Output feedback stabilization of multivariable systems with vector relative degree

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Abstract

A control strategy for stabilizing multi-input multi-output (MIMO) linear systems via feedback of the output and its derivatives is introduced. The feedback law is only based on structural properties of the system such as vector relative degree, stable zero dynamics and a positive definite high frequency gain matrix.

Keywords: linear systems, MIMO systems, vector relative degree, normal form, high gain output stabilization

Mathematical Subject Classification (2000): 93D15, 34D05

1 Introduction

The present note deals with stabilization by output derivative feedback of linear systems with m inputs and m outputs of the form

$$\left.\begin{aligned} \dot{x} &= Ax + \underbrace{\left[b_1^{(n)}, \ldots, b_m^{(n)}\right]}_{=B} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}}_{=u} \\ \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}}_{=y} &= \underbrace{\begin{bmatrix} c_{(n)}^1 \\ \vdots \\ c_{(n)}^m \end{bmatrix}}_{=C} x\,, \end{aligned}\right\} \tag{1.1}$$

where $n, m \in \mathbb{N}$ and $A \in \mathbb{R}^{n \times n}$, $B, C^T \in \mathbb{R}^{n \times m}$. System (1.1) has vector relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1 \times m}$ if, and only if, r_i is, for $i = 1, \ldots, m$, the least number one has to differentiate the i-th output y_i so that at least one of the m inputs $u_1, \ldots, u_m$ appears explicitly and the rows $c_{(n)}^1 A^{r_1 - 1} B$, …, $c_{(n)}^m A^{r_m - 1} B$ are linearly independent, see Definition 2.1(a).

Isidori [Isi95] gives a local definition of the vector relative degree for nonlinear multi-input multi-output (MIMO) systems and in [Isi99] he presents a normal form for nonlinear MIMO systems with given vector relative degree. Furthermore, this normal form is used to show the existence of feedback laws which achieve asymptotic stabilization. To design these feedback laws the system's data must be known explicitly. To the author's best knowledge, other stabilization results for linear MIMO systems with vector relative degree are not available in the literature.

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In the present paper, it is shown that, in case of strict relative degree, see Definition 2.1(c), for suitable design parameters $k_1, \ldots, k_m \in \mathbb{R}$, independent of the system's data, and sufficiently large $\kappa > 0$, the simple high-gain controller

$$u(t) = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1}\, y^{(i)}(t) \tag{1.2}$$

yields an exponentially stable closed-loop system (1.1), (1.2); the only structural assumptions – note that no system's data are required – are: (1.1) has stable zero dynamics and $CA^{r-1}B$ is positive definite. (In the recent paper a matrix $M \in \mathbb{C}^{n\times n}$ is called positive definite if, and only if, its Hermitian part $1/2(M + M^*)$ is positive definite. Thus it is not necessarily assumed that M is Hermitian.) In case of non-strict relative degree, the controller is more involved but still simple, see Theorem 3.2.

Similar stabilization results are well-known for the linear and nonlinear single-input single-output (SISO) system (1.1), i.e. $m = 1$: First, (1.1) with relative degree $r \in \mathbb{N}$ may be converted in the normal form

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0\\ \vdots & \ddots & \ddots & & \vdots\\ 0 & \ldots & 0 & 1 & 0\\ R_1 & \ldots & & R_r & S\\ \hline P & 0 & \ldots & 0 & Q \end{array}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \left[\begin{array}{c} 0\\ \vdots\\ 0\\ cA^{r-1}b\\ \hline 0 \end{array}\right] u\\ y &= \left[\,1,\ 0,\ 0,\ \ldots,\ 0\,\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix}, \end{aligned}\right\} \tag{1.3}$$

where $R_1, \ldots, R_r \in \mathbb{R}$, $S \in \mathbb{R}^{1\times(n-r)}$, $P \in \mathbb{R}^{n-r}$ and $Q \in \mathbb{R}^{(n-r)\times(n-r)}$ may be presented explicitly in terms of the system matrices A, b and c, see [Isi95] and [IRT07]. An application of the feedback $u(t) = \sum_{i=0}^{r-1} k_{i+1} y^{(i)}(t)$ to (1.3) yields the closed-loop system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \left[\begin{array}{cccc|c} 0 & 1 & & 0 & 0\\ \vdots & \ddots & \ddots & & \vdots\\ 0 & \ldots & 0 & 1 & 0\\ R_1 + cA^{r-1}bk_1 & \ldots & & R_r + cA^{r-1}bk_r & S\\ \hline P & 0 & \ldots & 0 & Q \end{array}\right]\begin{pmatrix}\xi\\ \eta\end{pmatrix},$$

and if (1.3) has stable zero dynamics, that is Q is stable, a Lyapunov-function argument shows that, for suitable $k_1, \ldots, k_r$, the above system is exponentially stable. Here the high frequency gain $cA^{r-1}b \in \mathbb{R}$ must be known. Isidori shows the existence of design parameters $\kappa, k_1, \ldots, k_r$ for a the stabilizing feedback $u(t) = -\sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1} y^{(i)}(t)$ for nonlinear SISO systems with known lower bound for the high frequency gain [Isi95, Th. 9.3.1.], [Isi99, Th. 12.1.1.]. If the high frequency gain is unknown but only the sign of $cA^{r-1}b$ is known, the present proof that the feedback law (1.2) is stabilizing for sufficiently large $\kappa > 0$ becomes much more involved.

In the present paper, the above described SISO result will be generalized to MIMO systems of the form (1.1). Although the stabilizing feedback strategies preserve the simplicity of (1.2), see the main results Theorems 3.1 and 3.2, the proof is much more subtle. On the one hand the proof is based on a generalization of normal form (1.3) for linear MIMO systems, see Proposition 2.2, which is implicitly contained in the nonlinear normal form [Isi99] and explicitly derived in [Mue08] and has similar structural properties as (1.3); on the other hand the proof is based on a root locus results on a sum of polynomials, see Lemma 4.1; finally, these findings allow for the design of a Lyapunov-function and an appropriate scaling of the states leads to the desired results. To the author's best knowledge, these results are even for linear SISO systems still not available in the literature.

The present paper is structured as follows. In Section 2 the normal form for linear MIMO systems is presented shortly and the system's zero dynamics are characterized. Section 3 contains the control strategies and in Section 4 all proofs are given. The introduction is closed with remarks on notation:

Nomenclature

$\left[l_1^{(n)},\dots,l_m^{(n)}\right] = L \in \mathbb{C}^{n\times m}$,	where $l_i^{(n)} \in \mathbb{C}^n$ denotes the i-th column of L and the superscript (n) remarks the dimension of the vector,
$\left[l_{(m)}^1 \,/\, \dots \,/\, l_{(m)}^n\right] = L \in \mathbb{C}^{n\times m}$,	where $l_{(m)}^j \in \mathbb{C}^{1\times m}$ denotes the j-th row of L and the subscript (m) remarks the dimension of the row-vector, moreover the skew lines denote that the matrix is a column matrix,
$e_k^{(n)} := \left[0_{1\times(k-1)}, 1, 0_{1\times(n-k)}\right]^T$,	the k-th column unit vector in $\mathbb{R}^n$,
$e_{(m)}^k := \left[0_{1\times(k-1)}, 1, 0_{1\times(m-k)}\right]$,	the k-th row unit vector in $\mathbb{R}^{1\times m}$,
$0_{n\times m} \in \mathbb{R}^{n\times m}$,	the 0-matrix of dimension $n\times m$,
$I_n \in \mathbb{R}^{n\times n}$,	the identity matrix of dimension $n\times n$,
$\mathcal{C}^m([0,\infty)\to\mathbb{R}^n)$,	the set of m-times continuously differentiable maps from $[0,\infty)$ to $\mathbb{R}^n$,
$\lfloor x\rfloor := \max\{n\in\mathbb{Z} \mid n\le x\}$,	the maximum integer less or equal $x\in\mathbb{R}$,
$\|x\| := \|x\|_2$,	the euclidian norm of $x\in\mathbb{C}^n$,
$\|A\| := \|A\|_2 := \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$,	the matrix norm of $A\in\mathbb{C}^{n\times m}$ induced by the euclidian norm,
$\operatorname{diag}(A_1,\dots,A_m)\in\mathbb{C}^{n\times n}$,	a matrix with $A_i\in\mathbb{C}^{j_i\times j_i}$, $i=1,\dots,m$, on the diagonal and zeros otherwise,
$\operatorname{spec}(A) := \{\lambda\in\mathbb{C} \mid \det(\lambda I_n - A)=0\}$,	the spectrum of the matrix $A\in\mathbb{C}^{n\times n}$,
$\mu(A) := \max\{\operatorname{Re} s \mid s\in\operatorname{spec}(A)\}$,	the largest real part of the eigenvalues of $A\in\mathbb{C}^{n\times n}$,
$\mathcal{Z}(p) := \{s\in\mathbb{C} \mid p(s)=0\}$,	the set of zeros of $p\in\mathbb{C}[s]$,
$\mu(p(\cdot)) := \max\{\operatorname{Re} s \mid s\in\mathcal{Z}(p)\}$,	the largest real part of the zeros of $p\in\mathbb{C}[s]$,
$\mathbb{R}^H[s] := \{p\in\mathbb{R}[s] \mid \mu(p)<0\}$,	the set of all Hurwitz polynomials,
$\mathcal{B}_\delta(s_0) := \{s\in\mathbb{C} \mid \lvert s-s_0\rvert<\delta\}$,	the ball in $\mathbb{C}$ of radius δ around $s_0\in\mathbb{C}$.

2 Normal form and zero dynamics

Consider, for $n,m\in\mathbb{N}$ and $A\in\mathbb{R}^{n\times n}$, $B, C^T\in\mathbb{R}^{n\times m}$, a linear MIMO-system (A,B,C) of form (1.1). The (vector) relative degree of a such a system is defined as follows.

Definition 2.1 A linear system (A,B,C) of form (1.1) with $A\in\mathbb{R}^{n\times n}$, $B, C^T\in\mathbb{R}^{n\times m}$ has

(a) *(vector) relative degree* $r=(r_1,\dots,r_m)\in\mathbb{N}^{1\times m}$ if, and only if,

(i) $\forall\, j\in\{1,\dots,m\}\ \forall\, k\in\{0,\dots,r_j-2\}\ :\ c_{(n)}^j A^k B = 0_{1\times m}$,

(ii) $\operatorname{rk}\left[c_{(n)}^1 A^{r_1-1}B \,/\, c_{(n)}^2 A^{r_2-1}B \,/\, \dots \,/\, c_{(n)}^m A^{r_m-1}B\right] = m$,

(b) *ordered (vector) relative degree* $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ if, and only if, (1.1) has (vector) relative degree $r = (r_1, \ldots, r_m)$ with $r_1 \geq r_2 \geq \ldots \geq r_m$,

(c) *strict relative degree* $\varrho \in \mathbb{N}$ if, and only if, (1.1) has (vector) relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ with $\varrho = r_1 = r_2 = \ldots = r_m$.

Note that for any linear system (A, B, C) of form (1.1) with relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$ it follows that $m = \operatorname{rk}\left[c^1_{(n)}A^{r_1-1}B \,/\, \ldots \,/\, c^m_{(n)}A^{r_m-1}B\right] \leq \operatorname{rk} B \leq \min\{n, m\}$. Thus $m \leq n$.

Next a normal form for linear systems (A, B, C) with ordered vector relative degree is presented.

