Regularity of distributional differential algebraic equations

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Abstract

Differential algebraic equations (DAEs) of the form $E\dot{x} = Ax + f$ are considered. The solutions x and the inhomogeneities f are assumed to be distributions (generalized functions). As a new approach, distributional entries in the coefficient matrices E and A are allowed, in particular, this encompasses the case where the coefficient matrices are time-varying but not continuous. Since a multiplication for general distributions is not possible, the smaller space of piecewise-smooth distributions is introduced. A restriction can be defined for the space of piecewise-smooth distributions, this restriction is used to study DAEs with inconsistent initial values; basically, it is assumed that some past trajectory for x is given and the DAE is activated at some initial time. If this initial trajectory problem always has a unique solution, then the DAE is called regular. This generalizes the regularity for classical DAEs (i.e. a DAE with constant coefficients).

Key words: differential algebraic equations, distributional solutions, regularity

1 Introduction

1.1 Aims and main results of the paper

A differential algebraic equation (DAE) is an equation of the form

$$E\dot{x} = Ax + f$$

where E, A are in general rectangular matrices and f is some inhomogeneity. The aim of this paper is to develop a solution theory for *distributional* DAEs, i.e. distributions as introduced by Schwartz [19] are considered as solutions x, as inhomogeneities f and as entries of the coefficient matrices E and A. For

general distributions a multiplication within the space of distributions is not possible; therefore, a smaller space, the space of *piecewise-smooth distributions* $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, is introduced (Definition 1). It is shown that it is possible to define a multiplication within $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ (Theorem 4) and also a distributional restriction (Definition 8), the latter is used to formulate initial trajectory problems (Definition 14) which allows to study solutions with inconsistent initial values.
For the constant coefficient case, it is well known that *regularity* of the matrix pair (E, A) is an important concept with respect to existence and uniqueness of solutions; this concept is generalized for distributional DAEs (Definition 15). Necessary conditions for regularity (Theorem 17, Theorem 18) as well as sufficient conditions (Theorem 20, Theorem 23) are given. The sufficient conditions can be summarized in the condition that the matrix pair (E, A) can be put into a generalized Weierstraß form (Corollary 25).

1.2 Motivation for studying distributional DAEs

DAEs of the form $E\dot{x} = Ax + f$ arise for example in modelling electrical circuits, mechanical and chemical systems (see e.g. [13, I.1.3]), in particular, if these models are generated automatically. If the inhomogeneity is not continuous (for example, if it is generated by a switching controller) then, even for constant coefficients, a classical solution does in general not exist and one has to consider distributional solutions (see e.g. [13, Remark 2.32]).
The motivation to consider distributional entries in the coefficients follows from the need to study switched DAEs, which appear in case of possible structural changes in the system. For an overview on classical switched systems and for further motivation, see e.g. [15]. Inconsistent initial values can also be interpreted as a result of switching. Switching yields that the coefficient matrices E and A are not continuous. Equivalent system description and normal forms play an important role for the analysis of DAEs. Two canonical transformations which do not change the qualitative solution behaviour are, firstly, multiplication of $E\dot{x} = Ax + f$ with some invertible matrix S from the left and, secondly, a coordinate transformation $x = Tz$ for some invertible matrix T and with the new variable z. This yields the equivalent DAE

$$SET\dot{z} = (SAT - SET')z + Sf.$$

If E and A are not continuous, then it is reasonable to assume that S and T may also be discontinuous. Hence T' only makes sense in the distributional sense. This motivates distributional entries in the coefficient matrices. Furthermore, linear impulsive systems (see e.g. [14]) can be rewritten as a distributional ODE $\dot{x} = Ax + f$ with distributional entries in A. For further motivation see also the switched electrical circuit example in Section 4.

