{2}} \times H \to H_{\frac{1}{2}} \times H$, is given by

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ -A_0 & -D \end{bmatrix},$$

$$\mathcal{D}(\mathcal{A}) = \left\{ \left(\begin{smallmatrix} z \\ w \end{smallmatrix} \right) \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \mid A_0 z + D w \in H \right\}.$$

In [23, Proof of Lemma 4.5] it is shown that $\mathcal{A}$ has a bounded inverse in $H_{\frac{1}{2}} \times H$, with

$$\mathcal{A}^{-1} = \begin{bmatrix} -A_0^{-1} D & -A_0^{-1} \\ I & 0 \end{bmatrix}, \tag{7}$$

where $A_0^{-1} D$ is considered as an operator acting in $H_{\frac{1}{2}}$.

Throughout this paper we will use the following notation. For a closed densely defined linear operator S on some Banach space X we denote by $\sigma_c(S)$, $\sigma_r(S)$, and $\sigma_p(S)$, the continuous spectrum, the residual spectrum, and the point spectrum, respectively. The *approximate point spectrum*, $\sigma_{ap}(S)$, consists of all λ for which there is a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $\mathcal{D}(S)$ such that

$$\|x_n\| = 1 \text{ and } \|(S - \lambda I) x_n\| \to 0 \quad \text{as} \quad n \to \infty$$

(see for example [8, page 242]). We point out that the point and continuous spectrum are subsets of the approximate point spectrum. Moreover, the boundary of the spectrum $\sigma(S)$ belongs to $\sigma_{ap}(S)$, see e.g. [8, §IV 1.10]. We set

$$r(S) := \mathbb{C} \backslash \sigma_{ap}(S).$$

3 Location of the spectrum of $\mathcal{A}$

The following theorem is well known, see e.g. [10].

Theorem 1 *The operator $\mathcal{A}$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of contractions on the state space $H_{\frac{1}{2}} \times H$.*

This guarantees that the spectrum of $\mathcal{A}$ is contained in the closed left half plane. However, otherwise the spectrum of $\mathcal{A}$ is quite arbitrary, see [16].

In the following theorem we give sufficient conditions guaranteeing that $\sigma(\mathcal{A})$ is contained in a smaller subset of $\mathbb{C}$. We define the following constants:

$$\beta := \inf_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_H^2},$$

$$\gamma := \sup_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_H^2},$$

$$\delta := \inf_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_{H_{\frac{1}{2}}}^2},$$

$$\eta := \sup_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_{H_{\frac{1}{2}}}^2},$$

$$\nu := \sup_{z\in H_{\frac{1}{2}}\setminus\{0\}} \frac{\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\|z\|_H \|z\|_{H_{\frac{1}{2}}}}.$$

By definition we have $\beta, \delta, \eta \in [0, \infty)$, and it is easy to see[1] that $a_0^2\delta \leq \beta \leq \gamma$ and

$$a_0^2\delta \leq a_0^2\eta \leq a_0\nu \leq \gamma, \tag{8}$$

where a_0 is as in (5).

Theorem 2 *The following assertion are true.*

1. *If $\beta > 0$ and if $\|D\|$ denotes the norm of the bounded operator $D \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$, then*
$$\left\{i\sigma \mid |\sigma| < \|D\|^{-1}\right\} \subset \rho(\mathcal{A}).$$
2. *If $\gamma < \infty$ then*
$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\gamma\} \subset r(\mathcal{A}).$$
In particular, if $\gamma < 2a_0$, where a_0 is given by (5), then
$$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\gamma\} \subset \rho(\mathcal{A}).$$

[1] We mention that in [16, page 51] the constant a_0^2 is missing; the correct inequality reads as $a_0^2\delta \leq \beta \leq \gamma$.

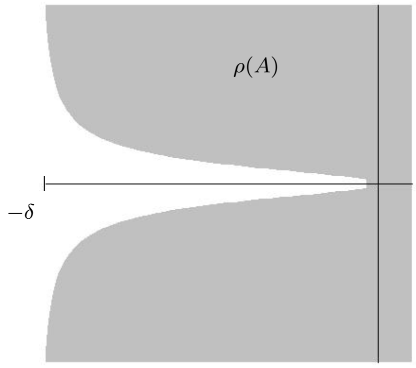

Fig. 1 Theorem 2, Part 3, $\delta > 0$ with $a_0 = 1$

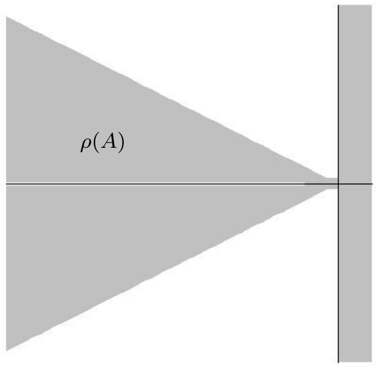

Fig. 2 Theorem 2: Part 5, $0 < \nu < 2$

3. If $\delta > 0$ *then*

$$M_\delta := \left\{\lambda \in \mathbb{C} \mid \delta > |\operatorname{Re}\lambda| \left(a_0^{-2} + |\lambda|^{-2}\right)\right\} \subset \rho(\mathcal{A}).$$

4. If $0 < \eta < \infty$ *then*

$$M_\eta := \left\{\lambda \in \mathbb{C} \mid \left|\lambda + \frac{1}{2\eta}\right| < \frac{1}{2\eta}\right\} \subset \rho(\mathcal{A}).$$

5. If $0 < \nu < 2$ *then*

$$M_\nu := \left\{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| < \sqrt{\frac{4}{\nu^2} - 1}|\operatorname{Re}\lambda|, \operatorname{Im}\lambda \neq 0\right\} \subset \rho(\mathcal{A})$$

and

$$\left(-\frac{a_0}{\nu} - \frac{4a_0}{\nu^3}, 0\right) \subset \rho(\mathcal{A}).$$

In particular, the open interval $(-a_0, 0)$ *belongs to* $\rho(\mathcal{A})$ *for all* $0 < \nu < 2$.

