Indirect sampled-data control with sampling period adaptation

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Abstract. It is known that if a continuous-time feedback system is exponentially stable, then the corresponding sampled-data system obtained by sample-hold discretization with constant sampling period is also exponentially stable, provided that the sampling period $\tau > 0$ is sufficiently small. In general it is difficult to estimate how small the sampling period has to be in order to achieve stability of the sampled-data system. In this paper, we present an adaptive mechanism for adjusting the sampling period. This mechanism has the properties that, for every initial state, (i) the adaptation of the sampling period terminates after finitely many time steps and (ii) the state of the adaptive sampled-data system is integrable and converges to zero as time goes to infinity.

Keywords. Adaptive control, feedback stabilization, indirect sampled-data control, variable sampling period.

1 Introduction

Consider the finite-dimensional continuous-time static output feedback system

$$\left.\begin{array}{rcl} \dot{x}(t) &=& Ax(t) + Bu(t)\,; \quad x(0) = x^0\,, \\ y(t) &=& Cx(t)\,, \\ u(t) &=& Fy(t)\,, \end{array}\right\} \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $F \in \mathbb{R}^{m\times p}$ and $x^0 \in \mathbb{R}^n$. System (1.1) is exponentially stable if, and only if, the matrix $A + BFC$ is exponentially stable, that is, all eigenvalues of $A + BFC$ have negative real parts.

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Digital implementation of the output feedback in (1.1) requires the application of sampling and (zero-order) hold, leading to the sampled-data feedback system

$$\left.\begin{array}{rcll} \dot{x}(t) &=& Ax(t)+Bu(t)\,; & x(0)=x^0\,, \\ y(t) &=& Cx(t)\,, & \\ u(t) &=& Fy(j\tau)\,, & \forall\, t\in[j\tau,(j+1)\tau)\,, \end{array}\right\} \tag{1.2}$$

where $\tau > 0$ is the sampling period. It is well-known that if system (1.1) is exponentially stable and if sampling period τ is sufficiently small, then system (1.2) is also exponentially stable in the sense that there exist $M \geq 1$ and $\alpha > 0$ such that

$$\|x(t;x^0,\tau)\| \leq Me^{-\alpha t}\|x^0\|\,, \quad \forall\, x^0\in\mathbb{R}^n,\ \forall\, t\geq 0\,,$$

where $x(\,\cdot\,;x^0,\tau)$ denotes the solution of (1.2) (for the proof and for related results, see [1, 2, 5, 6]).

Given that the continuous-time system (1.1) is exponentially stable, it is in general difficult to estimate how small the sampling period has to be in order to achieve stability of the sampled-data system (1.2) (see [10]). In this paper, we develop an adaptive strategy for adjusting the sampling period, so that, for every initial condition x^0, the adaptation of the sampling period terminates after finitely many time steps and the corresponding solution of (1.2) is integrable and tends to 0 as $t \to \infty$.

The use of adaptive strategies for the selection of "suitable" sampling periods has been considered before: see [4, 7] for results in a high-gain context and [8] for an application of sampling period adaptation to low-gain integral control. However, the general result on adaptive sampling in the context of indirect sampled-data control presented in this paper is new.

The statement of the main result of the paper is given in Section 2. A generalization of the result on static feedback in Section 2 to dynamic feedback is presented in Section 3. All proofs can be found in the Appendix (Section 4).

Nomenclature and terminology.

$\lfloor\sigma\rfloor$	$:=$	$\max\{n\in\mathbb{N}_0 \mid n\leq\sigma\}, \quad \sigma\in\mathbb{R}_+\,,$
$\ell^\infty(\mathbb{N}_0,\mathbb{R}^n)$		space of bounded $\mathbb{R}^n$-valued sequences $(s_j)_{j\in\mathbb{N}_0}$,
$\ell^1(\mathbb{N}_0,\mathbb{R}^n)$		space of $\mathbb{R}^n$-valued sequences $(s_j)_{j\in\mathbb{N}_0}$ with $\sum_{j=0}^\infty \|s_j\| < \infty$,
$L^1(\mathbb{R}_+,\mathbb{R}^n)$		space of measurable functions $f:\mathbb{R}_+\to\mathbb{R}^n$ with $\int_0^\infty \|f(t)\|\,\mathrm{d}t < \infty$.

A sequence $(s_j)_{j\in\mathbb{N}_0}$ is said to be *ultimately constant* if, and only if, there exists $N\in\mathbb{N}_0$ such that $s_{N+j}=s_N$ for all $j\in\mathbb{N}_0$.