1.3 Results in the literature

Distributional solutions for linear DAEs were considered already in [1] and [21], mainly to deal with inconsistent initial values, but no general distributional solution theory was introduced, problems like evaluations of distributions at a certain point (which is needed to speak of initial values) were not addressed. A first rigorous distributional solution theory was given by Cobb in [2], he introduced "piecewise continuous distributions" which encompass piecewise-smooth distributions. However, the space of piecewise continuous distributions is not closed under differentiation, and since Cobb seems to have overlooked this fact, some of the results in [2] might need a reformulation. The space of "impulsive smooth distributions" as defined in [18] is a subspace of piecewise-smooth distributions, where jumps and Dirac impulses (and its derivatives) can only occur at time $t = 0$. Piecewise-smooth distributions were used as an underlying solution space for time-invariant higher order Rosenbrock systems in [8], "time-varying" topics like inconsistent initial values and switched systems were not addressed. There is no literature on DAEs with distributional coefficient matrices.

There have been numerous approaches to define a multiplication for distributions. König [11] enlarged the space of distributions to define a multiplication, Fuchssteiner [6] introduced the space of "almost bounded" distributions, see also [7]. This space is very similar, but not identical, to the space of piecewise-smooth distributions and he defined an associative multiplication which ensures that the product rule for differentiation is fulfilled. This non-commutative multiplication is identical to the multiplication defined in this paper for piecewise-smooth distributions, although the approach is very different. In [22] a commutative but non-associative multiplication was defined for another subspace of distributions. Finally, there are several textbooks on the topic of multiplications of distributions [3,9,17], but the results are either too general (for example results on non-associative multiplications) or too restrictive (for example results on commutative multiplictaions) for the purpose of this paper. In Remark 7 an additional literature review is carried out with the focus on the definition of the square of the Dirac-impulse.

1.4 Organisation of the paper and notation

The paper is organized as follows. In Section 2 the classical distribution theory is revised and the space of piecewise-smooth distributions is introduced. For piecewise-smooth distributions, the Fuchssteiner multiplication and a distributional restriction are defined. Some calculation rules are presented. In Section 3 regularity of a distributional DAE is defined, this is strongly related to solvability of a DAE and the uniqueness of solutions. Necessary and sufficient conditions for regularity are presented. Finally, in Section 4 a simple switched electrical circuit is studied to illustrate the developed distributional solution theory.

To improve readability all proofs are carried out in the Appendix.

The following notation is used throughout the paper. $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ are the natural numbers, integers and real number, respectively. The space of functions $\alpha : \mathbb{R} \to \mathbb{R}$ which are smooth (i.e. arbitrarily often differentiable) is denoted by $\mathcal{C}^\infty$. The restriction $f_M : \mathbb{R} \to \mathbb{R}$ of some function $f : \mathbb{R} \to \mathbb{R}$ on some set $M \subseteq \mathbb{R}$ is defined by $f_M := \mathbb{1}_M f$, where $\mathbb{1}_M : \mathbb{R} \to \{0,1\}$ is the indicator function of M (i.e. $\mathbb{1}_M(t) = 1$ if, and only if, $t \in M$), in particular, the restricted function f_M is still defined on the whole of $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called locally integrable if, and only if, f is (Lebesgue-)measurable and for every compact set $K \subseteq \mathbb{R}$ the (Lebesgue-)integral $\int_K |f|$ is finite. The space of distributions is $\mathbb{D}$ and the space of piecewise-smooth distribution is $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ (see the later definitions).