Proposition 2.2 [Mue08] Consider a linear system (A, B, C) of form (1.1) with ordered vector relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$. Set $r^s := \sum_{j=1}^m r_j$. Then there exists an invertible matrix $U \in \mathbb{R}^{n\times n}$ such that the coordinate transformation

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} := Ux\,, \quad \xi(t) = \Big(y_1(t), \ldots, y_1^{(r_1-1)}(t)\Big|\ \ldots\ \Big|\,y_m(t), \ldots, y_m^{(r_m-1)}(t)\Big)^T \in \mathbb{R}^{r^s}\,,\ \eta(t) \in \mathbb{R}^{n-r^s}\,,$$

converts (A, B, C) into

$$\left.\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi\\ \eta\end{pmatrix} &= \widetilde{A}\begin{pmatrix}\xi\\ \eta\end{pmatrix} + \widetilde{B}u \\ y &= \widetilde{C}\begin{pmatrix}\xi\\ \eta\end{pmatrix} \end{aligned}\right\} \tag{2.1}$$

where

$$\left[\begin{array}{c|c}\widetilde{A} & \widetilde{B}\\ \hline \widetilde{C} & 0\end{array}\right] = \left[\begin{array}{cccc|cccc|c|cccc|c||c}
0 & 1 & & 0 & 0 & \cdots & & 0 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m} \\
\vdots & \ddots & \ddots & & \vdots & & & \vdots & \cdots & \vdots & & & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 1 & 0 & \cdots & & 0 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m} \\
R^1_{1,1} & \cdots & & R^1_{1,r_1} & R^1_{2,1} & \cdots & & R^1_{2,r_2} & & R^1_{m,1} & \cdots & & R^1_{m,r_m} & S^1 & c^1_{(n)}A^{r_1-1}B \\ \hline
0 & \cdots & & 0 & 0 & 1 & & 0 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m} \\
\vdots & & & \vdots & \vdots & \ddots & \ddots & & & \vdots & & & \vdots & \vdots & \vdots \\
0 & \cdots & & 0 & 0 & \cdots & 0 & 1 & & 0 & \cdots & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m} \\
R^2_{1,1} & \cdots & & R^2_{1,r_1} & R^2_{2,1} & \cdots & & R^2_{2,r_2} & & R^2_{m,1} & \cdots & & R^2_{m,r_m} & S^2 & c^2_{(n)}A^{r_2-1}B \\ \hline
& \vdots & & & & & & & \ddots & & & & & \vdots & \vdots \\ \hline
0 & \cdots & & 0 & 0 & \cdots & & 0 & & 0 & 1 & & 0 & 0_{1\times(n-r^s)} & 0_{1\times m} \\
\vdots & & & \vdots & \vdots & & & \vdots & & \vdots & \ddots & \ddots & & \vdots & \vdots \\
0 & \cdots & & 0 & 0 & \cdots & & 0 & & 0 & \cdots & 0 & 1 & 0_{1\times(n-r^s)} & 0_{1\times m} \\
R^m_{1,1} & \cdots & & R^m_{1,r_1} & R^m_{2,1} & \cdots & & R^m_{2,r_2} & & R^m_{m,1} & \cdots & & R^m_{m,r_m} & S^m & c^m_{(n)}A^{r_m-1}B \\ \hline
P_1 & 0 & \cdots & 0 & P_2 & 0 & \cdots & 0 & \cdots & P_m & 0 & \cdots & 0 & Q & 0_{(n-r^s)\times m} \\ \hline\hline
1 & 0 & \cdots & 0 & 0 & \cdots & & 0 & & 0 & \cdots & & 0 & & \\
0 & \cdots & & 0 & 1 & 0 & \cdots & 0 & & \vdots & & & \vdots & & \\
\vdots & & & \vdots & 0 & \cdots & & 0 & \cdots & & & & & 0_{m\times(n-r^s)} & 0 \\
 & & & & \vdots & & & \vdots & & 0 & \cdots & & 0 & & \\
0 & \cdots & & 0 & 0 & \cdots & & 0 & & 1 & 0 & \cdots & 0 & &
\end{array}\right] \tag{2.2}$$

and $R^j_{i,k} \in \mathbb{R}$, for $i, j \in \{1, \ldots, m\}$ and $k \in \{1, \ldots, r_i\}$, $S^1, \ldots, S^m \in \mathbb{R}^{1\times(n-r^s)}$, $P_1, \ldots, P_m \in \mathbb{R}^{n-r^s}$ and $Q \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$.

Note that input u and output y of the original system (A, B, C) and in normal form are the same. The coordinate transformation does not affect u and y.

The *zero dynamics* of a linear system (A, B, C) of form (1.1) are defined as the real vector space of trajectories

$$\mathcal{ZD}(A,B,C) := \Big\{ (x,u,y) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \times \mathcal{C}_{\mathrm{pw}}([0,\infty) \to \mathbb{R}^m) \times \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \;\Big|\; (x,u,y) \text{ solves (1.1) with } y \equiv 0 \text{ on } [0,\infty) \Big\} .$$

Using the normal form (2.1), (2.2) one can read off the zero dynamics of (1.1) very easily:

$$\mathcal{ZD}(A,B,C) = \Big\{ \big(\mathcal{V}\eta, -\Gamma^{-1} S\eta, 0\big) \in \mathcal{C}^1([0,\infty) \to \mathbb{R}^n) \times \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \times \mathcal{C}^1([0,\infty) \to \mathbb{R}^m) \;\Big|\; \dot\eta = Q\eta \Big\} ,$$

where $\Gamma := \Big[c^1_{(n)} A^{r_1 - 1} B \;/\; c^2_{(n)} A^{r_2 - 1} B \;/\; \ldots \;/\; c^m_{(n)} A^{r_m - 1} B \Big]$, $S := \Big[S^{1T}, \ldots, S^{mT} \Big]^T$, Q and $S^1, \ldots, S^m$ are given by (2.2) and $\mathcal{V} \in \mathbb{R}^{n \times (n - r^s)}$ is chosen such that

$$\operatorname{im} \mathcal{V} = \ker \left[\begin{bmatrix} c^1_{(n)} \\ \vdots \\ c^1_{(n)} A^{r_1 - 1} \end{bmatrix} \Big/ \begin{bmatrix} c^2_{(n)} \\ \vdots \\ c^2_{(n)} A^{r_2 - 1} \end{bmatrix} \Big/ \cdots \Big/ \begin{bmatrix} c^m_{(n)} \\ \vdots \\ c^m_{(n)} A^{r_m - 1} \end{bmatrix} \right] .$$

A linear system $\dot x = Ax$, for $A \in \mathbb{R}^{n \times n}$, is called *exponentially stable* if, and only if,

$$\exists\, M, \lambda > 0 \;\forall\, t \geq 0 \;:\; \|x(t)\| \leq M e^{-\lambda t} \|x(0)\| ,$$

for all solutions x of $\dot x = Ax$.

With the above characterization of the zero dynamics of linear MIMO systems (A, B, C), one can show, see [Mue08], that the zero dynamics are *exponentially stable*, i.e.

$$\exists\, M, \lambda > 0 \;\forall\, (x,u) \in \mathcal{ZD}(A,B,C) \;\forall\, t \geq 0 \;:\; \|(x(t), u(t))\| \leq M e^{-\lambda t} \|x(0)\| ,$$

if, and only if, the linear system $\dot\eta = Q\eta$ is an exponentially stable linear system.

3 Stabilization

In the following an output derivative feedback controller is designed for linear systems (A, B, C) of form (1.1) with (vector) relative degree $r = (r_1, \ldots, r_m) \in \mathbb{R}^{1 \times m}$, stable zero dynamics and unknown but positive definite high frequency gain $\Gamma \in \mathbb{R}^{m \times m}$.

First it is assumed that the MIMO systems has strict relative degree $r \in \mathbb{N}$. Note that in case of strict relative degree it follows that $\Gamma = CA^{r-1}B$.

Theorem 3.1 Suppose that system (A, B, C) of form (1.1) has strict relative degree $r \in \mathbb{N}$, positive definite $CA^{r-1}B$ and exponentially stable zero dynamics. Then for any monic Hurwitz polynomial $\Big(s \mapsto \sum_{i=0}^{r-1} k_{i+1}\, s^i \Big) \in \mathbb{R}[s]$, there exists $\kappa^* \geq 1$ such that, for all $\kappa > \kappa^*$, the feedback

$$u(t) = -\kappa \sum_{i=0}^{r-1} \kappa^{r-i} k_{i+1}\, y^{(i)}(t) \tag{3.1}$$

applied to (1.1) yields an exponentially stable closed-loop system.

Next a feedback is given for MIMO systems with non-strict relative degree.

Theorem 3.2 Suppose that system (A, B, C) of form (1.1) has vector relative degree $r = (r_1, \ldots, r_m) \in \mathbb{N}^{1\times m}$, exponentially stable zero dynamics and positive definite $\left[c^1_{(n)} A^{r_1-1} B \;/\; c^2_{(n)} A^{r_2-1} B \;/\; \ldots \;/\; c^m_{(n)} A^{r_m-1} B\right]$. Then for any m Hurwitz polynomials

$$\left(s \mapsto \sum_{i=0}^{r_j-1} k_{j,i+1}\, s^i\right) \in \mathbb{R}[s]\,, \quad j = 1, \ldots, m\,,$$

there exists $\kappa^* \geq 1$ such that, for all $\kappa > \kappa^*$, the feedback

$$u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix} = -\kappa \begin{pmatrix} \sum_{i=0}^{r_1-1} \kappa^{r_1-i} k_{1,i+1}\, y_1^{(i)}(t) \\ \vdots \\ \sum_{i=0}^{r_m-1} \kappa^{r_m-i} k_{m,i+1}\, y_m^{(i)}(t) \end{pmatrix} \tag{3.2}$$

applied to (1.1) yields an exponentially stable closed-loop system.

4 Proofs

First some useful properties of Hurwitz polynomials are shown. For a polynomial $p \in \mathbb{R}[s]$ write

$$p(s) = p_n \prod_{j=1}^{\ell} (s - s_j)^{m_j}\,, \quad \text{can. fact.}$$

for the canonical factorization of p with $s_1, \ldots, s_\ell \in \mathbb{C}$ pairwise distinct and $m_1, \ldots, m_\ell \in \mathbb{N}$. For the proof of Theorems 3.2 and 3.1, a root-locus result on a special sum of polynomials is required. This result is an generalization of [HP05, Th. 4.1.2].

Lemma 4.1 For $\delta > 0$, $d \in \mathbb{N}$ let

$$\big(s \mapsto p_i(s) = p_{i,n-i} s^{n-i} + p_{i,n-i-1} s^{n-i-1} + \cdots + p_{i,1} s + p_{i,0}\big) \in \mathbb{R}[s]\,, \quad i = 0, \ldots, d$$

such that

$$p_{d,n-d} > 0\,, \qquad p_{0,n} > 0\,, \qquad \mu(p_d(\cdot)) < -\delta$$

and

$$\left(\kappa \mapsto \widehat{p}(\kappa) = p_{0,n}\kappa^d + p_{1,n-1}\kappa^{d-1} + \cdots + p_{d-1,n-d+1}\kappa + p_{d,n-d}\right) \in \mathbb{R}[\kappa] \quad \text{with} \quad \mu(\widehat{p}(\cdot)) < -\delta\,.$$

Then

$$\exists\, \kappa_0 > 0\ \forall\, \kappa > \kappa_0\ :\ \mu\left(\sum_{k=0}^{d} \kappa^k p_k(\cdot)\right) < -\delta/2\,. \tag{4.1}$$

Proof. Write

$$p_d(s) = p_{d,n-d} \prod_{j=1}^{\ell_1} (s - \zeta_j)^{m_j}\,, \quad \text{can. fact.}$$

and

$$\widehat{p}(\kappa) = p_{0,n} \prod_{j=1}^{\ell_2} (\kappa - \xi_j)^{n_j}\,, \quad \text{can. fact.}$$

and, for $\gamma > 0$,

$$q[\gamma](s) := \sum_{k=0}^{d} \gamma^{d-k} p_k(s) = \gamma^d p_{0,n} \prod_{j=1}^{n} (s - s_j[\gamma])\,, \qquad \text{with} \quad s_1[\gamma], \ldots, s_n[\gamma] \in \mathbb{C}\,.$$

Suppose it is shown that, for $\alpha := \max_{j\in\{1,\ldots,\ell_1\}}\{|\zeta_j|\} + \delta$ and suitable numbering of the zeros $s_j[\gamma]$,

$$\begin{aligned} &\exists\, \gamma_0 > 0\ \forall\, \gamma \in (0, \gamma_0)\ : \\ &\qquad \{s_1[\gamma], \ldots, s_{n-d}[\gamma]\} \subset \bigcup_{j\in\{1,\ldots,\ell_1\}} \mathcal{B}_{\delta/2}(\zeta_j) \subset \mathbb{C}_-\,, \qquad (4.2a) \\ &\qquad \{s_{n-d+1}[\gamma], \ldots, s_n[\gamma]\} \subset \{s \in \mathbb{C} \,|\, \operatorname{Re} s < -\alpha\} \subset \mathbb{C}_-\,. \qquad (4.2b) \end{aligned}$$

Then there exists $\gamma_0 > 0$ such that for all $\gamma \in (0, \gamma_0)$ all zeros of $q[\gamma]$ are in $\mathbb{C}_{-\delta/2}$. Setting $\kappa = \gamma^{-1}$ yields $p[\kappa] = \kappa^d q[\kappa^{-1}] = \kappa^d q[\gamma]$. Hence, for all $\kappa, \gamma > 0$ and $s \in \mathbb{C}$, $p[\kappa](s) = 0$ if, and only if, $q[\gamma](s) = 0$. Thus setting $\kappa_0 = \gamma_0^{-1}$ yields (4.1).