2 Adaptation of the sampling period

The purpose of this section is to develop an adaptive feedback mechanism for adjusting the sampling period. The use of sampling and hold in (1.1), corresponding to the sampling points $(t_j)_{j\in\mathbb{N}_0}$, leads to the following sampled-data feedback system

$$\left.\begin{array}{rcll} \dot{x}(t) & = & Ax(t)+Bu(t)\,; & x(0)=x^0\,, \\ y(t) & = & Cx(t)\,, & \\ u(t) & = & Fy(t_j)\,, & \forall\, t\in[t_j,t_{j+1})\,. \end{array}\right\} \tag{2.1}$$

The sampling points t_j, or, equivalently, the sampling periods $\tau_j := t_{j+1}-t_j$, are determined by the following adaptive strategy:

$$\left.\begin{array}{rl} \text{for given} & \alpha\in(0,1) \text{ and } (\eta_j)_{j\in\mathbb{N}_0}\in\ell^\infty(\mathbb{N}_0,\mathbb{R}) \text{ with } \inf_{j\in\mathbb{N}_0}\eta_j>0\,, \\ \text{set} & t_0=0\,, \quad \sigma_0=0\,, \\ \text{and, for} & j=0,1,2,\ldots, \\ & k_j=\lfloor\sigma_j\rfloor\,, \\ & \tau_j=\max\left\{\eta_j/(j+1)^\alpha, \eta_{k_j}/(k_j+1)^\alpha\right\}\,, \\ & t_{j+1}=t_j+\tau_j\,, \\ & \sigma_{j+1}=\sigma_j+\|y(t_j)\|\,. \end{array}\right\} \tag{2.2}$$

The rationale for the adaptive strategy (2.2) is described in the following remark.

Remark 2.1. For simplicity, in the context of this remark, the reader may assume that $\eta_j=1$ for all $j\in\mathbb{N}_0$ (the role of the η_j will become clear later, see part (ii) of Remark 2.5). Obviously, the last definition in (2.2) is a discrete-time integrator with input $(\|y(t_j)\|)_{j\in\mathbb{N}_0}$, so that

$$\sigma_j=\sum_{l=0}^{j-1}\|y(t_l)\|\,, \quad \forall\, j\in\mathbb{N}\,. \tag{2.3}$$

The idea behind the adaptive strategy (2.2) is to decrease the sampling period as long as the norm of the sampled output values $y(t_j)$ is "large" in the sense that the partial sums σ_j has not "started to converge". It is readily verified that the following properties are equivalent:

(i) the sequence $(\tau_j)_{j\in\mathbb{N}_0}$ is ultimately constant;

(ii) the sequence $(k_j)_{j\in\mathbb{N}_0}$ is ultimately constant;

(iii) $(\sigma_j)_{j\in\mathbb{N}_0}\in\ell^\infty(\mathbb{N}_0,\mathbb{R})$;

(iv) $\big(y(t_j)\big)_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^p)$.

Also note that if $(\tau_j)_{j\in\mathbb{N}_0}$ is not ultimately constant, then $\lim_{j\to\infty}\tau_j=0$. $\diamond$

For the following, it is convenient to define

$$\delta_l := \eta_l/(l+1)^\alpha\,, \quad \forall\, l\in\mathbb{N}_0\,. \tag{2.4}$$

Note that, for each sampling period τ_j generated by (2.2), there exists $l_j\in\mathbb{N}_0$ such that $\tau_j=\delta_{l_j}$. We introduce the following detectability hypothesis.

(D) The pair $(C, e^{A\delta_l})$ is discrete-time detectable for every $l\in\mathbb{N}_0$.

We are now ready to state the main result of this contribution. The proof can be found in the Appendix.

Theorem 2.2. *Assume that the continuous-time feedback system* (1.1) *is exponentially stable and let* $x(\,\cdot\,;x^0)$ *denote the solution of the adaptive sampled-data system given by* (2.1) *and* (2.2)*. Then, for every initial state* $x^0 \in \mathbb{R}^n$*, the following statements hold:*

(i) *the sequence* $(\tau_j)_{j\in\mathbb{N}_0}$ *is ultimately constant, that is, the adaptation of the sampling period terminates in finite time;*

(ii) *if, additionally, hypothesis* (D) *is satisfied, then* $\lim_{t\to\infty} x(t;x^0) = 0$*,* $x(\,\cdot\,;x^0) \in L^1(\mathbb{R}_+,\mathbb{R}^n)$ *and* $(x(t_j;x^0))_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0,\mathbb{R}^n)$*.*

We note that in the case of state feedback (that is, $p = n$ and $C = I$), hypothesis (D) is trivially satisfied. In general however, the appearance of hypothesis (D) in statement (ii) of Theorem 2.2 is somewhat unsatisfactory, because it is formulated in discrete-time terms and not in terms of the original continuous-time data. The following definition will be useful in addressing this issue.