2 Distributions

2.1 Review of classical distribution theory

Basis knowledge of distribution theory as introduced by Schwartz [19] (see also textbooks like [10]) is assumed and is only summarized without proofs in the following paragraph.
The space of *test functions* $\mathcal{C}_0^\infty \subseteq \mathcal{C}^\infty$ is the set of all functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are *smooth* and which have *bounded support* (the support $\operatorname{supp} \varphi \subseteq \mathbb{R}$ of φ is the closure of $\{\ t \in \mathbb{R} \mid \varphi(t) \neq 0\ \}$). The space $\mathcal{C}_0^\infty$ is a topological vector space, and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^\infty$ converges to zero (in this topology) if, and only if, there exists a compact set $M \subseteq \mathbb{R}$ with $\operatorname{supp} \varphi_n \subseteq M$ for all $n \in \mathbb{N}$ and, for each $i \in \mathbb{N}$, the sequence of the i-th derivatives $\left(\varphi_n^{(i)}\right)_{n\in\mathbb{N}}$ converges uniformly to zero. The space of *distributions* $\mathbb{D}$ is the set of all *linear* and *continuous* operators $D : \mathcal{C}_0^\infty \to \mathbb{R}$. The *derivative* of a distribution $D \in \mathbb{D}$ is defined by $D'(\varphi) := -D(\varphi')$ for all test functions $\varphi \in \mathcal{C}_0^\infty$ and is itself a distribution. Every distribution $D \in \mathbb{D}$ has an *antiderivative* $H \in \mathbb{D}$, i.e. $H' = D$, and all antiderivatives of D only differ by a constant distribution (the constant distribution is given by $\varphi \mapsto c \int_{\mathbb{R}} \varphi$ for $c \in \mathbb{R}$). The space of locally integrable functions $\mathcal{L}_{1,\text{loc}}$ is injectively embedded into the space of distributions via the homomorphism

$$\mathcal{L}_{1,\text{loc}} \ni f \mapsto f_{\mathbb{D}} := \left(\varphi \mapsto \int_{\mathbb{R}} f\varphi\right) \in \mathbb{D}. \tag{1}$$

For differentiable functions f, the distributional derivative "equals" the standard derivative, i.e. $(f')_{\mathbb{D}} = (f_{\mathbb{D}})'$. Distributions induced by locally integrable functions via homomorphism (1) are called *regular distributions*. The well known *Dirac impulse* (also known as (Dirac-)δ-function) $\delta_t \in \mathbb{D}$ at some time $t \in \mathbb{R}$ is given by

$$\delta_t : \mathcal{C}_0^\infty \to \mathbb{R}, \varphi \mapsto \varphi(t)$$

and is the classical example for a distribution which is not regular. The *support* $\operatorname{supp} D \subseteq \mathbb{R}$ of a distribution $D \in \mathbb{D}$ is the complement of the largest open set on which D vanishes, i.e.

$$\operatorname{supp} D := \mathbb{R} \Big\backslash \bigcup \left\{ O \subseteq \mathbb{R} \;\middle|\; \begin{array}{l} O \text{ open and } \forall \varphi \in \mathcal{C}_0^\infty : \\ \operatorname{supp} \varphi \subseteq O \;\Rightarrow\; D(\varphi) = 0 \end{array} \right\}.$$

The support of the Dirac impulse δ_t at $t \in \mathbb{R}$ and of all its derivatives is $\{t\}$ and, conversely, the following implication holds for all $t \in \mathbb{R}$ and all $D \in \mathbb{D}$

$$\operatorname{supp} D = \{t\} \quad \Rightarrow \quad \exists n \in \mathbb{N}\ \exists a_0, a_1, \ldots, a_n \in \mathbb{R} : D = \sum_{i=0}^{n} a_i \delta_t^{(i)} \tag{2}$$

and the represantation is unique, i.e. $\sum_{i=0}^n a_i \delta_t^{(i)} = \sum_{i=0}^n b_i \delta_t^{(i)}$ if, and only if, $a_i = b_i$, $i = 1, \ldots, n$. Furthermore, one can show that the only regular distribution whose support has Lebesgue measure zero (for example any countable set) is the zero distribution or in other words the support of nontrivial regular distributions is essential. Distributions can be *multiplied by smooth functions*, i.e. for all $\alpha \in \mathcal{C}^\infty$ the product αD given by $(\alpha D)(\varphi) := D(\alpha\varphi)$, $\varphi \in \mathcal{C}_0^\infty$, is again a distribution. It is easy to see, that the multiplication rule for the derivative holds, i.e.