In the remainder of the proof (4.2) is shown. One may choose $\varepsilon^* > 0$ such that

$$\begin{aligned} &\forall\, i, j \in \{1, \ldots, \ell_1\},\ i \neq j\ :\ \mathcal{B}_{\varepsilon^*}(\zeta_i) \cap \mathcal{B}_{\varepsilon^*}(\zeta_j) = \emptyset\,, \\ &\forall\, i, j \in \{1, \ldots, \ell_2\},\ i \neq j\ :\ \mathcal{B}_{\varepsilon^*}(\xi_i) \cap \mathcal{B}_{\varepsilon^*}(\xi_j) = \emptyset\,. \end{aligned}$$

Then, an application of [HP05, Th. 4.1.2] to $q[\gamma](\cdot) = \sum_{k=0}^{d-1} \gamma^{d-k} p_k(\cdot) + p_d(\cdot)$ and suitable numbering of the zeros $s_j[\gamma]$ of $q[\gamma]$ implies

$$\begin{aligned} &\forall\, \varepsilon \in (0, \min\{\varepsilon^*, \delta/2\})\ \exists\, \gamma^* = \gamma^*(\varepsilon) > 0\ \forall\, \gamma \in (0, \gamma^*)\ : \\ &\qquad \{s_1[\gamma], \ldots, s_{n-d}[\gamma]\} \subset \bigcup_{j\in\{1,\ldots,\ell_1\}} \mathcal{B}_{\varepsilon}(\zeta_j)\,, \qquad \{s_{n-d+1}[\gamma], \ldots, s_n[\gamma]\} \subset \mathbb{C} \setminus \mathcal{B}_{1/\varepsilon}(0)\,. \end{aligned}$$

Now $\mu(p_d(\cdot)) < -\delta$ yields (4.2a).

For $\gamma > 0$ setting $x = \gamma s$ yields

$$\begin{aligned} q[\gamma](s) &= q[\gamma](\gamma^{-1}x) \\ &= \gamma^d \left(p_{0,n}\gamma^{-n}x^n + p_{0,n-1}\gamma^{-n+1}x^{n-1} + \cdots + p_{0,1}\gamma^{-1}x + p_{0,0}\right) \\ &\quad + \gamma^{d-1}\left(p_{1,n-1}\gamma^{-n+1}x^{n-1} + \cdots + p_{1,1}\gamma^{-1}x + p_{1,0}\right) \\ &\quad \vdots \\ &\quad + \gamma\left(p_{d-1,n-d+1}\gamma^{-n+d-1}x^{n-d+1} + \cdots + p_{d-1,1}\gamma^{-1}x + p_{d-1,0}\right) \\ &\quad + \left(p_{d,n-d}\gamma^{-n+d}x^{n-d} + \cdots + p_{d,1}\gamma^{-1}x + p_{d,0}\right) \\ &= \gamma^{-n+d}\Bigg(x^{n-d}\left(p_{0,n}x^d + p_{1,n-1}x^{d-1} + \cdots + p_{d-1,n-d+1}x + p_{d,n-d}\right) \\ &\qquad + \gamma\Big[\left(p_{0,n-1}x^{n-1} + \cdots + \gamma^{n-1}p_{0,0}\right) + \left(p_{1,n-2}x^{n-2} + \cdots + \gamma^{n-2}p_{1,0}\right) \\ &\qquad\quad + \cdots + \left(p_{d,n-d-1}x^{n-d-1} + \cdots + \gamma^{n-d-1}p_{d,0}\right)\Big]\Bigg)\,. \end{aligned}$$

Write

$$\widehat{q}[\gamma](x) := \gamma^{n-d} q[\gamma](\gamma^{-1}x) = x^{n-d}\widehat{p}(x) + \gamma \sum_{k=0}^{d} \sum_{i=0}^{n-k-1} \left(\gamma^{n-k-1-i} p_{k,i}\, x^i\right) = p_{0,n} \prod_{j=1}^{n} (x - x_j[\gamma]) \,.$$

Thus, by [HP05, Th. 4.1.2] and suitable numbering of the zeros $x_j[\gamma]$ of $\widehat{q}[\gamma]$,

$$\begin{aligned} &\forall\, \varepsilon \in (0, \min\{\varepsilon^*, \delta/2\})\ \exists\, \gamma_0 \in \left(0, \min\left\{\gamma^*, \varepsilon\alpha^{-1}, 1\right\}\right)\ \forall\, \gamma \in (0, \gamma_0)\ : \\ &\qquad \{x_1[\gamma], \ldots, x_{n-d}[\gamma]\} \subset \mathcal{B}_\varepsilon(0)\,, \qquad \{x_{n-d+1}[\gamma], \ldots, x_n[\gamma]\} \subset \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_\varepsilon(\xi_j)\,. \end{aligned}$$

Hence $\mu(\widehat{p}(\cdot)) < -\delta$ yields, for suitable numbering of the zeros $x_j[\gamma]$,

$$\forall\, \gamma \in (0, \gamma_0)\ :\ \{x_{n-d+1}[\gamma], \ldots, x_n[\gamma]\} \in \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_{\delta/2}(\xi_j) \subset \mathbb{C}_-\,. \tag{4.3}$$

Furthermore, $\mu(\widehat{p}(\cdot)) < -\delta$ and $\gamma_0 < 1$ yields

$$\forall\, \gamma \in (0, \gamma_0)\ \forall\, x_0 \in \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_{\delta/2}(\xi_j)\ :\ |\gamma^{-1}x_0| > |\gamma_0^{-1}x_0| > \varepsilon^{-1}\alpha\delta/2 > \alpha\,,$$

thus

$$\forall\, \gamma \in (0, \gamma_0)\ \forall\, x_0 \in \bigcup_{j\in\{1,\ldots,\ell_2\}} \mathcal{B}_{\delta/2}(\xi_j)\ :\ \gamma^{-1}x_0 \notin \bigcup_{j\in\{1,\ldots,\ell_1\}} \mathcal{B}_{\delta/2}(\zeta_j)$$

and by (4.3), for suitable numbering of the zeros $x_j[\gamma]$,

$$\forall\, \gamma \in (0, \gamma_0)\ :\ \{\gamma^{-1}x_{n-d+1}[\gamma], \ldots, \gamma^{-1}x_n[\gamma]\} \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) < -\alpha\}\,,$$

whence, noting that for every $\gamma > 0$ and $x_0 \in \mathbb{C}$, $\widehat{q}[\gamma](x_0) = 0$ if, and only if, $q[\gamma](\gamma^{-1}x_0) = 0$, (4.2b), which completes the proof. $\square$

Next some properties of matrices, which are required to prove Theorem 3.2, are derived. The principal minors of a matrix $A = [a_{i,j}]_{i,j=1,\ldots,n} \in \mathbb{R}^{n\times n}$ are defined as follows: for a set of indices $\{i_1, \ldots, i_k\}$ with $1 \le i_1 < \cdots < i_k \le n$, $k \in \{0, \ldots, n\}$, let

$$\mathrm{minor}(A\,;\,\{i_1, \ldots, i_k\}) := \det \begin{bmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_k} \\ a_{i_2,i_1} & a_{i_2,i_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{i_{k-1},i_k} \\ a_{i_k,i_1} & \cdots & a_{i_k,i_{k-1}} & a_{i_k,i_k} \end{bmatrix}, \qquad \mathrm{minor}(A\,;\,\emptyset) := 1\,.$$

Lemma 4.2 Let $A = [a_{i,j}]_{i,j=1,\ldots,m} \in \mathbb{R}^{m\times m}$ and $b_1, \ldots, b_m \in \mathbb{R}$. Then

$$\det(A + t\,\mathrm{diag}(b_1, \ldots, b_m)) = \sum_{k=0}^{m} \left(\sum_{1\le i_1<\cdots<i_k\le m} \mathrm{minor}\,(A\,;\,\{i_1, \ldots, i_k\}) \prod_{\substack{i\in\{1,\ldots,m\}\\ i\notin\{i_1,\ldots,i_k\}}} b_i \right) t^{m-k}\,. \tag{4.4}$$

Proof. Write, for $t \in \mathbb{C}$,

$$\det(A + t\,\mathrm{diag}(b_1, \ldots, b_m)) = p_m t^m + p_{m-1} t^{m-1} + \cdots + p_1 t + p_0$$

and define the function $f\colon \mathbb{R}^m \to \mathbb{R}$ by

$$
\begin{aligned}
f(t_1,\dots,t_m) &= \det(A + \operatorname{diag}(b_1t_1,\dots,b_mt_m)) \\
&= \det \begin{bmatrix} a_{1,1}+b_1t_1 & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2}+b_2t_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{m-1,m} \\ a_{m,1} & \cdots & a_{m,m-1} & a_{m,m}+b_mt_m \end{bmatrix} \\
&= \sum_{\sigma\in\mathcal{S}_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \left(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\right) ,
\end{aligned}
$$

where

$$
b_{i,\sigma(i)} := \begin{cases} b_i\,, & \text{if } i = \sigma(i) \\ 0\,, & \text{if } i \neq \sigma(i) \end{cases}
$$

and $\mathcal{S}_m$ is the set of all permutations of $\{1,\dots,m\}$. Then

$$
\begin{aligned}
f(t,\dots,t) &= \det(A + t\operatorname{diag}(b_1,\dots,b_m))\,, \\
p_0 = f(0,\dots,0) &= \det(A) = \operatorname{minor}(A\,;\,\{1,\dots,m\})\,.
\end{aligned}
$$

Furthermore, the coefficients p_k are given via the partial derivatives of f, that is

$$
p_k = \sum_{1\le i_1<\cdots<i_k\le m} \frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} f(t_1,\dots,t_m)\Big|_{t_1=\ldots=t_m=0}\,, \qquad k\in\{1,\dots,m\}\,.
$$