Definition 2.3. A number $\delta > 0$ is said to be *pathological* relative to $A \in \mathbb{R}^{n\times n}$ if, and only if, there exist $q \in \mathbb{Z}\setminus\{0\}$ and $\lambda, \mu \in \sigma(A)\cap\{s\in\mathbb{C} : \operatorname{Re} s \geq 0\}$ such that $\delta(\lambda-\mu) = 2q\pi i$. Otherwise, δ is said to be *non-pathological* relative to A. $\diamond$

We shall see that, in Theorem 2.2, hypothesis (D) can be replaced by the following hypothesis.

(D′) For every $l \in \mathbb{N}_0$, δ_l is non-pathological relative to A.

Corollary 2.4. *The conclusions of* Theorem 2.2 *remain valid if, in the statement of* Theorem 2.2*, hypothesis* (D) *is replaced by hypothesis* (D′)*.*

The proof of Corollary 2.4 can be found in the Appendix.

The following remark shows that hypothesis (D′) is not very restrictive.

Remark 2.5. (i) Let α and $(\eta_l)_{l\in\mathbb{N}_0}$ be given as in (2.2) and define $(\delta_l)_{l\in\mathbb{N}_0}$ by (2.4). Then it can be shown that the set

$$\{A \in \mathbb{R}^{n\times n} : \delta_l \text{ is non-pathological relative to } A \text{ for every } l \in \mathbb{N}_0\}$$

is open and dense in $\mathbb{R}^{n\times n}$ (see [5, Appendix A.1]). Consequently, the probability that, for a randomly chosen matrix $A \in \mathbb{R}^{n\times n}$, there exists $l \in \mathbb{N}_0$ such that δ_l is pathological relative to A is zero.

(ii) It is clear that, for every $A \in \mathbb{R}^{n\times n}$ and every $\alpha \in (0,1)$, there exists a bounded sequence $(\eta_l)_{l\in\mathbb{N}_0}$ with $\inf_{l\in\mathbb{N}_0}\eta_l > 0$ and such that δ_l (defined in (2.4)) is non-pathological relative to A for every $l \in \mathbb{N}_0$ (that is, hypothesis (D′) holds). $\diamond$

3 Generalization to dynamic output feedback

Consider a dynamic output feedback system with plant given by

$$\begin{aligned} \dot{x}_{\mathrm{p}} &= A_{\mathrm{p}} x_{\mathrm{p}} + B_{\mathrm{p}} u_{\mathrm{p}}\,; \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}}^0\,, \\ y_{\mathrm{p}} &= C_{\mathrm{p}} x_{\mathrm{p}}\,, \end{aligned} \tag{3.1}$$

controller given by

$$\begin{aligned} \dot{x}_{\mathrm{c}} &= A_{\mathrm{c}} x_{\mathrm{c}} + B_{\mathrm{c}} u_{\mathrm{c}}\,; \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}}^0\,, \\ y_{\mathrm{c}} &= C_{\mathrm{c}} x_{\mathrm{c}} + D_{\mathrm{c}} u_{\mathrm{c}}\,, \end{aligned} \tag{3.2}$$

and feedback interconnection equations

$$u_{\mathrm{c}} = y_{\mathrm{p}}\,, \quad u_{\mathrm{p}} = y_{\mathrm{c}}\,, \tag{3.3}$$

where $A_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}} \times n_{\mathrm{p}}}$, $B_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}} \times m}$, $C_{\mathrm{p}} \in \mathbb{R}^{p \times n_{\mathrm{p}}}$, $A_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}} \times n_{\mathrm{c}}}$, $B_{\mathrm{c}} \in \mathbb{R}^{n_{\mathrm{c}} \times p}$, $C_{\mathrm{c}} \in \mathbb{R}^{m \times n_{\mathrm{c}}}$, $D_{\mathrm{c}} \in \mathbb{R}^{m \times p}$, $x_{\mathrm{p}}^0 \in \mathbb{R}^{n_{\mathrm{p}}}$ and $x_{\mathrm{c}}^0 \in \mathbb{R}^{n_{\mathrm{c}}}$. Defining

$$A := \operatorname{diag}(A_{\mathrm{p}}, A_{\mathrm{c}}), \quad B := \operatorname{diag}(B_{\mathrm{p}}, B_{\mathrm{c}}), \quad C := \begin{pmatrix} C_{\mathrm{p}} & 0 \\ D_{\mathrm{c}} C_{\mathrm{p}} & C_{\mathrm{c}} \end{pmatrix}, \quad F := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{3.4}$$

a routine calculation shows that the continuous-time dynamic feedback system given by (3.1)–(3.3) can be written as

$$\dot{x} = (A + BFC)x\,; \quad x(0) = x^0 = \begin{pmatrix} x_{\mathrm{p}}^0 \\ x_{\mathrm{c}}^0 \end{pmatrix}, \quad \text{where } x := \begin{pmatrix} x_{\mathrm{p}} \\ x_{\mathrm{c}} \end{pmatrix}. \tag{3.5}$$