The following lemma is needed for the proof of Theorem 2.

Lemma 1 *Let* $\lambda = \mu + i\sigma$ *with* $\sigma \in \mathbb{R}$, $\mu \leq 0$ *and* $\lambda \neq 0$. *Assume that there exists a sequence* $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ *in* $\mathcal{D}(\mathcal{A})$ *with*

$$\|\tbinom{x_n}{y_n}\|_{H_{\frac{1}{2}}\times H} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|(\lambda I - \mathcal{A})\tbinom{x_n}{y_n}\|_{H_{\frac{1}{2}}\times H} = 0. \tag{9}$$

Then we have

1. $\|y_n - \lambda x_n\|_{H_{\frac{1}{2}}} \to 0$ *as* $n \to \infty$.
2. $\liminf_{n\to\infty} \|x_n\|_{H_{\frac{1}{2}}} > 0$.

3. *If $\sigma \neq 0$, then we have as $n \to \infty$,*

$$\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \sigma\|x_n\|^2 - \frac{\sigma}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \to 0, \tag{10}$$

$$\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\|x_n\|^2 + \frac{\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \to 0, \tag{11}$$

$$\mu\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \frac{2\sigma\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \to 0, \tag{12}$$

and

$$\mu\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\sigma\mu\|x_n\|^2 \to 0. \tag{13}$$

4. *If $\sigma = 0$, then we have* $\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \to 0$ *and*

$$\|x_n\|^2_{H_{\frac{1}{2}}} + \mu\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\|x_n\|^2 \to 0, \quad n \to \infty. \tag{14}$$

Proof (9) implies

$$\|y_n - \lambda x_n\|_{H_{\frac{1}{2}}} \to 0 \text{ and} \tag{15}$$

$$\|A_0x_n + Dy_n + \lambda y_n\| \to 0 \text{ as } n \to \infty. \tag{16}$$

It follows from (15) that $(x_n)_{n\in\mathbb{N}}$ has no subsequence which converges to zero in $H_{\frac{1}{2}}$. Thus Part 1 and Part 2 are shown. Combining (15) and (16) we get

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu+i\sigma)\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu+i\sigma)^2\langle x_n, x_n\rangle \to 0, \tag{17}$$

as $n \to \infty$. This implies the result for $\sigma = 0$. It remains to show Part 3. Let $\sigma \neq 0$. Then the imaginary part of (17) tends to zero, i.e.

$$\sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu\sigma\|x_n\|^2 \to 0, \tag{18}$$

as $n \to \infty$, which proves (13). Further, the real part tends to zero, i.e.

$$\|x_n\|^2_{H_{\frac{1}{2}}} + \mu\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \sigma\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu^2-\sigma^2)\|x_n\|^2 \to 0, \tag{19}$$

as $n \to \infty$. Combining (18) and (19), we obtain (10), (11) and (12). □

Proof (of Theorem 2)

1. As $0 \in \rho(\mathcal{A})$ and the boundary of the spectrum $\sigma(\mathcal{A})$ belongs to $\sigma_{ap}(\mathcal{A})$, it is sufficient to show that the intersection $\{i\sigma \mid |\sigma| < \|D\|^{-1}\} \cap \sigma_{ap}(\mathcal{A}) = \emptyset$. Assume $i\sigma$, $\sigma \neq 0$, with $|\sigma| < \|D\|^{-1}$ belongs to $\sigma_{ap}(\mathcal{A})$. Then there exists a sequence $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathcal{A})$ which satisfies (9). Then (11) implies

$$\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \to 0, \quad n \to \infty,$$

and, as $\beta > 0$, also $\lim_{n\to\infty}\|x_n\| = 0$. We have

$$\sigma\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \|x_n\|^2_{H_{\frac{1}{2}}} \leq (|\sigma|\|D\| - 1)\,\|x_n\|^2_{H_{\frac{1}{2}}},$$

a contradiction to (10) and Part 2 of Lemma 1.

2. Let $\lambda = \mu + i\sigma$ and assume that λ belongs to the approximate point spectrum of $\mathcal{A}$. Then there exists a sequence $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathcal{A})$ which satisfies (9). Let $\mu \leq -\gamma < 0$. If $\sigma \neq 0$, then for $\|x_n\|_{H_{\frac{1}{2}}} \neq 0$

$$\begin{aligned} 0 > \frac{\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} &\geq (\gamma+\mu)\|x_n\|^2 + \frac{\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \\ &\geq \operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\|x_n\|^2 + \frac{\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}}, \end{aligned}$$

which contradicts (11) and Part 2 of Lemma 1.
If $\sigma = 0$, then for $\|x_n\|_{H_{\frac{1}{2}}} \neq 0$

$$\begin{aligned} 0 < \|x_n\|^2_{H_{\frac{1}{2}}} &\leq \|x_n\|^2_{H_{\frac{1}{2}}} + (\mu\gamma+\mu^2)\|x_n\|^2 \\ &\leq \|x_n\|^2_{H_{\frac{1}{2}}} + \mu\operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\|x_n\|^2, \end{aligned}$$

which contradicts (14) and Part 2 of Lemma 1.
In the case $\gamma < 2a_0$ we deduce from (8) that $\nu < 2$. Then the sets $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\gamma\}$ and M_ν have a nonempty intersection and, part 5 (which is proved below) implies that M_ν belongs to the resolvent set. Thus the set $\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leq -\gamma\}$ belongs also to the resolvent set of $\mathcal{A}$.