Moreover, it follows that

$$
\begin{aligned}
&\frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} f(t_1,\dots,t_m)\Big|_{t_1=\ldots=t_m=0} \\
&= \frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} \left(\sum_{\sigma\in\mathcal{S}_m} \operatorname{sgn}(\sigma) \prod_{i=1}^m \left(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\right) \right)\Bigg|_{t_1=\ldots=t_m=0} \\
&= \sum_{\sigma\in\mathcal{S}_m} \operatorname{sgn}(\sigma) \frac{\partial^k}{\partial t_{i_1}\cdots\partial t_{i_k}} \prod_{i=1}^m \left(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\right) \Bigg|_{t_1=\ldots=t_m=0} \\
&= \sum_{\sigma\in\mathcal{S}_m} \operatorname{sgn}(\sigma) \frac{\partial^{k-1}}{\partial t_{i_1}\cdots\partial t_{i_{k-1}}} \left(\begin{cases} 0\,, & \text{if } \sigma(i_k)\neq i_k \\ b_{i_k} \prod\limits_{\substack{i=1\\ i\neq i_k}}^m \left(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\right) \\ + \left(a_{i_k,i_k} + b_{i_k} t_{i_k}\right) \\ \underbrace{\cdot \frac{\partial}{\partial t_{i_k}} \prod\limits_{\substack{i=1\\ i\neq i_k}}^m \left(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\right)}_{=0}, & \text{if } \sigma(i_k) = i_k \end{cases} \right)\Bigg|_{t_1=\ldots=t_m=0} \\
&= \sum_{\sigma\in\mathcal{S}_m} \operatorname{sgn}(\sigma) \begin{cases} 0\,, & \text{if } \exists\, j=1,\dots,k \;:\; \sigma(i_j)\neq i_j \\ \prod\limits_{j=1}^k b_{i_j} \prod\limits_{\substack{i=1\\ i\neq i_1,\dots,i_k}}^m \left(a_{i,\sigma(i)} + b_{i,\sigma(i)} t_i\right)\,, & \text{if } \forall\, j=1,\dots,k \;:\; \sigma(i_j) = i_j \end{cases}\Bigg|_{t_1=\ldots=t_m=0} \\
&= \sum_{\substack{\sigma\in\mathcal{S}_m\\ \{i_1,\dots,i_k\}\subset\sigma}} \operatorname{sgn}(\sigma) \prod_{j=1}^k b_{i_j} \prod_{\substack{i=1\\ i\neq i_1,\dots,i_k}}^m a_{i,\sigma(i)} = \prod_{j=1}^k b_{i_j} \operatorname{minor}(A\,;\,\{1,\dots,m\}\setminus\{i_1,\dots,i_k\})\,.
\end{aligned}
$$

Hence, for $k = 1, \dots, m$,

$$p_k = \sum_{1 \le i_1 < \cdots < i_k \le m} \prod_{j=1}^{k} b_{i_j} \operatorname{minor}(A\,;\, \{1, \dots, m\} \setminus \{i_1, \dots, i_k\}),$$

and thus

$$p_{m-k} = \sum_{1 \le i_1 < \cdots < i_k \le m} \prod_{\substack{i \in \{1,\dots,m\} \\ i \notin \{i_1,\dots,i_k\}}} b_i \operatorname{minor}(A\,;\, \{i_1, \dots, i_k\}),$$

which shows (4.4) and completes the proof. $\square$

Lemma 4.3 Suppose $A\colon (\kappa_0, \infty) \to \mathbb{R}^{m\times m}$, $\kappa_0 \in \mathbb{R}$, satisfies

$$\exists\, \delta > 0 \;\forall\, t > t_0 \;:\; \mu(A(t)) < -\delta\,.$$

Then the unique symmetric, positive definite matrix

$$P(t) \in \mathbb{R}^{m\times m} \;:\; P(\kappa)A(\kappa) + A(\kappa)^T P(\kappa) = -I_m$$

satisfies

$$\forall\; \kappa > \kappa_0 \;:\; \|P(\kappa)\| \le \frac{2e}{\delta}\left(1 + \frac{(2m)!}{(2\delta)^{2m}}\right). \tag{4.5}$$

Proof. For every $\kappa > \kappa_0$ choose

$$\begin{aligned}
&U(\kappa) \in \mathbb{C}^{m\times m} \quad \text{invertible with} \quad \|U(\kappa)\| = 1\\
&\Lambda(\kappa) = \operatorname{diag}(\lambda_1(\kappa), \dots, \lambda_m(\kappa)) \in \mathbb{C}^{m\times m}\\
&N(\kappa) = \begin{bmatrix} 0 & \psi_1(\kappa) & & 0\\ \vdots & \ddots & \ddots & \\ & & & \psi_{m-1}(\kappa)\\ 0 & \dots & & 0 \end{bmatrix} \quad \text{with} \quad \psi_i(\kappa) \in \{0,1\}\,,\; i = 1, \dots, m-1\,,
\end{aligned}$$

such that

$$A(\kappa) = U(\kappa)^{-1} \underbrace{[\Lambda(\kappa) + N(\kappa)]}_{=:\,J(\kappa)} U(\kappa)$$

and

$$\forall\, i \in \{1, \dots, m-1\} \;:\; \lambda_i(\kappa) \ne \lambda_{i+1}(\kappa) \;\Rightarrow\; \psi_i(\kappa) = 0\,.$$

Then

$$\forall\, \kappa > \kappa_0 \;\forall\, i \in \{1, \dots, m-1\} \;:\; \lambda_i(\kappa)\psi_i(\kappa) = \lambda_{i+1}(\kappa)\psi_i(\kappa)\,,$$

and thus

$$\begin{aligned}
&\forall\, \kappa > \kappa_0 \;:\\
&\Lambda(\kappa)N(\kappa) = \begin{bmatrix} 0 & \lambda_1(\kappa)\psi_1(\kappa) & & 0\\ \vdots & \ddots & \ddots & \\ & & & \lambda_{m-1}(\kappa)\psi_{m-1}(\kappa)\\ 0 & \dots & & 0 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_2(\kappa)\psi_1(\kappa) & & 0\\ \vdots & \ddots & \ddots & \\ & & & \lambda_m(\kappa)\psi_{m-1}(\kappa)\\ 0 & \dots & & 0 \end{bmatrix}\\
&\qquad = N(\kappa)\Lambda(\kappa)\,,
\end{aligned}$$

whence

$$\forall\, \kappa > \kappa_0 \;:\; e^{\Lambda(\kappa)+N(\kappa)} = e^{\Lambda(\kappa)} e^{N(\kappa)}\,.$$

Then

$$
\begin{aligned}
\|P(\kappa)\| &= \Big\| \int_0^\infty e^{A(\kappa)^T s} e^{A(\kappa)\,s}\, \mathrm{d}s \Big\| \\
&= \Big\| \int_0^\infty U(\kappa)^T\, e^{J(\kappa)^T s}\, (U(\kappa)^T)^{-1}\, U(\kappa)^{-1}\, e^{J(\kappa)\,s}\, U(\kappa)\, \mathrm{d}s \Big\| \\
&\le \int_0^\infty \|e^{J(\kappa)\,s}\|^2\, \mathrm{d}s \\
&\le \int_0^\infty \|e^{\Lambda(\kappa)\,s}\|^2\, \|e^{N(\kappa)\,s}\|^2\, \mathrm{d}s \\
&\le \int_0^\infty e^{2\mu(A(\kappa))\,s} \left(\sum_{j=0}^m \frac{s^j}{j!} \right)^2 \mathrm{d}s \\
&\le \int_0^\infty e^{-2\delta\,s} \underbrace{\left(\sum_{j=0}^m \frac{1}{j!} + \sum_{j=0}^m \frac{1}{j!} s^m \right)^2}_{\le e(1+s^m)^2 = e(1+2s^m+s^{2m})} \mathrm{d}s \\
&\le 4e \int_0^\infty e^{-2\delta\,s}(1+s^{2m})\, \mathrm{d}s \\
&\le 4e \left(\left[\frac{1}{-2\delta} e^{-2\delta\,s} \right]_0^\infty + \int_0^\infty e^{-2\delta\,s}\, s^{2m}\, \mathrm{d}s \right) \\
&\le 4e \left(\frac{1}{2\delta} + \left[e^{-2\delta\,s} \sum_{i=0}^{2m} (-1)^{2m-i} \frac{1}{(-2\delta)^{2m-i+1}} \frac{(2m)!}{i!}\, s^i \right]_0^\infty \right) \\
&= 4e \left(\frac{1}{2\delta} + \frac{(2m)!}{(2\delta)^{2m+1}} \right),
\end{aligned}
$$

which shows (4.5) and completes the proof. □

Proof of Theorem 3.1. Let $x(\cdot)$ be a solution of (1.1).

Step 1: Representation of (1.1) in normal form.

By e.g. [IRT07, Lemma 3.5] or [Isi95, Section 4] there exists an invertible $V \in \mathbb{R}^{n\times n}$ such that the coordinate transformation

$$
\begin{pmatrix} \xi \\ \eta \end{pmatrix}(t) := Vx(t)
$$

converts (1.1) into $(\widehat{A}, \widehat{B}, \widehat{C})$ with

$$
\left.
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \xi \\ \eta \end{pmatrix} &= \underbrace{\begin{bmatrix} 0 & I_m & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I_m & 0 \\ R_1 & \dots & & R_r & S \\ P & 0 & \dots & 0 & Q \end{bmatrix}}_{=:\widehat{A}=VAV^{-1}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{pmatrix}}_{=:\widehat{B}=VB} u\,, \\
y = \xi_1 &= \underbrace{(I_m, 0, \dots, 0)}_{=:\widehat{C}=CV^{-1}} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \\
P \in \mathbb{R}^{(n-rm)\times m},\ & Q \in \mathbb{R}^{(n-rm)\times(n-rm)},\ S \in \mathbb{R}^{m\times(n-rm)},\ R_1, \dots, R_r \in \mathbb{R}^{m\times m}.
\end{aligned}
\right\} \quad (4.6)
$$

In the following let $\Gamma := CA^{r-1}B$.

Step 2: Scaling of the state vector.

Setting $\zeta_i = \kappa^{-i+1}\xi_i$, for $i = 1, \ldots, r$, yields

$$\dot{\zeta}_i \;=\; \kappa^{-i+1}\dot{\xi}_i = \kappa^{-i+1}\xi_{i+1} = \kappa\zeta_{i+1}\,, \qquad \text{for } i = 1, \ldots, r-1\,,$$

and

$$\begin{aligned}
\dot{\zeta}_r &= \kappa^{-r+1}\dot{\xi}_r \\
&= \kappa^{-r+1}\big((R_1 - \kappa k_1 \kappa^r \Gamma)\xi_1 + \cdots + (R_{r-1} - \kappa k_{r-1}\kappa^2\Gamma)\xi_{r-1} + (R_r - \kappa k_r \kappa\Gamma)\xi_r\big) + \kappa^{-r+1}S\eta \\
&= \kappa\left(\left(\frac{1}{\kappa^r}R_1 - \kappa k_1\Gamma\right)\zeta_1 + \cdots + \left(\frac{1}{\kappa^2}R_{r-1} - \kappa k_{r-1}\Gamma\right)\zeta_{r-1} + \left(\frac{1}{\kappa}R_r - \kappa k_r\Gamma\right)\zeta_r\right) + \kappa^{-r+1}S\eta\,.
\end{aligned}$$

Thus, for $\kappa \geq 1$, additional scaling

$$\begin{pmatrix}\zeta\\ \eta\end{pmatrix} = \underbrace{\operatorname{diag}\left(I_m, \kappa^{-1}I_m, \ldots, \kappa^{-r+1}I_m, I_{n-rm}\right)}_{=:\,U_\kappa}\begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

leads to

$$\left.\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t) &= \underbrace{\left[\begin{array}{c|c}
\kappa\begin{bmatrix} 0 & I_m & & & \\ & \ddots & \ddots & & \\ & & 0 & & I_m \\ \frac{R_1}{\kappa^r} - \kappa k_1\Gamma & \ldots & \frac{R_{r-1}}{\kappa^2} - \kappa k_{r-1}\Gamma & & \frac{R_r}{\kappa} - \kappa k_r\Gamma \end{bmatrix} & \begin{matrix} 0 \\ \vdots \\ 0 \\ \kappa^{-r+1}S \end{matrix} \\ \hline
P \quad 0 \quad \ldots \quad 0 & Q
\end{array}\right]}_{=:\,A_{\Gamma,k,\kappa}}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t) \\
y(t) &= (I_m, 0, \ldots, 0)\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t)\,.
\end{aligned}\right\} \tag{4.7}$$

Step 3: Design of positive definite solutions of two Lyapunov-equations.