Let $(t_j)_{j \in \mathbb{N}_0}$ be the sampling points to be determined adaptively. As before, we define the associated sampling periods $\tau_j := t_{j+1} - t_j$ for $j \in \mathbb{N}_0$. Consider the corresponding sample-hold discretization of (3.2)

$$\begin{aligned} x_{\mathrm{c}}^{\mathrm{d}}(j+1) &= e^{A_{\mathrm{c}} \tau_j} x_{\mathrm{c}}^{\mathrm{d}}(j) + \textstyle\int_0^{\tau_j} e^{A_{\mathrm{c}} s} \mathrm{d}s B_{\mathrm{c}} u_{\mathrm{c}}^{\mathrm{d}}(j)\,; \quad x_{\mathrm{c}}^{\mathrm{d}}(0) = x_{\mathrm{c}}^0 \in \mathbb{R}^{n_{\mathrm{c}}}\,, \\ y_{\mathrm{c}}^{\mathrm{d}}(j) &= C_{\mathrm{c}} x_{\mathrm{c}}^{\mathrm{d}}(j) + D_{\mathrm{c}} u_{\mathrm{c}}^{\mathrm{d}}(j)\,, \end{aligned} \tag{3.6}$$

together with the feedback interconnection equations

$$u_{\mathrm{c}}^{\mathrm{d}}(j) = y_{\mathrm{p}}(t_j)\,, \quad u_{\mathrm{p}}(t_j + \theta) = y_{\mathrm{c}}^{\mathrm{d}}(j)\,, \quad \forall\, \theta \in [0, \tau_j)\,,\ \forall\, j \in \mathbb{N}_0\,. \tag{3.7}$$

The adaptive strategy for determining the sampling points is very similar to that in the case of static feedback, the only difference being in the equation for $(\sigma_j)_{j \in \mathbb{N}_0}$:

$$\left.\begin{array}{rl} \text{for given} & \alpha \in (0,1) \text{ and } (\eta_j)_{j \in \mathbb{N}_0} \in \ell^\infty(\mathbb{N}_0, \mathbb{R}) \text{ with } \inf_{j \in \mathbb{N}_0} \eta_j > 0\,, \\ \text{set} & t_0 = 0\,, \quad \sigma_0 = 0\,, \\ \text{and, for} & j = 0, 1, 2, \ldots, \\ & k_j = \lfloor \sigma_j \rfloor\,, \\ & \tau_j = \max\left\{ \eta_j/(j+1)^\alpha, \eta_{k_j}/(k_j+1)^\alpha \right\}\,, \\ & t_{j+1} = t_j + \tau_j\,, \\ & \sigma_{j+1} = \sigma_j + \left\| \left(y_{\mathrm{p}}(t_j), y_{\mathrm{c}}^{\mathrm{d}}(j) \right) \right\|\,. \end{array}\right\} \tag{3.8}$$

Remark 3.1. Remark 2.1 remains true in the context of the adaptive strategy (3.8), provided that, in (2.3), $\|y(t_j)\|$ is replaced by $\|(y_{\rm p}(t_j), y_{\rm c}^{\rm d}(j))\|$ and, in item (iv), $(y(t_j))_{j\in\mathbb{N}_0}$ and $\ell^1(\mathbb{N}_0,\mathbb{R}^p)$ are replaced by $(y_{\rm p}(t_j), y_{\rm c}^{\rm d}(j))_{j\in\mathbb{N}_0}$ and $\ell^1(\mathbb{N}_0,\mathbb{R}^{p+m})$, respectively. $\diamond$

The sampled-data feedback system given by (3.1), (3.6), (3.7) and (3.8) has a unique solution which will be denoted by

$$\begin{pmatrix} x_{\rm p}(t_j+\theta; x^0) \\ x_{\rm c}^{\rm d}(j; x^0) \end{pmatrix}, \quad \forall\, \theta \in [0,\tau_j)\,,\ \forall\, j\in\mathbb{N}_0\,. \tag{3.9}$$

The corollary below is the main result of this section. The proof can be found in the Appendix.