3. Note that $M_\delta \cup \{0\}$ is connected. Due to the fact that $0 \in \rho(\mathcal{A})$ it is enough to show that the intersection $M_\delta \cap \sigma_{ap}(\mathcal{A})$ is empty. Assume that there is a $\lambda = \mu + i\sigma$ belonging to $M_\delta \cap \sigma_{ap}(\mathcal{A})$. Let $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{A})$ which satisfies (9). We have for $\|x_n\|_{H_{\frac{1}{2}}} \neq 0$ and $\sigma \neq 0$

$$\begin{aligned} 0 &< \left(\delta + \frac{\mu}{\mu^2+\sigma^2} + \frac{\mu}{a_0^2}\right)\|x_n\|^2_{H_{\frac{1}{2}}} \\ &\leq \operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\|x_n\|^2 + \frac{\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}}, \end{aligned}$$

where we used (5). This contradicts (11) and Part 2 of Lemma 1.
If $\sigma = 0$, then for $\|x_n\|_{H_{\frac{1}{2}}} \neq 0$ and $\mu < 0$ with $1 + \mu\delta + \mu^2 a_0^{-2} < 0$ (or equivalently $\delta > |\mu|(a_0^{-2} + |\mu|^{-2})$) we have

$$0 > (1+\mu\delta+\mu^2 a_0^{-2})\|x_n\|^2_{H_{\frac{1}{2}}} \geq \|x_n\|^2_{H_{\frac{1}{2}}} + \mu\operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\|x_n\|^2,$$

which contradicts (14) and Part 2 of Lemma 1.

4. The set M_η intersected with the open lower (upper) half-plane is connected. Due to the fact that $0 \in \rho(\mathcal{A})$ it is enough to show that the intersection with $\sigma_{ap}(\mathcal{A})$ is empty. Let $\lambda = \mu + i\sigma$ be in M_η, i.e.

$$\left(\mu + \frac{1}{2\eta}\right)^2 + \sigma^2 < \frac{1}{4\eta^2}.$$

Hence,

$$\eta\mu^2 + \eta\sigma^2 + \mu < 0$$

and this yields

$$\eta + \frac{\mu}{|\lambda|^2} < 0. \tag{20}$$

Assume that λ belongs to the approximate point spectrum of $\mathcal{A}$. Then there exists a sequence $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathcal{A})$ which satisfies (9). For $\|x_n\|_{H_{\frac{1}{2}}} \neq 0$ and $\sigma \neq 0$ we have with (20)

$$\begin{aligned} 0 &> \left(\eta + \frac{\mu}{|\lambda|^2}\right)\|x_n\|^2_{H_{\frac{1}{2}}} \\ &\geq \operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\|x_n\|^2 + \frac{\mu}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}}, \end{aligned}$$

which contradicts (11) and Part 2 of Lemma 1.
If $\sigma = 0$, then (20) gives

$$\mu\eta + 1 > 0$$

and for $\|x_n\|_{H_{\frac{1}{2}}} \neq 0$

$$0 < (\mu\eta+1)\|x_n\|^2_{H_{\frac{1}{2}}} \leq \|x_n\|^2_{H_{\frac{1}{2}}} + \mu\operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\|x_n\|^2,$$

which contradicts (14) and Part 2 of Lemma 1.

5. Note that the set M_ν intersected with the open lower (upper) half-plane is connected. Due to the fact that $0 \in \rho(\mathcal{A})$ it is enough to show that the intersection with $\sigma_{ap}(\mathcal{A})$ is empty. The set M_ν can be written in the following way:

$$M_\nu = \left\{\lambda \in \mathbb{C} \mid \frac{\nu}{2}|\lambda| < |\operatorname{Re}\lambda|, \operatorname{Im}\lambda \neq 0\right\}.$$

Assume that there is a $\lambda = \mu + i\sigma$, $\mu \neq 0$, $\sigma \neq 0$, in $\sigma_{ap}(\mathcal{A})$ with $\frac{\nu}{2}|\lambda| \leq |\mu|$. Let $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{A})$ which satisfies (9). We claim

$$\liminf_{n\to\infty}\left(\frac{\|x_n\|_{H_{\frac{1}{2}}}}{|\lambda|} - \|x_n\|\right)^2 > 0 \tag{21}$$

Assume that (21) is not true. Then it is no restriction to assume that

$$\lim_{n\to\infty}\frac{\|x_n\|_{H_{\frac{1}{2}}}}{|\lambda|} - \|x_n\| = 0 \tag{22}$$

Then, (11) and (22) imply

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left(\operatorname{Re}\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu\|x_n\|^2\right) \\ &\leq \lim_{n\to\infty}\left(\nu\|x_n\|\|x_n\|_{H_{\frac{1}{2}}} + 2\mu\|x_n\|^2\right) = \lim_{n\to\infty} 2\left(\frac{\nu}{2}|\lambda| + \mu\right)\|x_n\|^2 \leq 0. \end{aligned}$$

This implies $\lim_{n\to\infty}\|x_n\| = 0$, which contradicts (22) and Part 2 of Lemma 1, hence (21) holds.