Similar to showing that the last row of the companion matrix contains the negative coefficients of its characteristic polynomial, it follows that

$$\det\left(sI_n - \begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ -\kappa k_1\Gamma & \ldots & -\kappa k_{r-1}\Gamma & -\kappa k_r\Gamma \end{bmatrix}\right) = \det\left(s^r I_m + \left(\sum_{i=0}^{r-1}\kappa k_{i+1}s^i\right)\Gamma\right)$$

and, by Lemma 4.2,

$$\begin{aligned}
&\det\left(s^r I_m + \left(\sum_{i=0}^{r-1}\kappa k_{i+1}s^i\right)\Gamma\right) \\
&= \det\left(s^r\Gamma^{-1} + \left(\sum_{i=0}^{r-1}\kappa k_{i+1}s^i I_m\right)\right)\det(\Gamma) \\
&= \det(\Gamma)\sum_{j=0}^{m}\left(\sum_{1\leq i_1<\cdots<i_j\leq m}\operatorname{minor}\left(\Gamma^{-1};\{i_1,\ldots,i_j\}\right)\right)\kappa^{m-j}\left(s^{jr}\left(\sum_{i=0}^{r-1}k_{i+1}s^i\right)^{m-j}\right)\,.
\end{aligned}$$

Let $\{\gamma_1,\dots,\gamma_m\} = \operatorname{spec}(\Gamma) \subset \mathbb{C}$ and $J \in \mathbb{C}^{m\times m}$ be a Jordan canonical form of Γ^{-1}. Then Lemma 4.2 yields

$$\sum_{j=0}^{m}\left(\sum_{1\le i_1<\dots<i_j\le m} \operatorname{minor}\left(\Gamma^{-1};\{i_1,\dots,i_j\}\right)\right)\kappa^{m-j} = \det\left(\Gamma^{-1}+\kappa I_m\right) = \det\left(J+\kappa I_m\right) = \prod_{i=1}^{m}(\gamma_i^{-1}+\kappa)\,.$$

Since Γ and Γ^{-1} are positive definite, $\operatorname{spec}(\Gamma^{-1}) \subset \mathbb{C}_+$, thus $\left(\kappa \mapsto \widehat{p}(\kappa) := \prod_{i=1}^{m}(\gamma_i^{-1}+\kappa)\right) \in \mathbb{R}^H[\kappa]$, and since $\left(s \mapsto p(s) := \sum_{i=0}^{r-1} k_{i+1}\, s^i\right) \in \mathbb{R}^H[s]$ by assumption, setting

$$\delta := -\max\left\{\mu(p(\cdot)), \mu(\widehat{p}(\cdot))\right\} > 0\,,$$

and Lemma 4.1 yields

$$\exists\, \kappa^* > 0\ \forall\, \kappa > \kappa^*\ :\ \mu\left(s \mapsto \det\left(s^r I_m + \left(\sum_{i=0}^{r-1}\kappa k_{i+1} s^i\right)\Gamma\right)\right) < -\delta/2\,.$$

Thus, and since $\operatorname{spec}(Q) \subset \mathbb{C}_-$, one may choose, for all $\kappa > \kappa^*$, positive definite matrices $N_\zeta(\kappa) = N_\zeta(\kappa)^T \in \mathbb{R}^{mr\times mr}$ and $N_\eta = N_\eta^T \in \mathbb{R}^{(n-mr)\times(n-mr)}$ such that

$$\left.\begin{array}{rcl} N_\zeta(\kappa)\begin{bmatrix} 0_{(r-1)m\times m} & I_{(r-1)m} \\ -\kappa k_1\Gamma & \dots\ -\kappa k_r\Gamma\end{bmatrix} + \begin{bmatrix} 0_{(r-1)m\times m} & I_{(r-1)m} \\ -\kappa k_1\Gamma & \dots\ -\kappa k_r\Gamma\end{bmatrix}^T N_\zeta(\kappa) &=& -I_{mr}\,, \\ N_\eta Q + Q^T N_\eta &=& -I_{n-mr}\,. \end{array}\right\} \tag{4.8}$$

Moreover, by Lemma 4.3 for all $\kappa > \kappa^*$

$$\|N_\zeta(\kappa)\| \le C_0 := \frac{4e}{\delta}\left(1 + \frac{(2mr)!}{\delta^{2mr}}\right)\,.$$

Step 4: Design of a Lyapunov-function to show exponential stability.
The derivative of

$$t \mapsto V(t) \ :=\ \tfrac{1}{2}\,\zeta(t)^T N_\zeta(\kappa)\zeta(t) + \tfrac{1}{2}\,\eta(t)^T N_\eta \eta(t)$$

along the solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t) = A_{\Gamma,k,\kappa}\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t)$$

yields, for all $t \ge 0$, and omitting the argument t for brevity,

$$\begin{aligned}
\dot V(t) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\tfrac{1}{2}\,\zeta^T N_\zeta(\kappa)\zeta + \tfrac{1}{2}\,\eta^T N_\eta \eta\right) \\
&= \zeta^T N_\zeta(\kappa)\left(\kappa\begin{bmatrix} 0 & I_m & & \\ & \ddots & \ddots & \\ & & 0 & I_m \\ \frac{R_1}{\kappa^r} - \kappa k_1\Gamma & \cdots & \frac{R_{r-1}}{\kappa^2} - \kappa k_{r-1}\Gamma & \frac{R_r}{\kappa} - \kappa k_r\Gamma\end{bmatrix}\zeta + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \kappa^{-r+1}S\eta\end{bmatrix}\right) \\
&\quad + \eta^T N_\eta\left(Q\eta + P\zeta_1\right) \\
&\overset{(4.8)}{\le} -\tfrac{\kappa}{2}\|\zeta\|^2 + \kappa\zeta N_\zeta(\kappa)\begin{bmatrix} 0_{m(r-1)\times mr} \\ \frac{R_1}{\kappa^r}\ \cdots\ \frac{R_r}{\kappa}\end{bmatrix}\zeta + \frac{1}{\kappa^{r-1}}\|N_\zeta(\kappa)\|\,\|S\|\,\|\zeta\|\,\|\eta\| \\
&\quad -\tfrac{1}{2}\|\eta\|^2 + \|N_\eta P\|\,\|\eta\|\,\|\zeta_1\| \\
&\overset{\kappa\ge 1}{\le} -\tfrac{\kappa}{2}\|\zeta\|^2 + \|N_\zeta(\kappa)\|\,\|(R_1,\dots,R_r)\|\,\|\zeta\|^2 + \tfrac{1}{\kappa^{r-1}}\|N_\zeta(\kappa)\|\,\|S\|\,\|\zeta\|^2 \\
&\quad + \tfrac{1}{\kappa^{r-1}}\|N_\zeta(\kappa)\|\,\|S\|\,\|\eta\|^2 - \tfrac{1}{2}\|\eta\|^2 + \tfrac{1}{4}\|\eta\|^2 + 4\|N_\eta P\|\,\|\zeta_1\|^2 \\
&\le -\left(\tfrac{\kappa}{2} - \|N_\zeta(\kappa)\|\,\|(R_1,\dots,R_r)\| - \|N_\zeta(\kappa)\|\,\|S\| - 4\|N_\eta P\|\right)\|\zeta\|^2 \\
&\quad -\left(\tfrac{1}{4} - \tfrac{\|N_\zeta(\kappa)\|\,\|S\|}{\kappa^{r-1}}\right)\|\eta\|^2\,.
\end{aligned}$$

Setting

$$
\begin{aligned}
\kappa^{**} &:= \max\left\{\tfrac{1}{4} + 2(C_0\,\|(R_1,\dots,R_r)\| - C_0\,\|S\| - 4\|N_\eta P\|), (8\,C_0\,\|S\|)^{-r+1}, \kappa^*\right\}\,,\\
\alpha &:= \min\left\{\frac{1}{8\,C_0}, \frac{1}{8\|N_\eta\|}\right\}\,,
\end{aligned}
$$

yields, for all $t \geq 0$ and $\kappa > \kappa^{**}$,

$$
\dot V(t) \leq -\tfrac{1}{8}\|\zeta(t)\|^2 - \tfrac{1}{8}\|\eta(t)\|^2 \leq -\tfrac{1}{8\|N_\zeta(\kappa)\|}\zeta(t)^T N_\zeta(\kappa)\zeta(t) - \tfrac{1}{8\|N_\eta\|}\eta(t)^T N_\eta \eta(t) \leq -\alpha V(t)\,,
$$

whence, since the initial value $x(t_0) = x^0$ for (1.1) leads to the initial value $\begin{pmatrix}\zeta\\ \eta\end{pmatrix}(t_0) = U_\kappa V x^0$ for (4.7),

$$
\forall\, t \geq t_0\ \forall\, t_0 \geq 0\ :\ \left\|\begin{pmatrix}\zeta(t)\\ \eta(t)\end{pmatrix}\right\| \leq \exp\left(-\alpha(t-t_0)\right)\sqrt{\frac{\max\operatorname{spec}\left[\begin{smallmatrix}N_\zeta(\kappa) & 0\\ 0 & N_\eta\end{smallmatrix}\right]}{\min\operatorname{spec}\left[\begin{smallmatrix}N_\zeta(\kappa) & 0\\ 0 & N_\eta\end{smallmatrix}\right]}}\ \left\|\begin{pmatrix}\zeta(t_0)\\ \eta(t_0)\end{pmatrix}\right\|\,,
$$

which completes the proof of the theorem. □

Proof of Theorem 3.2. Without loss of generality suppose that (1.1) has ordered relative degree $r = (r_1,\dots,r_m) \in \mathbb{N}^{1\times m}$, otherwise note that for a linear system (A,B,C) of form (1.1) with vector relative degree $r = (r_1,\dots,r_m) \in \mathbb{N}^{1\times m}$ there exists a permutation matrix $P \in \mathbb{R}^{m\times m}$ such that the system (A,B,PC) has ordered vector relative degree $rP = (\widetilde r_1,\dots,\widetilde r_m)$. Thus it is sufficient to prove the statement of Theorem 3.2 for systems with ordered vector relative degree.

Step 1: Next it is shown that, for

$$
\begin{bmatrix}\Gamma_1^1 & \Gamma_2^1 & \dots & \Gamma_m^1\\ \Gamma_1^2 & \Gamma_2^2 & \dots & \Gamma_m^2\\ \vdots & \vdots & \ddots & \vdots\\ \Gamma_1^m & \Gamma_2^m & \dots & \Gamma_m^m\end{bmatrix} := \Gamma = \begin{bmatrix} c_{(n)}^1 A^{r_1-1}B\\ c_{(n)}^2 A^{r_2-1}B\\ \vdots\\ c_{(n)}^m A^{r_m-1}B\end{bmatrix}\,,
$$

$R_{i,k}^j \in \mathbb{R}$, for $i,j \in \{1,\dots,m\}$ and $k \in \{1,\dots,r_i\}$, $S^1,\dots,S^m \in \mathbb{R}^{1\times(n-r^s)}$, $P_1,\dots,P_m \in \mathbb{R}^{n-r^s}$ and $Q \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$, as in (2.1) and

$$
A_{\Gamma,k,\kappa} :=
$$

$$
\left[\begin{array}{c|c|c|c}
\kappa\begin{bmatrix} 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1\\ \frac{R_{1,1}^1}{\kappa^{r_1}} - \Gamma_1^1\kappa\,k_{1,1} & \dots & & \frac{R_{1,r_1}^1}{\kappa} - \Gamma_1^1\kappa\,k_{1,r_1}\end{bmatrix} & \dots & \kappa\begin{bmatrix} 0 & \dots & 0\\ \vdots & & \vdots\\ 0 & \dots & 0\\ \frac{R_{m,1}^1}{\kappa^{r_m}} - \Gamma_m^1\kappa\,k_{m,1} & \dots & \frac{R_{m,r_m}^1}{\kappa} - \Gamma_m^1\kappa\,k_{m,r_m}\end{bmatrix} & \kappa\begin{bmatrix}0\\ \vdots\\ 0\\ \frac{1}{\kappa^{r_m}}S^1\end{bmatrix}\\
\hline
\vdots & & \vdots & \vdots\\
\hline
\kappa\begin{bmatrix} 0 & \dots & 0\\ \vdots & & \vdots\\ 0 & \dots & 0\\ \frac{R_{1,1}^m}{\kappa^{r_1}} - \Gamma_1^m\kappa\,k_{1,1} & \dots & \frac{R_{1,r_1}^m}{\kappa} - \Gamma_1^m\kappa\,k_{1,r_1}\end{bmatrix} & \dots & \kappa\begin{bmatrix} 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1\\ \frac{R_{m,1}^m}{\kappa^{r_m}} - \Gamma_m^m\kappa\,k_{m,1} & \dots & & \frac{R_{m,r_m}^m}{\kappa} - \Gamma_m^m\kappa\,k_{m,r_m}\end{bmatrix} & \kappa\begin{bmatrix}0\\ \vdots\\ 0\\ \frac{1}{\kappa^{r_m}}S^m\end{bmatrix}\\
\hline
\begin{bmatrix}\kappa^{-r_1+r_m}P_1 & 0 & \dots & 0\end{bmatrix} & \dots & \begin{bmatrix}\kappa^{-r_m+r_m}P_m & 0 & \dots & 0\end{bmatrix} & Q
\end{array}\right],
$$

the closed-loop system (1.1), (3.2) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\zeta\\ \vartheta\end{pmatrix} = A_{\Gamma,k,\kappa}\begin{pmatrix}\zeta\\ \vartheta\end{pmatrix} \tag{4.9}$$

in the sense that

$$\exists\, W \in \mathbb{R}^{n\times n} \;:\; \begin{pmatrix}\zeta\\ \vartheta\end{pmatrix} = Wx\,.$$

By Proposition 2.2 there exists an invertible $V \in \mathbb{R}^{n\times n}$ such that the coordinate transformation $\binom{\xi}{\eta} = Vx$ converts (1.1) into normal form (2.1), (2.2). Thus the closed-loop system (1.1), (3.2) is equivalent to (2.1), (2.2), (3.2).