Corollary 3.2. *Assume that the continuous-time dynamic feedback system given by* (3.1)–(3.3) (*or, equivalently, system* (3.5)) *is exponentially stable. Then, for every initial state* $x^0\in\mathbb{R}^{n_{\rm p}+n_{\rm c}}$*, the sampled-data feedback system given by* (3.1), (3.6), (3.7) *and* (3.8) *has the following properties:*

(i) *the sequence* $(\tau_j)_{j\in\mathbb{N}_0}$ *is ultimately constant, that is, the adaptation of the sampling period terminates in finite time;*

(ii) *if, additionally,* $\eta_l/(l+1)^\alpha$ *is non-pathological relative to* $A=\mathrm{diag}(A_{\rm p},A_{\rm c})$ *for every* $l\in\mathbb{N}_0$*, then* $\lim_{t\to\infty} x_{\rm p}(t;x^0)=0$, $x_{\rm p}(\,\cdot\,;x^0)\in L^1(\mathbb{R}_+,\mathbb{R}^{n_{\rm p}})$, $(x_{\rm p}(t_j;x^0))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^{n_{\rm p}})$ *and* $(x_{\rm c}^{\rm d}(j;x^0))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^{n_{\rm c}})$.

4 Appendix

To facilitate the proofs of the results in Sections 2 and 3, it is convenient to first state and prove a technical lemma. To this end, consider the sampled-data feedback system (2.1) with a prespecified sequence $\boldsymbol{t}:=(t_j)_{j\in\mathbb{N}_0}$ of sampling points satisfying

$$t_0=0\,,\qquad t_{j+1}>t_j\,,\ \forall j\in\mathbb{N}_0\,,\qquad t_j\to\infty \ \text{ as } \ j\to\infty\,.$$

Let $x(\,\cdot\,;x^0,\boldsymbol{t})$ denote the corresponding solution of system (2.1).

The following lemma shows that if the continuous-time system (1.1) is exponentially stable and if the sampling periods $\tau_j:=t_{j+1}-t_j$ converge to 0 as $j\to\infty$, with rate of convergence sufficiently small in a suitable sense, then the sequence $(x(t_j;x^0,\boldsymbol{t}))_{j\in\mathbb{N}_0}$ is summable.

Lemma 4.1. *Assume that the continuous-time feedback system* (1.1) *is exponentially stable. Let the sequence* $\boldsymbol{t}=(t_j)_{j\in\mathbb{N}_0}$ *be such that* $t_0=0$ *and* $t_{j+1}>t_j$ *for all* $j\in\mathbb{N}_0$*. Set* $\tau_j:=t_{j+1}-t_j$ *and assume that*

$$\lim_{j\to\infty}\tau_j=0 \quad \textit{and} \quad \inf_{j\in\mathbb{N}}\tau_j j^\alpha>0 \quad \textit{for some } \alpha\in(0,1)\,. \tag{4.1}$$

Then, for every $x^0\in\mathbb{R}^n$*, the sequence* $(x(t_j;x^0,\boldsymbol{t}))_{j\in\mathbb{N}_0}$ *is in* $\ell^1(\mathbb{N}_0,\mathbb{R}^n)$*.*

Proof. The variation-of-parameters formula yields

$$x(t_{j+1}; x^0, \boldsymbol{t}) = \left(e^{A\tau_j} + \int_0^{\tau_j} e^{As} \mathrm{d}s\, BFC \right) x(t_j; x^0, \boldsymbol{t}), \quad \forall\, j \in \mathbb{N}_0 . \tag{4.2}$$

Writing

$$\Delta_j := e^{A\tau_j} + \int_0^{\tau_j} e^{As} \mathrm{d}s\, BFC \qquad \text{and} \qquad x_j := x(t_j; x^0, \boldsymbol{t}); \quad \forall\, j \in \mathbb{N}_0 ,$$

(4.2) becomes

$$x_{j+1} = \Delta_j x_j , \quad \forall\, j \in \mathbb{N}_0 ; \quad x_0 = x^0 . \tag{4.3}$$

It follows from the exponential stability of (1.1) that there exists a unique matrix $P = P^T > 0$, such that

$$(A + BFC)^T P + P(A + BFC) = -I \tag{4.4}$$

(see, for example, [9, Theorem 18, p. 231]). Let $\| \cdot \|_P$ be the norm on $\mathbb{R}^n$ defined by

$$\|z\|_P^2 := \langle z, Pz \rangle , \quad \forall\, z \in \mathbb{R}^n .$$

Using the power series expansion of e^{At}, we may decompose

$$\Delta_j = I + \tau_j(A + BFC) + \tau_j^2 \Gamma(\tau_j) , \quad \forall\, j \in \mathbb{N}_0 , \tag{4.5}$$

where

$$\Gamma(\tau) := \sum_{l=0}^{\infty} \frac{\tau^l}{(l+2)!} A^{l+1}(A + BFC) , \quad \forall\, \tau \geq 0 .$$