We have

$$
\begin{aligned}
&-\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + |\mu|\|x_n\|^2 + \frac{|\mu|}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \\
&\geq -2\frac{|\mu|}{|\lambda|}\|x_n\|\|x_n\|_{H_{\frac{1}{2}}} + |\mu|\|x_n\|^2 + \frac{|\mu|}{\mu^2+\sigma^2}\|x_n\|^2_{H_{\frac{1}{2}}} \\
&= \left(\sqrt{|\mu|}\|x_n\| - \frac{\sqrt{|\mu|}}{|\lambda|}\|x_n\|_{H_{\frac{1}{2}}}\right)^2
\end{aligned}
$$

and this is, with (21), a contradiction to (11).
If $\sigma = 0$ and $\mu \in \left(-\frac{a_0}{\nu} - \frac{4a_0}{\nu^3}, 0\right)$, we have $a_0^{-1}\mu\nu + 1 + \frac{4}{\nu^2} > 0$. Let $(\left(\begin{smallmatrix} x_n \\ y_n \end{smallmatrix}\right))_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{A})$ which satisfies (9). Then

$$
\begin{aligned}
&\|x_n\|^2_{H_{\frac{1}{2}}} + \mu\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\|x_n\|^2 \\
&\geq \|x_n\|^2_{H_{\frac{1}{2}}} + \mu\nu\|x_n\|\|x_n\|_{H_{\frac{1}{2}}} + \frac{\nu^2\mu^2}{4}\|x_n\|^2 \\
&= \left(\|x_n\|_{H_{\frac{1}{2}}} + \frac{\nu\mu}{2}\|x_n\|\right)^2 \geq 0,
\end{aligned}
$$

hence, by (14),

$$
\lim_{n\to\infty}\left(\|x_n\|_{H_{\frac{1}{2}}} + \frac{\nu\mu}{2}\|x_n\|\right) = 0.
$$

Moreover, this gives with (14)

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\left(\|x_n\|^2_{H_{\frac{1}{2}}} + \mu\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\|x_n\|^2\right) \\
&\geq \lim_{n\to\infty}\left(\frac{\mu\nu}{a_0}\|x_n\|^2_{H_{\frac{1}{2}}} + \left(1+\frac{4}{\nu^2}\right)\|x_n\|^2_{H_{\frac{1}{2}}}\right),
\end{aligned}
$$

which contradicts Part 2 of Lemma 1.

□

4 Accretive Damping

First, we present the defining properties of accretive operators.

Let $\omega \in [0, \pi/2)$. An operator $S : \mathcal{D}(S) \subset H \to H$ is called m-ω-accretive if $\mathrm{Ran}(S+I)$ is dense in H and if

$$
|\mathrm{Im}\,\langle Sx, x\rangle| \leq (\tan\omega)\mathrm{Re}\,\langle Sx, x\rangle, \quad x \in \mathcal{D}(S).
$$

This means that the numerical range of S is contained in the closure of the sector S_ω. Here S_θ, for some $\theta \in [0, \pi]$, denotes the sector of angle 2θ symmetric about $(0, \infty)$,

$$
S_\theta := \begin{cases} \{z \in \mathbb{C} \mid z \neq 0, \text{ and } |\arg z| < \theta\} & \text{if } \theta \in (0, \pi], \\ (0, \infty) & \text{if } \theta = 0. \end{cases}
$$

For $\omega \in [0, \pi/2)$ it is shown in [9, Section 7.1] that an operator S is m-ω-accretive if and only if $-S$ generates an analytic C_0-semigroup $(T(z))_{z \in S_{\pi/2-\omega}}$ on H such that $\|T(z)\| \le 1$ for every $z \in S_{\pi/2-\omega}$.

We call $D \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ m-ω-accretive if the operator $A_0^{-1/2} D A_0^{-1/2} \in \mathcal{L}(H)$ is m-ω-accretive. Thus $D \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ is m-ω-accretive if and only if

$$|\mathrm{Im}\,\langle Dx, x\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}| \le (\tan\omega)\mathrm{Re}\,\langle Dx, x\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}, \quad x \in H_{\frac{1}{2}}. \tag{23}$$

Note that if $D \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ satisfies (23) then $\mathrm{Ran}(A_0^{-1/2} D A_0^{-1/2} + I) = H$.

The following lemma relates the property $\delta > 0$ to the m-ω-accretivity of the operator D.

Lemma 2 *If $\delta > 0$, then the damping operator D is m-ω-accretive. The converse direction is in general not true.*

Proof $D = 0$ shows directly that the converse direction in general does not hold. We next suppose that $\delta > 0$. Assuming that D is not m-ω-accretive, there exists a sequence (z_n) in $H_{\frac{1}{2}}$ such that

$$\mathrm{Re}\,\langle Dz_n, z_n\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \to 0 \text{ and } |\mathrm{Im}\,\langle Dz_n, z_n\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}| = 1.$$

The assumption $\delta > 0$ implies that $\|z_n\|_{H_{\frac{1}{2}}} \to 0$ as $n \to \infty$. Thus

$$1 = |\mathrm{Im}\,\langle Dz_n, z_n\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}| \le \|Dz_n\|_{H_{-\frac{1}{2}}} \|z_n\|_{H_{\frac{1}{2}}} \le c\|z_n\|^2_{H_{\frac{1}{2}}},$$

which is in contradiction to $\|z_n\|_{H_{\frac{1}{2}}} \to 0$ as $n \to \infty$. □

Theorem 3 *Assume that D is m-$(\arctan k)$-accretive with $k > 0$.*

1. *If $\beta > 0$ then*

$$M_\beta := \left\{\mu + i\sigma \in \mathbb{C} \mid -\beta < 2\mu < 0, |\sigma| > \frac{k\beta|\mu|}{\beta - 2|\mu|}\right\} \subset \rho(\mathcal{A}).$$

2. *If $\delta > 0$ then*

$$M_\beta \cup \left\{\mu + i\sigma \in \mathbb{C} \mid |\sigma| > \frac{1}{\delta} + k|\mu|\right\} \subset \rho(\mathcal{A}).$$

We note that the constants β and δ have been defined in the previous section, and that we have $a_0^2\delta \le \beta$.

Proof Let $\lambda = \mu + i\sigma$ and assume that λ belongs to the approximate point spectrum of $\mathcal{A}$. Let $(\binom{x_n}{y_n})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{A})$ which satisfies (9).