Split ξ as follows:

$$\xi = \begin{pmatrix}\xi^1\\ \vdots\\ \xi^m\end{pmatrix} \qquad \text{and} \qquad \xi^j = \begin{pmatrix}\xi_1^j\\ \vdots\\ \xi_{r_j}^j\end{pmatrix} \in \mathbb{R}^{r_j}\,,\; j \in \{1,\ldots,m\}\,.$$

For ease of notation define vectors

$$R_\mu^j := \left[R_{\mu,1}^j,\ldots,R_{\mu,r_\mu}^j\right] \in \mathbb{R}^{1\times r_\mu}\,,\quad j,\mu \in \{1,\ldots,m\}\,,$$

where, for $i \in \{1,\ldots,r_\mu\}$, $R_{\mu,i}^j$, is defined by (2.2).

Setting $\zeta_i^j = \kappa^{r_j-r_1-i+1}\xi_i^j$, for $j \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,r_j\}$, yields

$$\xi_i^j = \kappa^{-r_j+r_1+i-1}\zeta_i^j\,,\qquad \text{for } j \in \{1,\ldots,m\} \text{ and } i \in \{1,\ldots,r_j\}\,,$$

$$\xi^j = \begin{pmatrix}\xi_1^j\\ \vdots\\ \xi_{r_j}^j\end{pmatrix} = \begin{pmatrix}\kappa^{-r_j+r_1+1-1}\zeta_1^j\\ \vdots\\ \kappa^{-r_j+r_1+r_j-1}\zeta_{r_j}^j\end{pmatrix} = \begin{bmatrix}\kappa^{r_1-r_j} & 0 & \cdots & 0\\ 0 & \kappa^{r_1-r_j+1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \kappa^{r_1-1}\end{bmatrix}\underbrace{\begin{pmatrix}\zeta_1^j\\ \vdots\\ \zeta_{r_j}^j\end{pmatrix}}_{=:\zeta^j},$$

$$\text{for } j \in \{1,\ldots,m\}\,,$$

and

$$\begin{aligned}
\dot\zeta_i^j &= \kappa^{r_j-r_1-i+1}\dot\xi_i^j\\
&= \kappa^{r_j-r_1-i+1}\xi_{i+1}^j\\
&= \kappa^{r_j-r_1-i+1}\kappa^{r_1-r_j+i+1-1}\zeta_{i+1}^j = \kappa\zeta_{i+1}^j\,,\qquad \text{for } j \in \{1,\ldots,m\} \text{ and } i \in \{1,\ldots,r_j-1\},\\
\dot\zeta_{r_j}^j &= \kappa^{-r_1+1}\dot\xi_{r_j}^j\\
&= \kappa^{-r_1+1}\left[\sum_{\mu=1}^m R_\mu^j\xi^\mu + S^j\eta + c_{(n)}^j A^{r_j-1}Bu\right]\,,\qquad \text{for } j \in \{1,\ldots,m\}\,.
\end{aligned}$$

Thus by (3.2) and $y_j^{(i-1)} = \xi_i^j$, for all $j \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,r_j\}$, it follows that, for

$j \in \{1, \ldots, m\}$,

$$\begin{aligned}
\dot{\zeta}^j_{r_j} &= \kappa \left[\sum_{i=1}^{m} \kappa^{-r_1} R^j_i \begin{bmatrix} \kappa^{r_1 - r_i} & 0 & \ldots & 0 \\ 0 & \kappa^{r_1-r_i+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \kappa^{r_1-1} \end{bmatrix} \zeta^i + \kappa^{-r_1} S^j \eta \right. \\
&\qquad \left. - \kappa\, \kappa^{-r_1} c^j_{(n)} A^{r_j-1} B \begin{pmatrix} \sum_{i=0}^{r_1-1} \kappa^{r_1-i} k_{1,i+1}\, \xi^1_{i+1} \\ \vdots \\ \sum_{i=0}^{r_m-1} \kappa^{r_m-i} k_{m,i+1}\, \xi^m_{i+1} \end{pmatrix} \right] \\
&= \kappa \left[\sum_{i=1}^{m} \left[\kappa^{-r_i} R^j_{i,1}, \kappa^{-r_i+1} R^j_{i,2}, \ldots, \kappa^{-1} R^j_{i,r_i} \right] \zeta^i + \kappa^{-r_1} S^j \eta \right. \\
&\qquad \left. - \kappa\, \kappa^{-r_1} \left[\Gamma^j_1, \Gamma^j_2, \ldots, \Gamma^j_m \right] \begin{pmatrix} \sum_{i=0}^{r_1-1} \kappa^{r_1-i} \kappa^{r_1-r_1+i} k_{1,i+1}\, \zeta^1_{i+1} \\ \vdots \\ \sum_{i=0}^{r_m-1} \kappa^{r_m-i} \kappa^{r_1-r_m+i} k_{m,i+1}\, \zeta^m_{i+1} \end{pmatrix} \right] \\
&= \kappa \left[\sum_{i=1}^{m} \left(\left[\frac{R^j_{i,1}}{\kappa^{r_i}}, \frac{R^j_{i,2}}{\kappa^{r_i-1}}, \ldots, \frac{R^j_{i,r_i}}{\kappa} \right] - \Gamma^j_i \kappa \left[k_{i,1}, k_{i,2}, \ldots, k_{i,r_i} \right] \right) \zeta^i + \frac{1}{\kappa^{r_1}} S^j \eta \right].
\end{aligned}$$

Setting $\vartheta = \kappa^{r_m - r_1} \eta$ yields

$$\begin{aligned}
\dot{\vartheta} = \kappa^{r_m-r_1} \dot{\eta} &= \kappa^{r_m-r_1} \left(\sum_{i=1}^{m} P_i \xi^i_1 + Q\eta \right) \\
&= \kappa^{r_m-r_1} \left(\sum_{i=1}^{m} \kappa^{r_1-r_i} P_i \zeta^1_i + \kappa^{r_1-r_m} Q \vartheta \right) = \sum_{i=1}^{m} \kappa^{r_m-r_i} P_i \zeta^1_i + Q\vartheta \,.
\end{aligned}$$

Thus, in view of the coordinate transformation

$$\begin{aligned}
&\zeta^j_i = \kappa^{r_j - r_1 - i + 1} \xi^j_i \,, \qquad \text{for } j \in \{1, \ldots, m\}\,,\ i \in \{1, \ldots, r_j\}\,, \\
&\vartheta = \kappa^{r_m - r_1} \eta \,,
\end{aligned}$$

and setting $\zeta = \left[\zeta^1 / \zeta^2 / \ldots / \zeta^m \right]$ the closed-loop system (2.1), (2.2), (3.2) is equivalent to (4.9).

Step 2: For $r^s := \sum_{i=1}^{m} r_i$ and

$$\mathfrak{A}_{\Gamma,k,\kappa} := \left[\begin{array}{c|c|c|c}
\begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1 \\ -\kappa\, \Gamma^1_1 [k_{1,1} & \ldots & & k_{1,r_1}] \end{bmatrix} &
\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ -\kappa\, \Gamma^1_2 [k_{2,1} & \ldots & k_{2,r_2}] \end{bmatrix} & \ldots &
\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ -\kappa\, \Gamma^1_m [k_{m,1} & \ldots & k_{m,r_m}] \end{bmatrix} \\ \hline
\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ -\kappa\, \Gamma^2_1 [k_{1,1} & \ldots & k_{1,r_1}] \end{bmatrix} &
\begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1 \\ -\kappa\, \Gamma^2_2 [k_{2,1} & \ldots & & k_{2,r_2}] \end{bmatrix} & \ldots &
\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ -\kappa\, \Gamma^2_m [k_{m,1} & \ldots & k_{m,r_m}] \end{bmatrix} \\ \hline
\vdots & \vdots & & \vdots \\ \hline
\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ -\kappa\, \Gamma^m_1 [k_{1,1} & \ldots & k_{1,r_1}] \end{bmatrix} &
\begin{bmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ 0 & \ldots & 0 \\ -\kappa\, \Gamma^m_2 [k_{2,1} & \ldots & k_{2,r_2}] \end{bmatrix} & \ldots &
\begin{bmatrix} 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1 \\ -\kappa\, \Gamma^m_m [k_{m,1} & \ldots & & k_{m,r_m}] \end{bmatrix}
\end{array} \right].$$

it is shown that

$$\exists\, \delta > 0 \ \exists\, \kappa^* > 0 \ \forall\, \kappa > \kappa^* \ : \ \mu\left(\mathfrak{A}_{\Gamma,k,\kappa}\right) < -\delta/2\,. \tag{4.10}$$

Set, for $i = 1, \ldots, r_1$, $m_i := \#\left\{ r_j \,\middle|\, r_j \geq i, j \in \{1, \ldots, m\} \right\}$, the number of r_j's, $j \in \{1, \ldots, m\}$, such that $r_j \geq i$. Define the permutation matrices

$$\Pi_R := \left[e_1^{(r^s)}, e_{r_1+1}^{(r^s)}, \ldots, e^{(r^s)}_{\sum_{i=1}^{m_1-1} r_i+1} \,\middle|\, e_2^{(r^s)}, e_{r_1+2}^{(r^s)}, \ldots, e^{(r^s)}_{\sum_{i=1}^{m_2-1} r_i+2} \,\middle|\, \cdots \,\middle|\, e_{r_1}^{(r^s)}, e_{r_1+r_2}^{(r^s)}, \ldots, e^{(r^s)}_{\sum_{i=1}^{m_{r_1}-1} r_i+r_1} \right],$$

$$\Pi_L := \left[\begin{bmatrix} e^1_{(r^s)} \\ e^{r_1+1}_{(r^s)} \\ \vdots \\ e^{\sum_{i=1}^{m_2-1} r_i+1}_{(r^s)} \end{bmatrix} \Big/ \begin{bmatrix} e^2_{(r^s)} \\ e^{r_1+2}_{(r^s)} \\ \vdots \\ e^{\sum_{i=1}^{m_3-1} r_i+2}_{(r^s)} \end{bmatrix} \Big/ \cdots \Big/ \begin{bmatrix} e^{r_1-1}_{(r^s)} \\ e^{r_1+r_2-1}_{(r^s)} \\ \vdots \\ e^{\sum_{i=1}^{m_{r_1}-1} r_i+(r_1-1)}_{(r^s)} \end{bmatrix} \Big/ \begin{bmatrix} e^{r_1}_{(r^s)} \\ e^{r_1+r_2}_{(r^s)} \\ \vdots \\ e^{\sum_{i=1}^{m_1} r_i}_{(r^s)} \end{bmatrix} \right],$$

and

$$\begin{aligned} \Pi_D := \Bigg[& e_1^{(r^s)}, e_2^{(r^s)}, \ldots, e_{m_2}^{(r^s)} \,\Big|\, e_{m_1}^{(r^s)}, \ldots, e_{m_1+m_3}^{(r^s)} \,\Big|\, \cdots \,\Big|\, e^{(r^s)}_{\sum_{i=1}^{r_1-2} m_i+1}, \ldots, e^{(r^s)}_{\sum_{i=1}^{r_1-2} m_i+m_{r_1}} \,\Big| \\ & e^{(r^s)}_{r^s-m_{r_1}+1}, \ldots, e^{(r^s)}_{r^s-m_{r_1}+m_{r_1}} \,\Big| \\ & e^{(r^s)}_{\sum_{i=1}^{r_1-2} m_i+m_{r_1}+1}, \ldots, e^{(r^s)}_{\sum_{i=1}^{r_1-1} m_i} \,\Big|\, \cdots \,\Big|\, e^{(r^s)}_{\sum_{i=1}^{r_1-r_1} m_i+m_{r_1-(r_1-2)}+1}, \ldots, e^{(r^s)}_{\sum_{i=1}^{r_1-(r_1-1)} m_i} \Bigg] \end{aligned}$$