The boundedness of $(\tau_j)_{j\in\mathbb{N}_0}$ implies the boundedness of the sequence $(\Gamma(\tau_j))_{j\in\mathbb{N}_0}$ and hence, invoking (4.3) and (4.5), we conclude that there exists a constant $L \geq 0$ such that

$$\begin{aligned} \|x_{j+1}\|_P^2 - \|x_j\|_P^2 &= \langle \Delta_j x_j, P\Delta_j x_j \rangle - \langle x_j, Px_j \rangle \\ &\leq \tau_j \langle x_j, \left[(A + BFC)^T P + P(A + BFC) \right] x_j \rangle + L\tau_j^2 \|x_j\|^2, \quad \forall\, j \in \mathbb{N}_0 . \end{aligned}$$

Combining this with (4.4) shows that

$$\|x_{j+1}\|_P^2 - \|x_j\|_P^2 \leq (-\tau_j + L\tau_j^2)\|x_j\|^2, \quad \forall\, j \in \mathbb{N}_0 ,$$

and therefore, in view of $\lim_{j\to 0} \tau_j = 0$, we obtain that there exists $N \in \mathbb{N}$ such that

$$\|x_{j+1}\|_P^2 - \|x_j\|_P^2 \leq -\frac{\tau_j}{2} \|x_j\|^2 , \quad \forall\, j \geq N .$$

Consequently,

$$\|x_{j+1}\|_P^2 \leq \|x_j\|_P^2 - \frac{\tau_j}{2}\|x_j\|^2 \leq \left(1 - \frac{\tau_j}{2\|P\|} \right) \|x_j\|_P^2 , \quad \forall\, j \geq N , \tag{4.6}$$

and hence,

$$\|x_j\|_P^2 \leq \left[\prod_{l=N}^{j-1} \left(1 - \frac{\tau_l}{2\|P\|} \right) \right] \|x_N\|_P^2 , \quad \forall\, j \geq N + 1 . \tag{4.7}$$

If $x_{j_0} = 0$ for some $j_0 \geq N$, then it follows from (4.6) that $x_j = 0$ for all $j \geq j_0$, and thus $(x_j)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$. Assume now that $x_j \neq 0$ for all $j \geq N$. Then, by (4.6), $1 - \tau_j/(2\|P\|) > 0$ for all $j \geq N$. Moreover, since $M := \inf_{j\in\mathbb{N}}\{\tau_j j^\alpha\} > 0$, we have that $\tau_j \geq M/j^\alpha$ for all $j \in \mathbb{N}$, and thus

$$0 < 1 - \frac{\tau_j}{2\|P\|} \leq 1 - \frac{M}{2\|P\|j^\alpha}\,, \quad \forall\, j \geq N\,.$$

Combining this with (4.7) yields

$$\|x_j\|_P \leq \left[\prod_{l=N}^{j-1}\left(1 - \frac{M}{2\|P\|l^\alpha}\right)^{1/2}\right]\|x_N\|_P\,, \quad \forall j \geq N+1\,. \tag{4.8}$$

Define a positive sequence $(v_j)_{j\in\mathbb{N}_0}$ by

$$v_j := \prod_{l=N}^{N+j}\left(1 - \frac{M}{2\|P\|l^\alpha}\right)^{1/2} = \prod_{l=N}^{N+j}\left(1 - \frac{\gamma}{l^\alpha}\right)^{1/2}\,,$$

where $\gamma := M/(2\|P\|)$. By (4.8), to show that $(x_j)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$, it suffices to prove that $(v)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R})$. Invoking the inequality $1 - t \leq e^{-t}$ (which holds for all $t \in \mathbb{R}$), we have

$$\sum_{j=0}^{k} v_j \leq \sum_{j=0}^{k}\exp\left(-\frac{\gamma}{2}\sum_{l=N}^{N+j}\frac{1}{l^\alpha}\right) \leq \sum_{j=0}^{k}\exp\left(-\frac{\gamma(j+1)}{2(N+j)^\alpha}\right)\,,\ \forall\, k \in \mathbb{N}_0\,. \tag{4.9}$$

Since $\alpha \in (0,1)$, it follows that

$$\exp\left(-\frac{\gamma(j+1)}{2(N+j)^\alpha}\right) \leq \frac{1}{j^2} \qquad \text{for all sufficiently large } j.$$

Hence, the right-hand side of (4.9) converges to a finite limit as $k \to \infty$, showing that $(v_j)_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R})$. □

Proof of Theorem 2.2. Let $x^0 \in \mathbb{R}^n$ be fixed, but arbitrary.