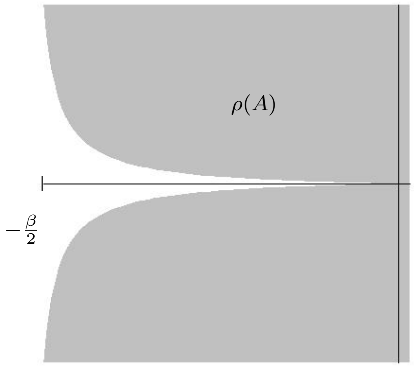

Fig. 3 Theorem 3, Part 1, $\beta > 0$

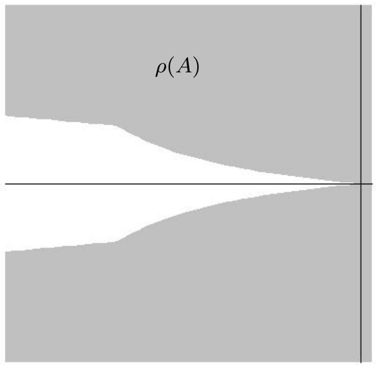

Fig. 4 Theorem 3, Part 2, $\delta > 0$

1. Let $\mu < 0$, $-\beta < 2\mu < 0$ and $|\sigma| > \frac{k\beta|\mu|}{\beta - 2|\mu|} > 0$. This implies

$$|\sigma|\beta - 2|\sigma||\mu| > k\beta|\mu| \quad \text{and} \quad k|\mu| - |\sigma| < -\frac{2|\sigma||\mu|}{\beta} < 0. \tag{24}$$

For $\|x_n\| \neq 0$ we have

$$\begin{aligned} 0 &> ((k|\mu| - |\sigma|)\beta + 2|\sigma||\mu|)\, \|x_n\|^2 \\ &\geq (k|\mu| - |\sigma|)\mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + 2|\sigma|\, |\mu| \|x_n\|^2 \\ &\geq |\mu| \left| \mathrm{Im}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \right| - |\sigma| \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \\ &\quad +2|\sigma|\, |\mu| \|x_n\|^2. \end{aligned} \tag{25}$$

If $\sigma > 0$, we obtain

$$0 > -\mu \mathrm{Im}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} - \sigma \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} - 2\sigma\, \mu \|x_n\|^2 \tag{26}$$

and if $\sigma < 0$,

$$0 > \mu \mathrm{Im}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \sigma \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + 2\sigma\, \mu \|x_n\|^2. \tag{27}$$

Then (26) and (27), together with (13) and (25), imply $\lim_{n \to \infty} \|x_n\| = 0$, hence

$$\mu \mathrm{Im}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \sigma \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \to 0, \quad n \to \infty. \tag{28}$$

If $\sigma > 0$, we have

$$\begin{aligned} &-\mu \mathrm{Im}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} - \sigma \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \\ &\leq |\mu| \left| \mathrm{Im}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \right| - |\sigma| \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \\ &\leq (|\mu| k - |\sigma|)\, \mathrm{Re}\, \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \end{aligned}$$

and if $\sigma < 0$,

$$\begin{aligned}&\mu\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\\ &\leq |\mu|\left|\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\right| - |\sigma|\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\\ &\leq (|\mu|k - |\sigma|)\,\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\end{aligned}$$

From (24) and (28), we conclude $\lim_{n\to\infty}\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = 0$. Then (28) implies $\lim_{n\to\infty}\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = 0$, a contradiction to (12) and Part 2 of Lemma 1. Hence $\lambda = \mu + i\sigma \notin \sigma_{ap}(\mathcal{A})$ if $-\beta < 2\mu < 0$ and $|\sigma| > \frac{k\beta|\mu|}{\beta - 2|\mu|} > 0$. But, $0 \in \rho(\mathcal{A})$ and the boundary of $\sigma(\mathcal{A})$ belongs to $\sigma_{ap}(\mathcal{A})$, Part 1 of Theorem 3 is proved.

2. Let $\frac{1}{\delta} < |\sigma| - |\mu|k$. If $\sigma > 0$, we have $(|\mu|k - \sigma)\,\delta < -1$ and

$$\begin{aligned}&\mu\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \frac{2\sigma\mu}{\sigma^2+\mu^2}\|x_n\|^2_{H_{\frac{1}{2}}}\\ &\leq |\mu|\left|\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\right| - \sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \frac{2\sigma|\mu|}{\sigma^2+\mu^2}\|x_n\|^2_{H_{\frac{1}{2}}}\\ &\leq ((|\mu|k - \sigma)\delta + 1)\,\|x_n\|^2_{H_{\frac{1}{2}}}.\end{aligned}$$

If $\sigma < 0$ we have $(-|\mu|k + |\sigma|)\,\delta > 1$ and

$$\begin{aligned}&\mu\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \sigma\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - \frac{2\sigma\mu}{\sigma^2+\mu^2}\|x_n\|^2_{H_{\frac{1}{2}}}\\ &\geq -|\mu|\left|\mathrm{Im}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\right| + |\sigma|\mathrm{Re}\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\\ &\quad - \frac{2|\sigma||\mu|}{\sigma^2+\mu^2}\|x_n\|^2_{H_{\frac{1}{2}}}\\ &\geq ((-|\mu|k + |\sigma|)\delta - 1)\,\|x_n\|^2_{H_{\frac{1}{2}}},\end{aligned}$$

a contradiction to (12) and Part 2 of Lemma 1. Hence $\lambda = \mu + i\sigma \in r(\mathcal{A})$ if $1/\delta < |\sigma| - |\mu|k$. Due to the fact that $\beta \geq a_0^2\delta > 0$, Part 1 of this theorem implies the statement.