Then, for $s \in \mathbb{C}$,

$$\Pi_D \Pi_L \left(s I_{r^s} - \mathfrak{A}_{\Gamma,k,\kappa}\right) \Pi_R =$$

$$\left[\begin{array}{c|c|c|c|c||}
sI_{m_2} & \begin{bmatrix} -I_{m_3} \\ 0_{(m_2-m_3)\times m_3} \end{bmatrix} & 0_{m_2\times m_4} & \cdots & 0_{m_2\times m_{r_1}} \\ \hline
0_{m_3\times m_2} & sI_{m_3} & \begin{bmatrix} -I_{m_4} \\ 0_{(m_3-m_4)\times m_4} \end{bmatrix} & 0_{m_3\times m_5} & \cdots \\ \hline
\vdots & 0_{m_4\times m_3} & sI_{m_4} & \ddots & \ddots \\ \hline
 & \vdots & \ddots & \ddots & \begin{bmatrix} -I_{m_{r_1}} \\ 0_{(m_{r_1-1}-m_{r_1})\times m_{r_1}} \end{bmatrix} \\ \hline
0_{m_{r_1}\times m_2} & 0_{m_{r_1}\times m_3} & \cdots & 0_{m_{r_1}\times m_{r_1-1}} & sI_{m_{r_1}} \\ \hline\hline
\Gamma \left[\begin{array}{c} \begin{matrix} \kappa k_{1,1} & & 0 \\ & \ddots & \\ 0 & & \kappa k_{m_2,1} \end{matrix} \\ \hline 0_{(m_1-m_2)\times m_2} \end{array}\right] & \Gamma \left[\begin{array}{c} \begin{matrix} \kappa k_{1,2} & & 0 \\ & \ddots & \\ 0 & & \kappa k_{m_3,2} \end{matrix} \\ \hline 0_{(m_1-m_3)\times m_3} \end{array}\right] & \Gamma \left[\begin{array}{c} \begin{matrix} \kappa k_{1,3} & & 0 \\ & \ddots & \\ 0 & & \kappa k_{m_4,3} \end{matrix} \\ \hline 0_{(m_1-m_4)\times m_4} \end{array}\right] & \cdots & \Gamma \left[\begin{array}{c} \begin{matrix} \kappa k_{1,r_1-1} & & 0 \\ & \ddots & \\ 0 & & \kappa k_{m_{r_1},r_1-1} \end{matrix} \\ \hline 0_{(m_1-m_{r_1})\times m_{r_1}} \end{array}\right]
\end{array}\right.$$

$$\left.\begin{array}{c|c|c|c|c}
0_{m_2\times m_{r_1}} & \cdots & & \begin{bmatrix} 0_{m_3\times(m_2-m_3)} \\ -I_{m_2-m_3} \end{bmatrix} & 0_{m_2\times(m_1-m_2)} \\ \hline
\vdots & & \cdot^{\cdot^{\cdot}} & & \vdots \\ \hline
0_{m_{r_1-1}\times m_{r_1}} & \begin{bmatrix} 0_{m_{r_1}\times(m_{r_1-1}-m_{r_1})} \\ -I_{m_{r_1-1}-m_{r_1}} \end{bmatrix} & & \vdots & \\ \hline
\left[-I_{m_{r_1}}\right] & 0_{m_{r_1}\times(m_{r_1-1}-m_{r_1})} & \cdots & 0_{m_{r_1}\times(m_2-m_3)} & 0_{m_{r_1}\times(m_1-m_2)} \\ \hline\hline
\multicolumn{5}{c}{sI_{m_1} + \Gamma\, \underbrace{\operatorname{diag}\left(\kappa k_{1,r_1}, \ \ldots, \ \kappa k_{m_{r_1},r_1}, \kappa k_{m_{r_1}+1,r_1-1}, \ \ldots, \ \kappa k_{m_{r_1-1},r_1-1}, \ \ldots, \kappa k_{m_2+1,1}, \ \ldots, \ \kappa k_{m_1,1}\right)}_{=:\,\mathfrak{K}_{k,\kappa}}}
\end{array}\right], \tag{4.11}$$

and

$$
\begin{aligned}
&\det(\Pi_D \Pi_L \left(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\kappa}\right) \Pi_R) \\
&= \det(sI_{m_2}) \\
&\quad \cdot \det \left[\begin{array}{c|c|c||c}
sI_{m_3} & \begin{bmatrix} -I_{m_4} \\ 0_{(m_3-m_4)\times m_4} \end{bmatrix} & & 0_{m_3\times m_4} \; \begin{bmatrix} 0_{m_3\times(m_3-m_4)} \\ I_{m_3-m_4} \end{bmatrix} \; 0_{m_3\times(m_1-m_3)} \\
\hline
0_{\left(\sum\limits_{i=4}^{r_1} m_i\right)\times m_3} & & \cdots & \cdots \\
\hline\hline
\Gamma \left[\begin{array}{c} \begin{matrix} \kappa\, k_{1,2} + \frac{1}{s}\kappa\, k_{1,1} & & 0 \\ & \ddots & \\ 0 & & \kappa\, k_{m_3,2} + \frac{1}{s}\kappa\, k_{m_3,1} \end{matrix} \\ \hline 0_{(m_1-m_3)\times m_3} \end{array}\right] & \Gamma \left[\begin{array}{c} \begin{matrix} \kappa\, k_{1,3} & & 0 \\ & \ddots & \\ 0 & & \kappa\, k_{m_4,3} \end{matrix} \\ \hline 0_{(m_1-m_4)\times m_4} \end{array}\right] & & sI_{m_1} + \Gamma \mathfrak{K}_{k,\kappa} + \dfrac{\Gamma}{s} \left[\begin{array}{c|c} 0_{m_3\times(m_2-m_3)} & \\ \hline \begin{matrix} \kappa\, k_{m_3+1,1} & & 0 \\ & \ddots & \\ 0 & & \kappa\, k_{m_2,1} \end{matrix} & 0 \\ \hline 0_{(m_1-m_2)\times(m_2-m_3)} & \end{array}\right]
\end{array}\right] \\
&= \ldots \\
&= s^{\sum\limits_{i=2}^{r_1-1} m_i} \det(sI_{m_{r_1}}) \\
&\quad \cdot \det \Bigg[sI_{m_1} + \Gamma \operatorname{diag} \Bigg(\sum_{i=1}^{r_1} \kappa\, k_{1,i} s^{-(r_1-i)}, \ldots, \sum_{i=1}^{r_1} \kappa\, k_{m_{r_1},i} s^{-(r_1-i)}, \\
&\qquad \sum_{i=1}^{r_1-1} \kappa\, k_{m_{r_1}+1,i} s^{-(r_1-1-i)}, \ldots, \sum_{i=1}^{r_1-1} \kappa\, k_{m_{r_1-1},i} s^{-(r_1-1-i)}, \\
&\qquad \ldots, \\
&\qquad \sum_{i=1}^{1} \kappa\, k_{m_2+1,i} s^{-(1-i)}, \ldots, \sum_{i=1}^{1} \kappa\, k_{m_1,i} s^{-(1-i)} \Bigg) \Bigg] \\
&= \det \left[\operatorname{diag}\left(s^{r_1}, \ldots, s^{r_m}\right) + \Gamma \operatorname{diag}\left(\sum_{i=1}^{r_1} \kappa\, k_{1,i} s^{i-1}, \sum_{i=1}^{r_2} \kappa\, k_{2,i} s^{i-1}, \ldots, \sum_{i=1}^{r_{m_1}} \kappa\, k_{m_1,i} s^{i-1} \right) \right].
\end{aligned}
$$

Recall that $m = m_1$. Setting, for $j \in \{1, \ldots, m\}$, $k^j(s) := \sum_{i=1}^{r_j} k_{j,i} s^{i-1}$ an application of Lemma 4.2 leads to

$$
\begin{aligned}
&\det(\Gamma) \det(\Pi_D \Pi_L \left(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\kappa}\right) \Pi_R) \\
&= \det \begin{bmatrix}
s^{r_1} \left(\Gamma^{-1}\right)_{1,1} + \kappa\, k^1(s) & s^{r_2} \left(\Gamma^{-1}\right)_{1,2} & \cdots & s^{r_m} \left(\Gamma^{-1}\right)_{1,m} \\
s^{r_1} \left(\Gamma^{-1}\right)_{2,1} & s^{r_2} \left(\Gamma^{-1}\right)_{2,2} + \kappa\, k^2(s) & \ddots & \vdots \\
\vdots & \ddots & \ddots & s^{r_m} \left(\Gamma^{-1}\right)_{m-1,m} \\
s^{r_1} \left(\Gamma^{-1}\right)_{m,1} & \cdots & s^{r_m-1} \left(\Gamma^{-1}\right)_{m,m-1} & s^{r_m} \left(\Gamma^{-1}\right)_{m,m} + \kappa\, k^m(s)
\end{bmatrix} \\
&= \sum_{j=0}^{m} \left(\sum_{1\le i_1 < \cdots < i_j \le m} s^{\left(\sum_{l=1}^{j} r_{i_l}\right)} \operatorname{minor}\left(\Gamma^{-1};\, \{i_1, \ldots, i_j\}\right) \prod_{\substack{i\in\{1,\ldots,m\} \\ i \notin \{i_1,\ldots,i_j\}}} k^i(s) \right) \kappa^{m-j}. \qquad (4.12)
\end{aligned}
$$

Observe that for fixed $j = 0, \ldots, m$ every summand in (4.12) is a polynomial in $\mathbb{R}[s]$ with degree $r^s - m + j$. Moreover, for $j = 0$, the summand in (4.12) equals $p(s) := \prod_{i=1}^{m} k^i(s)$ which is the product of m Hurwitz polynomials and thus $p \in \mathbb{R}^H[s]$, hence

$$\exists\, \delta^* > 0 \;:\; \mu(p(\cdot)) < -\delta^*.$$

Let $\{\gamma_1, \ldots, \gamma_m\} = \operatorname{spec}(\Gamma) \subset \mathbb{C}$ and $J \in \mathbb{C}^{m\times m}$ be a Jordan canonical form of Γ^{-1}. Since $\det(\Gamma^{-1} + \kappa I_m) = \det(J + \kappa I_m) = \prod_{j=1}^{m} (\kappa + \gamma_j^{-1})$ and Γ and thus Γ^{-1} are positive definite, whence $\operatorname{spec}(\Gamma^{-1}) \subset \mathbb{C}_+$, Lemma 4.2 yields

$$\exists\, \delta^{**} > 0 \ : \ \mu\left(\kappa \mapsto \sum_{j=0}^{m}\left(\sum_{1\le i_1<\cdots<i_j\le m} \operatorname{minor}\left(\Gamma^{-1}\,;\,\{i_1,\ldots,i_j\}\right)\right)\kappa^{m-j}\right) < -\delta^{**}$$

Setting $\delta = \min\{\delta^*, \delta^{**}\}$ and since

$$\det(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\kappa}) = \underbrace{\det(\Pi_D)\det(\Pi_L)\det(\Pi_R)}_{\in\{-1,+1\}}\det(\Pi_D\Pi_L\,(sI_{r^s} - \mathfrak{A}_{\Gamma,k,\kappa})\,\Pi_R)\,, \qquad s \in \mathbb{C}\,,$$

Lemma 4.1 and (4.12) yield (4.10).