To prove statement (i), we adopt a contradiction argument and suppose that the sequence of sampling periods $(\tau_j)_{j\in\mathbb{N}_0}$ is not ultimately constant. Then, by Remark 2.1, $\lim_{j\to\infty}\tau_j = 0$. Moreover, invoking the definition of τ_j in (2.2), we obtain

$$\tau_j j^\alpha \geq \eta_j\left(\frac{j}{j+1}\right)^\alpha\,, \quad \forall\, j \in \mathbb{N}\,.$$

By assumption, $\inf_{j\in\mathbb{N}_0}\eta_j > 0$, and thus,

$$\inf_{j\in\mathbb{N}}\tau_j j^\alpha > 0\,.$$

Therefore, (4.1) is satisfied and Lemma 4.1 yields that $(x(t_j; x^0))_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^n)$, and hence, $(y(t_j))_{j\in\mathbb{N}_0} \in \ell^1(\mathbb{N}_0, \mathbb{R}^p)$. Invoking again Remark 2.1 shows that $(\tau_j)_{j\in\mathbb{N}_0}$ is ultimately constant, contradicting the supposition that $(\tau_j)_{j\in\mathbb{N}_0}$ is not ultimately constant.

To prove statement (ii), we first note that, by the variation-of-parameter formula,

$$x(t_j+\theta;x^0)=\left(e^{A\theta}+\int_0^\theta e^{As}\mathrm{d}s\,BFC\right)x(t_j;x^0)\,,\quad \forall\,\theta\in[0,\tau_j],\ \forall\,j\in\mathbb{N}_0\,. \tag{4.10}$$

By statement (i), there exists $N\in\mathbb{N}_0$ such that

$$\tau_j=\tau_N=:\tau\,,\quad \forall\,j\geq N\,.$$

Hypothesis (D) guarantees that the pair $(C,e^{A\tau})$ is discrete-time detectable. Hence there exists $H\in\mathbb{R}^{n\times p}$ such that $e^{A\tau}+HC$ is power stable, i.e., all eigenvalues of $e^{A\tau}+HC$ are in the open unit disc $\{s\in\mathbb{C}:|s|<1\}$. Setting $B_\tau:=\int_0^\tau e^{As}\mathrm{d}s\,B$, it follows from (4.10) with $\theta=\tau$ that

$$\begin{aligned}x(t_{j+1};x^0)&=e^{A\tau}x(t_j;x^0)+B_\tau FCx(t_j;x^0)\\&=(e^{A\tau}+HC)x(t_j;x^0)+(B_\tau F-H)y(t_j)\,,\quad \forall j\geq N\,.\end{aligned}$$

Combining this with the power stability of $e^{A\tau}+HC$ and the fact that $(y(t_j))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^p)$ (guaranteed by Remark 2.1), we conclude that $(x(t_j;x^0))_{j\in\mathbb{N}_0}\in\ell^1(\mathbb{N}_0,\mathbb{R}^n)$. This implies in particular that

$$\lim_{j\to\infty}x(t_j;x^0)=0\,. \tag{4.11}$$

Setting

$$\bar\tau:=\sup_{j\in\mathbb{N}_0}\tau_j<\infty\qquad\text{and}\qquad M:=\sup_{\theta\in[0,\bar\tau]}\left\|e^{A\theta}+\int_0^\theta e^{As}\mathrm{d}s\,BFC\right\|\,,$$

we obtain from (4.10) that

$$\|x(t_j+\theta;x^0)\|\leq M\|x(t_j;x^0)\|\,,\quad \forall\,\theta\in[0,\tau_j],\ \forall\,j\in\mathbb{N}_0\,.$$

Consequently, by (4.11),

$$\lim_{t\to\infty}x(t;x^0)=0\,.$$

Finally,

$$\int_0^\infty\|x(t)\|\mathrm{d}t=\sum_{j=0}^\infty\int_{t_j}^{t_{j+1}}\|x(t;x^0)\|\mathrm{d}t\leq M\bar\tau\sum_{j=0}^\infty\|x(t_j;x^0)\|<\infty\,,$$

showing that $x\in L^1(\mathbb{R}_+,\mathbb{R}^n)$ and completing the proof of statement (ii). □