□

The following theorem shows that an m-ω-accretive damping implies that the operator $\mathcal{A}$ is m-$\tilde{\omega}$-accretive. Here and in the following, E denotes the identity on $H_{\frac{1}{2}}$ regarded as an operator into $H_{-\frac{1}{2}}$.

Theorem 4 *If* $\delta > 0$ *then* $\omega_0 I - \mathcal{A}$ *is m-*$\arctan\left(\max\left\{\frac{1}{\omega_0}, \frac{1}{\delta} + \tan\omega\right\}\right)$*-accretive for all* $\omega_0 > 0$*, where* ω *is as in Lemma 2.*

Proof

Let $\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in \mathcal{D}(\mathcal{A})$. An easy calculation shows that

$$\mathrm{Re}\,\langle(\omega_0 I - \mathcal{A})\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\rangle_{H_{\frac{1}{2}}\times H} = \omega_0\|z\|^2_{H_{\frac{1}{2}}} + \mathrm{Re}\,\langle(D+\omega_0 I)w, w\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$$

and

$$\mathrm{Im}\,\langle(\omega_0 I - \mathcal{A})\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\rangle_{H_{\frac{1}{2}}\times H} = 2\mathrm{Im}\,\langle z, w\rangle_{H_{\frac{1}{2}}} + \mathrm{Im}\,\langle(D+\omega_0 I)w, w\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}.$$

This implies

$$\begin{aligned}
&\left|\mathrm{Im}\,\langle(\omega_0 I - \mathcal{A})\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\rangle_{H_{\frac{1}{2}}\times H}\right| \\
&\le \|z\|^2_{H_{\frac{1}{2}}} + \|w\|^2_{H_{\frac{1}{2}}} + (\tan\omega)\mathrm{Re}\,\langle(D+\omega_0 I)w, w\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \\
&\le \|z\|^2_{H_{\frac{1}{2}}} + \left(\frac{1}{\delta} + \tan\omega\right)\mathrm{Re}\,\langle(D+\omega_0 I)w, w\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \\
&\le \max\left\{\frac{1}{\omega_0}, \frac{1}{\delta} + \tan\omega\right\}\mathrm{Re}\,\langle(\omega_0 I - \mathcal{A})\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right), \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right)\rangle_{H_{\frac{1}{2}}\times H}.
\end{aligned}$$

□

For the next condition guaranteeing that $\mathcal{A}$ generates an analytic semigroup we need in addition the following assumption.

(A3) There exist constants $M_0, M_1 > 0$ and $\omega_0 \ge 0$ such that

$$\|(D + (\omega_0 + z)E)^{-1}x\|_{H_{\frac{1}{2}}} \le M_0\|x\|_{H_{-\frac{1}{2}}}, \quad x \in H_{-\frac{1}{2}}, \quad z \in S_{\frac{\pi}{2}}, \tag{29}$$

and

$$\|(\omega_0 + z)(D + (\omega_0 + z)E)^{-1}x\| \le M_1\|x\|, \quad x \in H, \quad z \in S_{\frac{\pi}{2}}. \tag{30}$$

Note that (29) is equivalent to

$$\|A_0(D + (\omega_0 + z)E)^{-1}x\|_{H_{-\frac{1}{2}}} \le M_0\|x\|_{H_{-\frac{1}{2}}}, \quad x \in H_{-\frac{1}{2}}, \quad z \in S_{\frac{\pi}{2}}. \tag{31}$$

We remark that the example $D = A_0^{1/2}$ shows that (30) does not imply (29) or $\delta > 0$. The following lemma shows that Property **(A3)** is implied by the condition $\delta > 0$.

Lemma 3 *If $\delta > 0$ then* **(A3)** *is satisfied with $\omega_0 = 0$. In particular, $D + \lambda E : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$ is an isomorphism for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \ge 0$.*

Proof 1. For $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \ge 0$ and $x \in H$ we have

$$\mathrm{Re}\,\langle(A_0^{-1/2}DA_0^{-1/2} + \lambda A_0^{-1})x, x\rangle \ge \delta\|A_0^{-1/2}x\|^2_{H_{\frac{1}{2}}} = \delta\|x\|^2.$$

Therefore zero does not belong to the numerical range of the bounded operator $A_0^{-1/2}DA_0^{-1/2} + \lambda A_0^{-1}$, hence it is boundedly invertible in H. As

$$D + \lambda E = A_0^{1/2}\left(A_0^{-1/2}DA_0^{-1/2} + \lambda A_0^{-1}\right)A_0^{1/2},$$

$D + \lambda E$ is an isomorphism for every $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \ge 0$.

2. Assume that (29) does not hold for $\omega_0 = 0$. By the uniform boundedness theorem, there exists a vector $x \in H$ and a sequence (λ_n) in $S_{\pi/2}$ such that

$$s_n = \|A_0^{1/2}(D + \lambda_n E)^{-1} A_0^{1/2} x\| \to \infty.$$

We define

$$z_n = \frac{1}{s_n}(D + \lambda_n E)^{-1} A_0^{1/2} x \in H_{\frac{1}{2}}.$$

Then $\|z_n\|_{H_{\frac{1}{2}}} = 1$ and

$$\frac{1}{s_n} x = A_0^{-1/2} D z_n + \lambda_n A_0^{-1/2} z_n \to 0 \quad \text{in } H.$$

Taking inner products with $A_0^{1/2} z_n$ we obtain

$$\frac{1}{s_n}\langle x, A_0^{1/2} z_n \rangle = \langle D z_n, z_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \lambda_n \|z_n\|^2 \to 0,$$

Looking here at the real part only we get $\mathrm{Re}\langle D z_n, z_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \to 0$, which is in contradiction to $\delta > 0$.