Step 3: It is shown that, for any initial value $x(t_0) = x^0$ for (1.1) or, equivalently, $\begin{pmatrix}\zeta\\ \vartheta\end{pmatrix}(t_0) = Wx^0$ for (4.9),

$$\begin{aligned}&\exists\,\alpha > 0\ \exists\,\kappa^{**} > 0\ \forall\,\kappa > \kappa^{**}\ \exists\,M = M(\kappa) > 0\ \forall\,t \ge t_0\ \forall\,t_0 \ge 0\ :\\ &\qquad\left\|\begin{pmatrix}\zeta(t)\\ \vartheta(t)\end{pmatrix}\right\| \le \exp\left(-\alpha(t-t_0)\right)M\left\|\begin{pmatrix}\zeta(t_0)\\ \vartheta(t_0)\end{pmatrix}\right\|\,. \end{aligned} \tag{4.13}$$

By Step 2 and Lemma 4.3 and since system (1.1) has stable zero dynamics, i.e. $\operatorname{spec}(Q) \subset \mathbb{C}_-$, one may choose symmetric positive definite matrices $N_\zeta(\kappa) = N_\zeta(\kappa)^T \in \mathbb{R}^{r^s\times r^s}$, for all $\kappa > \kappa^*$, and $N_\vartheta = N_\vartheta^T \in \mathbb{R}^{(n-r^s)\times(n-r^s)}$ such that

$$N_\zeta(\kappa)\,\mathfrak{A}_{\Gamma,k,\kappa} + \mathfrak{A}_{\Gamma,k,\kappa}^T\,N_\zeta(\kappa) = -I_{r^s}\,, \qquad N_\vartheta\,Q + Q^T\,N_\vartheta = -I_{n-r^s}\,, \tag{4.14}$$

and moreover

$$\forall\ \kappa > \kappa^*\ :\ \|N_\zeta(\kappa)\| \le C_0 := \frac{4e}{\delta}\left(1 + \frac{(2r^s)!}{\delta^{2r^s}}\right).$$

where $\zeta = \left[\zeta^1/\zeta^2/\ldots/\zeta^m\right]$.
Let $(\zeta^T, \vartheta^T)^T$ be an arbitrary solution of the closed system (2.1), (2.2), (3.2). Set

$$\mathfrak{R}_\kappa := \kappa\left[\begin{array}{c|c|c}\begin{bmatrix}0 & 1 & \ldots & 0\\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1\\ \frac{R^1_{1,1}}{\kappa^{r_1}} & \ldots & & \frac{R^1_{1,r_1}}{\kappa}\end{bmatrix} & \ldots & \begin{bmatrix}0 & \ldots & 0\\ \vdots & & \vdots\\ 0 & \ldots & 0\\ \frac{R^1_{m,1}}{\kappa^{r_m}} & \ldots & \frac{R^1_{m,r_m}}{\kappa}\end{bmatrix}\\ \hline \vdots & & \vdots\\ \hline \begin{bmatrix}0 & \ldots & 0\\ \vdots & & \vdots\\ 0 & \ldots & 0\\ \frac{R^m_{1,1}}{\kappa^{r_1}} & \ldots & \frac{R^m_{1,r_1}}{\kappa}\end{bmatrix} & \ldots & \begin{bmatrix}0 & 1 & \ldots & 0\\ \vdots & & \ddots & \\ 0 & \ldots & 0 & 1\\ \frac{R^m_{m,1}}{\kappa^{r_m}} & \ldots & & \frac{R^m_{m,r_m}}{\kappa}\end{bmatrix}\end{array}\right], \qquad \mathfrak{S}_\kappa := \kappa\left[\begin{array}{c}\begin{bmatrix}0_{1\times(n-r^s)}\\ \vdots\\ 0_{1\times(n-r^s)}\\ \frac{1}{\kappa^{r_m}}S^1\end{bmatrix}\\ \hline \vdots\\ \hline \begin{bmatrix}0_{1\times(n-r^s)}\\ \vdots\\ 0_{1\times(n-r^s)}\\ \frac{1}{\kappa^{r_m}}S^m\end{bmatrix}\end{array}\right]. \tag{4.15}$$

Then differentiation of

$$t \mapsto V(t) := \frac{1}{2}\zeta(t)^T N_\zeta(\kappa)\zeta(t) + \frac{1}{2}\vartheta(t)^T N_\vartheta\vartheta(t)$$

along $(\zeta^T, \vartheta^T)^T$ yields, for all $t \geq 0$, and omitting the argument t for brevity,

$$\begin{aligned}
\dot{V}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\zeta^T N_\zeta(\kappa)\zeta + \frac{1}{2}\vartheta^T N_\vartheta \vartheta\right) \\
&= \frac{1}{2}\left(\dot{\zeta}^T N_\zeta(\kappa)\zeta + \zeta^T N_\zeta(\kappa)\dot{\zeta}\right) + \frac{1}{2}\left(\dot{\vartheta}^T N_\vartheta \vartheta + \vartheta^T N_\vartheta \dot{\vartheta}\right) \\
&= \frac{1}{2}\left(\left[\kappa\,\mathfrak{A}_{\Gamma,k,\kappa}\zeta + \mathfrak{R}_\kappa \zeta + \mathfrak{S}_\kappa \vartheta\right]^T N_\zeta(\kappa)\zeta + \zeta^T N_\zeta(\kappa)\left[\kappa\,\mathfrak{A}_{\Gamma,k,\kappa}\zeta + \mathfrak{R}_\kappa \zeta + \mathfrak{S}_\kappa \vartheta\right]\right. \\
&\quad + \frac{1}{2}\left(\left(Q\vartheta + \sum_{j=1}^m \frac{P_j}{\kappa^{r_1 - r_m}}\zeta_1^j\right)^T N_\vartheta \vartheta + \vartheta^T N_\vartheta\left(Q\vartheta + \sum_{j=1}^m \frac{P_j}{\kappa^{r_1 - r_m}}\zeta_1^j\right)\right) \\
&= \frac{1}{2}\Big(\kappa\,\zeta^T\left(\mathfrak{A}_{\Gamma,k,\kappa}^T N_\zeta(\kappa) + N_\zeta(\kappa)\mathfrak{A}_{\Gamma,k,\kappa}\right)\zeta + \zeta^T \mathfrak{R}_\kappa^T N_\zeta(\kappa)\zeta + \zeta^T N_\zeta(\kappa)\mathfrak{R}_\kappa \zeta \\
&\qquad + \vartheta^T \mathfrak{S}_\kappa^T N_\zeta(\kappa)\zeta + \zeta^T N_\zeta(\kappa)\mathfrak{S}_\kappa \vartheta\Big) \\
&\quad + \frac{1}{2}\left(\vartheta^T\left(Q^T N_\vartheta + N_\vartheta Q\right)\vartheta + \sum_{j=1}^m\left[(\zeta_1^j)^T \frac{P_j^T}{\kappa^{r_1 - r_m}} N_\vartheta \vartheta + \vartheta^T N_\vartheta \frac{P_j}{\kappa^{r_1 - r_m}}\zeta_1^j\right]\right) \\
&\overset{(4.14)}{=} -\frac{1}{2}\kappa\,|\zeta|^2 + \zeta^T N_\zeta(\kappa)\mathfrak{R}_\kappa \zeta + \zeta^T N_\zeta(\kappa)\mathfrak{S}_\kappa \vartheta - \frac{1}{2}|\vartheta|^2 + \sum_{j=1}^m \vartheta^T N_\vartheta \frac{P_j}{\kappa^{r_1 - r_m}}\zeta_1^j\,,
\end{aligned}$$

and since $r_1 \geq \cdots \geq r_m \geq 1$ and $\kappa \geq 1$ it follows that

$$\begin{aligned}
\dot{V}(t) \leq &- \frac{1}{2}\kappa\,\|\zeta\|^2 + \|N_\zeta(\kappa)\|\,\|\mathfrak{R}_1\|\,\|\zeta\|^2 + \|N_\zeta(\kappa)\|\,\|\mathfrak{S}_1\|\,\|\zeta\|\,\|\vartheta\| \\
&- \frac{1}{2}\|\vartheta\|^2 + \sum_{j=1}^m \frac{\|N_\vartheta\|\,\|P_j\|}{\kappa^{r_1 - r_m}}\|\zeta_1^j\|\,\|\vartheta\| \\
\leq &- \frac{1}{2}\kappa\,\|\zeta\|^2 + \|N_\zeta(\kappa)\|\,\|\mathfrak{R}_1\|\,\|\zeta\|^2 + 2\|N_\zeta(\kappa)\|^2\|\mathfrak{S}_1\|^2\|\zeta\|^2 + \frac{1}{8}\|\vartheta\|^2 \\
&- \frac{1}{2}\|\vartheta\|^2 + \sum_{j=1}^m\left(2m\|N_\vartheta\|^2\|P_j\|^2\|\zeta_1^j\|^2 + \frac{1}{8m}\|\vartheta\|^2\right) \\
\leq &- \frac{1}{2}\kappa\,\|\zeta\|^2 + \|N_\zeta(\kappa)\|\,\|\mathfrak{R}_1\|\,\|\zeta\|^2 + 2\|N_\zeta(\kappa)\|^2\|\mathfrak{S}_1\|^2\|\zeta\|^2 + 2m\|N_\vartheta\|^2\sum_{j=1}^m\|P_j\|^2\|\zeta\|^2 \\
&- \frac{1}{2}\|\vartheta\|^2 + \frac{1}{8}\|\vartheta\|^2 + m\frac{1}{8m}\|\vartheta\|^2 \\
\leq &- \left(\frac{1}{2}\kappa - \|N_\zeta(\kappa)\|\,\|\mathfrak{R}_1\| - 2\|N_\zeta(\kappa)\|^2\|\mathfrak{S}_1\|^2 - 2m\|N_\vartheta\|^2\sum_{j=1}^m\|P_j\|^2\right)\|\zeta\|^2 - \frac{1}{4}\|\vartheta\|^2\,,
\end{aligned}$$

where $\mathfrak{R}_1$, $\mathfrak{S}_1$ are defined setting $\kappa = 1$ in (4.15). Setting

$$\begin{aligned}
\kappa^{**} &:= \max\left\{\frac{1}{4} + 2\left(C_0\|\mathfrak{R}_1\| - 2\,C_0^2\|\mathfrak{S}_1\|^2 - 2m\|N_\vartheta\|^2\sum_{j=1}^m\|P_j\|^2\right), \kappa^*\right\}\,, \\
\alpha &:= \min\left\{\frac{1}{8\,C_0}, \frac{1}{8\|N_\vartheta\|}\right\}\,, \\
M &:= \sqrt{\frac{\max\operatorname{spec}\begin{bmatrix} N_\zeta(\kappa) & 0 \\ 0 & N_\vartheta \end{bmatrix}}{\min\operatorname{spec}\begin{bmatrix} N_\zeta(\kappa) & 0 \\ 0 & N_\vartheta \end{bmatrix}}}
\end{aligned}$$

yields, for all $t \geq 0$ and all $\kappa > \kappa^{**}$,

$$\dot{V}(t) \leq -\frac{1}{8}\|\zeta(t)\|^2 - \frac{1}{8}\|\vartheta(t)\|^2 \leq -\frac{1}{8\|N_\zeta(\kappa)\|}\zeta(t)^T N_\zeta(\kappa)\zeta(t) - \frac{1}{8\|N_\vartheta\|}\vartheta(t)^T N_\vartheta \vartheta(t) \leq -\alpha V(t)\,,$$

hence (4.13) which shows the exponential stability of the closed-loop system and the proof is complete. □

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