Proof of Corollary 2.4. Since the continuous-time feedback system (1.1) is exponentially stable, the pair (C,A) is continuous-time detectable. By assumption, δ_l is non-pathological relative to A for all $l\in\mathbb{N}_0$, and therefore, by a standard result (see [3, Lemma 8]), the pair $(C,e^{A\delta_l})$ is discrete-time detectable for all $l\in\mathbb{N}_0$, showing that hypothesis (D) holds. The claim now follows from Theorem 2.2. □

Proof of Corollary 3.2. Let $x^0\in\mathbb{R}^{n_\mathrm{p}+n_\mathrm{c}}$ be fixed, but arbitrary. Moreover, let the matrices B, C and F be defined as in (3.4). Invoking the variation-of-parameters formula, we conclude that

$$\begin{pmatrix}x_\mathrm{p}(t_j+\theta;x^0)\\x_\mathrm{c}^\mathrm{d}(j+1;x^0)\end{pmatrix}=\left[\begin{pmatrix}e^{A_\mathrm{p}\theta}&0\\0&e^{A_\mathrm{c}\tau_j}\end{pmatrix}+\begin{pmatrix}\int_0^\theta e^{A_\mathrm{p}s}\mathrm{d}s&0\\0&\int_0^{\tau_j}e^{A_\mathrm{c}s}\mathrm{d}s\end{pmatrix}BFC\right]\begin{pmatrix}x_\mathrm{p}(t_j;x^0)\\x_\mathrm{c}^\mathrm{d}(j;x^0)\end{pmatrix},$$
$$\forall\,\theta\in[0,\tau_j)\,,\ \forall\,j\in\mathbb{N}_0\,. \tag{4.12}$$

Since, by continuity of $x_{\rm p}(\,\cdot\,;x^0)$, $x_{\rm p}(t_j+\theta;x^0)\to x_{\rm p}(t_{j+1};x^0)$ as $\theta\uparrow\tau_j$, we obtain from (4.12), as $\theta\uparrow\tau_j$,

$$\begin{pmatrix} x_{\rm p}(t_{j+1};x^0) \\ x_{\rm c}^{\rm d}(j+1;x^0)\end{pmatrix} = \Delta_j \begin{pmatrix} x_{\rm p}(t_j;x^0) \\ x_{\rm c}^{\rm d}(j;x^0)\end{pmatrix}, \quad \forall\, j\in\mathbb{N}_0\,; \quad \begin{pmatrix} x_{\rm p}(0;x^0) \\ x_{\rm c}^{\rm d}(0;x^0)\end{pmatrix} = x^0\,, \tag{4.13}$$

where $\Delta_j := e^{A\tau_j}+\int_0^{\tau_j} e^{As}{\rm d}s\,BFC$ with A, B, C and F given by (3.4). Now consider the adaptive sampled-data system defined by (2.1) and (2.2), where again A, B, C and F are given by (3.4) and, furthermore, $n=n_{\rm p}+n_{\rm c}$. Denoting its solution by $x(\,\cdot\,;x^0)$, it follows that

$$x(t_{j+1};x^0)=\Delta_j x(t_j;x^0), \quad \forall\, j\in\mathbb{N}_0\,; \quad x(0;x^0)=x^0.$$

Combining this with (4.13) shows that

$$x(t_j;x^0)=\begin{pmatrix} x_{\rm p}(t_j;x^0) \\ x_{\rm c}^{\rm d}(j;x^0)\end{pmatrix}, \quad \forall\, j\in\mathbb{N}_0\,.$$

An application of Corollary 2.4 to the sampled-data system defined by (2.1) and (2.2), with A, B, C and F given by (3.4), then shows that $(\tau_j)_{j\in\mathbb{N}_0}$ is ultimately constant and the sequence $(x(t_j;x^0))_{j\in\mathbb{N}_0}$ is in $\ell^1(\mathbb{N}_0,\mathbb{R}^n)$. In particular,

$$\lim_{j\to\infty} x(t_j;x^0)=\lim_{j\to\infty}\begin{pmatrix} x_{\rm p}(t_j;x^0) \\ x_{\rm c}^{\rm d}(j;x^0)\end{pmatrix}=0\,. \tag{4.14}$$

Finally, we note that by using (4.12) and (4.14) in combination with an argument similar to that adopted at the end of the proof of Theorem 2.2 (after equation (4.11)), it follows that $\lim_{t\to\infty} x_{\rm p}(t;x^0)=0$ and $x_{\rm p}(\,\cdot\,;x^0)\in L^1(\mathbb{R}_+,\mathbb{R}^{n_{\rm p}})$, completing the proof. □

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