3. We assume next that (30) does not hold for $\omega_0 = 0$. Again by the uniform boundedness theorem, there exists a vector $x \in H$ and a sequence (λ_n) in $S_{\pi/2}$ such that

$$s_n = \|\lambda_n (D + \lambda_n E)^{-1} x\| \to \infty.$$

We define

$$z_n = \frac{1}{s_n}(D + \lambda_n E)^{-1} x \in H_{\frac{1}{2}}.$$

Then $\|\lambda_n z_n\| = 1$ and

$$\frac{1}{s_n} x = D z_n + \lambda_n z_n \to 0 \quad \text{in } H.$$

Taking inner products with z_n we obtain

$$\frac{1}{s_n}\langle x, z_n \rangle = \langle D z_n, z_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \lambda_n \|z_n\|^2 \to 0,$$

Again, this is in contradiction with $\delta > 0$.

□

The following theorem is our main result concerning analyticity of $\mathcal{A}$.

Theorem 5 *If* **(A3)** *is satisfied, then the operator $\mathcal{A}$ is the generator of an analytic C_0-semigroup.*

Proof By Theorem 1 each λ with $|\arg\lambda| < \frac{\pi}{2}$ is in $\rho(A)$. It suffices to show that there exists a constant $K > 0$ such that

$$\|(\mathcal{A} - \lambda I)^{-1}\| \leq \frac{K}{|\lambda|}, \qquad \lambda \in \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega_0\},$$

see for example [8, Theorem II.4.6]. For every $\lambda \in \rho(A)$ we have (see e.g. [21, Prop. 5.3], [15]) that the operator $\lambda^2 E + D\lambda + A_0$ has a bounded inverse $V(\lambda) : H_{-\frac{1}{2}} \to H_{\frac{1}{2}}$,

$$V(\lambda) = \left(\lambda^2 E + D\lambda + A_0\right)^{-1},$$

such that

$$\lambda(\mathcal{A} - \lambda I)^{-1} = \begin{bmatrix} V(\lambda)A_0 - I & -\lambda V(\lambda) \\ \lambda V(\lambda)A_0 & -\lambda^2 V(\lambda) \end{bmatrix}. \tag{32}$$

We obtain for $x \in H_{-\frac{1}{2}}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0$

$$\begin{aligned} &\lambda V(\lambda)x = (D + \lambda E + \lambda^{-1}A_0)^{-1}x \\ &= \left((I + \lambda^{-1}A_0(D + \lambda E)^{-1})(D + \lambda E)\right)^{-1} x \\ &= (D + \lambda E)^{-1}\sum_{k=0}^{\infty}(-1)^k(\lambda^{-1}A_0(D + \lambda E)^{-1})^k x. \end{aligned} \tag{33}$$

Thus, with (29) and (31),

$$\|\lambda V(\lambda)x\|_{H_{\frac{1}{2}}} \leq M_0 \sum_{k=0}^{\infty} \frac{M_0^k}{|\lambda|^k}\|x\|_{H_{-\frac{1}{2}}},$$

for $|\lambda| > M_0$. It follows from (33) and (31) that the operator norms of $V(\lambda)A_0$, considered as an operator from $H_{\frac{1}{2}}$ into $H_{\frac{1}{2}}$, of $\lambda V(\lambda)$, considered as an operator from H into $H_{\frac{1}{2}}$, and of $\lambda V(\lambda)A_0$, considered as an operator from $H_{\frac{1}{2}}$ into H remains uniformly bounded as λ varies over the set $\{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_0\}$.

Using (33) we obtain for $x \in H$

$$\lambda V(\lambda)x = (D + \lambda E)^{-1}x + (D + \lambda E)^{-1}\sum_{k=1}^{\infty}(-1)^k(\lambda^{-1}A_0(D + \lambda E)^{-1})^k x. \tag{34}$$

It follows from (34), (30) and (31) that the operator norms of $\lambda^2 V(\lambda)$, considered as an operator from H into H remains uniformly bounded as λ varies over the set $\{z \in \mathbb{C} \mid \operatorname{Re} z > \omega_0\}$. This, together with (32), concludes the proof of Theorem 5. □

Next we show that (30) is equivalent to the fact that the restriction of D to $(D - zE)^{-1}H$, denoted by $\widetilde{D}$, generates an analytic semigroup, that is,

$$\|(\omega_0 + z)(\widetilde{D} + (\omega_0 + z)I)^{-1}x\| \leq M_1\|x\|, \quad x \in H, z \in S_{\frac{\pi}{2}}. \tag{35}$$

Assume that there exists an $z \in \mathbb{C}$ such that $D - zE$ is an isomorphism. Then we have for all $\lambda \in \mathbb{C}$ such that $D - \lambda E$ is an isomorphism

$$(D - zE)^{-1}H = (D - \lambda E)^{-1}H.$$

We define an operator $\widetilde{D}$ in H via

$$\mathcal{D}(\widetilde{D}) := (D - zE)^{-1}H$$

and

$$\widetilde{D}y := Dy, \quad y \in \mathcal{D}(\widetilde{D}),$$

that is, $\widetilde{D}$ is the restriction of D to $(D - zE)^{-1}H$ and is considered as an operator in H. Then $\widetilde{D}$ is a densely defined operator in H. The following lemma is known, however, we give a short proof.

Lemma 4 *Assume there exists a $z \in \mathbb{C}$ such that $D - zE$ is an isomorphism. Then $\widetilde{D}$ is a closed operator in H, and the operator $D - \lambda E$ is an isomorphism if and only if $\lambda \in \rho(\widetilde{D})$.*

Proof 1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $(D - zE)^{-1}H$, $x_n = (D - zE)^{-1}w_n$ for some $w_n \in H$, $n \in \mathbb{N}$, which converges to some x in H and $(\widetilde{D}x_n)_{n\in\mathbb{N}}$ converges to some y in H. We have

$$y - zx = \lim_{n\to\infty}(\widetilde{D}x_n - zx_n) = \lim_{n\to\infty} w_n,$$

therefore

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty}(D - zE)^{-1}w_n = (D - zE)^{-1}(y - zx) \in \mathcal{D}(\widetilde{D}).$$

Moreover, we obtain

$$\widetilde{D}(D - zE)^{-1}(y - zx) = y - zx + z(D - zE)^{-1}(y - zx) = y$$

and $\widetilde{D}$ is a closed operator in H.

2. Obviously, $\widetilde{D} - \lambda I$ is injective if and only if $D - \lambda E$ is injective. Moreover, if $D - \lambda E$ is an isomorphism, then, by the definition of $\widetilde{D}$, the range of $\widetilde{D} - \lambda I$ is H, hence $\lambda \in \rho(\widetilde{D})$.
3. Assume $\lambda \in \rho(\widetilde{D})$. Then $D - \lambda E$ is injective and has a dense range. We will show that the range of $D - \lambda E$ is $H_{-\frac{1}{2}}$. Let $y \in H_{-\frac{1}{2}}$. By assumption $D - zE$ is an isomorphism, hence, the first and second part of the proof imply $z \in \rho(\widetilde{D})$. Therefore there exists $x \in \mathcal{D}(\widetilde{D})$ with

$$(\widetilde{D} - \lambda I)x = (D - zE)^{-1}y$$

and

$$(D - \lambda E)(\widetilde{D} - \lambda I)x = (D - \lambda E)(D - zE)^{-1}y =$$
$$y + (z - \lambda)(D - zE)^{-1}y = y + (z - \lambda)(\widetilde{D} - \lambda I)x.$$

Hence, y belongs to the range of $D - \lambda E$.

□

This lemma immediately implies the equivalence of (30) and (35).

5 Example: Small transverse oscillations of ideal incompressible fluid in a pipe

The small transverse oscillations of a horizontal pipe of length 1, carrying steady-state fluid of ideal incompressible fluid are described by the equation [20]

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial r^2}\left[E\frac{\partial^2 u}{\partial r^2} + C\frac{\partial^3 u}{\partial r^2 \partial t}\right] + K\frac{\partial^2 u}{\partial t \partial x} = 0, \qquad r \in (0,1), t > 0. \tag{36}$$

Let $u(r,t)$ denote the transverse oscillations at time t and position r, and E, C, K are positive physical constants. The fourth term in the left hand side of (36) is called the gyroscopic term. Assuming that the pipe is pinned at the endpoints, we have for all $t > 0$ the following boundary conditions:

$$u\big|_{r=0} = 0, \quad \frac{\partial^2 u}{\partial r^2}\bigg|_{r=0} = 0, \quad u\big|_{r=1} = 0, \quad \frac{\partial^2 u}{\partial r^2}\bigg|_{r=1} = 0, \tag{37}$$

We consider the partial differential equation (36)-(37) as a second order problem in the Hilbert space $H = L^2(0,1)$. In H we define the operator A_0 by

$$A_0 = E\frac{d^4}{dr^4}, \quad \mathcal{D}(A_0) = \left\{z \in H^4(0,1) \mid z(0) = z(1) = z''(0) = z''(1) = 0\right\}.$$

It is easy to see that the operator A_0 satisfies assumption (A1) and that A_0^{-1} is a compact operator. We have

$$H_{\frac{1}{2}} = \left\{z \in H^2(0,1) \mid z(0) = z(1) = 0\right\}$$

with inner product $\langle z, v\rangle_{H_{\frac{1}{2}}} = E\langle z'', v''\rangle$. The operator $A_0^{1/2}$ is given by

$$A_0^{1/2} = -E^{1/2}\frac{d^2}{dr^2} \quad \text{and} \quad \|z\|^2_{H_{\frac{1}{2}}} \geq \pi^4 E\|z\|^2 \quad \text{for } z \in H_{\frac{1}{2}}. \tag{38}$$

Let $x(t) = (u(\cdot,t), \dot{u}(\cdot,t))$. Then $\|x(t)\|^2_{H_{\frac{1}{2}} \times H} = \|u''(\cdot,t)\|^2 + \|\dot{u}(\cdot,t)\|^2$ corresponds to the energy of the pipe which justifies the choice of $L^2(0,1)$ as the Hilbert space for the analysis of the boundary value problem (36)-(37).

We define the damping operator as

$$Dy = \frac{C}{E}A_0 y + Ky' \quad \text{for } y \in H_{\frac{1}{2}}.$$

For $z \in H_{\frac{1}{2}}$ we have

$$\operatorname{Re}\langle Dz, z\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle Cz'', z''\rangle_H = \frac{C}{E}\|z\|^2_{H_{\frac{1}{2}}} \geq \pi^4 C\|z\|^2, \tag{39}$$

and thus the assumption (A2) holds as well. Furthermore, we have $\delta = C/E$ and therefore each solution of the abstract problem $\ddot{z}(t) + A_0 z(t) + D\dot{z}(t) =$

0 corresponds to a solution of the boundary value problem (36)-(37). The damping operator is also m-ω-accretive, as

$$|\mathrm{Im}\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}| = |K\langle z', z\rangle| \le K\|z'\|\,\|z\| \le M\mathrm{Re}\,\langle Dz, z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$$

for some constant $M > 0$ and every $z \in H_{\frac{1}{2}}$. Thus Theorem 4 and [9, Section 7.1] imply that $\mathcal{A}$ generates an exponentially stable analytic semigroup.

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