$$\forall \alpha \in \mathcal{C}^\infty\ \forall D \in \mathbb{D} : \quad (\alpha D)' = \alpha' D + \alpha D' \tag{3}$$

A simple consequence of [12, Folg. 3.24] is the following property for all $\alpha \in \mathcal{C}^\infty$ and $D \in \mathbb{D}$

$$\Big[\forall i \in \mathbb{N}\ \forall t \in \operatorname{supp} D : \ \alpha^{(i)}(t) = 0\Big] \quad \Rightarrow \quad \alpha D = 0. \tag{4}$$

Note that in property (4) it is not assumed that $\operatorname{supp} \alpha \cap \operatorname{supp} D = \emptyset$.
Convergence of a sequence of distributions is defined "pointwise", i.e. a sequences $(D_n)_{n\in\mathbb{N}} \in \mathbb{D}^{\mathbb{N}}$ converges to $D \in \mathbb{D}$ if, and only if, $D_n(\varphi) \to D(\varphi)$ for all test functions $\varphi \in \mathcal{C}_0^\infty$. The space $\mathbb{D}$ is closed with respect to this convergence, i.e. if for a sequences $(D_n)_{n\in\mathbb{D}}$ of distributions the pointwise limit exists then this limit is a distribution or, more formally,

$$\Big[\forall \varphi \in \mathcal{C}_0^\infty : \lim_{n\to\infty} D_n(\varphi) \in \mathbb{R}\Big] \quad \Rightarrow \quad \lim D_n := (\varphi \mapsto \lim_{n\to\infty} D_n(\varphi)) \in \mathbb{D}. \tag{5}$$

Furthermore, for all sequences $(D_n)_{n\in\mathbb{N}}$ of distributions,

$$\lim_{n\to\infty} D_n = D \in \mathbb{D} \quad \Rightarrow \quad \forall i \in \mathbb{N} : \lim_{n\to\infty} D_n^{(i)} = D^{(i)}. \tag{6}$$

Definition 1 (Piecewise-smooth functions and distributions) *Let the space of* piecewise-smooth functions *be given by*

$$\mathcal{C}^\infty_{\text{pw}} := \left\{ \alpha = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})}\alpha_i \;\middle|\; \begin{array}{l} \{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}\ \textit{locally finite,} \\ (\alpha_i)_{i\in\mathbb{Z}} \in (\mathcal{C}^\infty)^{\mathbb{Z}} \end{array} \right\}.$$

The space of piecewise-smooth distributions is defined as

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}},\ T \subset \mathbb{R}\ \textit{locally finite,} \\ \forall\, t \in T : D_t \in \mathbb{D} \wedge \operatorname{supp} D_t \subseteq \{t\} \end{array} \right\}$$

For a piecewise-smooth distribution $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ with representation $D = f_{\mathbb{D}} + \sum_{t\in T} D_t$ the left *and* right sided evaluation at $t \in \mathbb{R}$ *is defined by*

$$D(t+) := \lim_{\varepsilon\searrow 0} f(t+\varepsilon), \quad D(t-) := \lim_{\varepsilon\searrow 0} f(t-\varepsilon)$$

The impulsive part at $t \in \mathbb{R}$ *of the above D is defined by*

$$D[t] := \begin{cases} D_t, & t \in T \\ 0, & \textit{otherwise} \end{cases}$$

and the impulsive part *of D is defined by*

$$D[\cdot] := \sum_{t\in T} D[t] = \sum_{t\in T} D_t.$$

Finally, $D_{\text{reg}} := f_{\mathbb{D}} = D - D[\cdot]$ is called the regular part *of D.*

It is easy to show that the representation of piecewise-smooth distributions is unique, i.e. two piecewise-smooth distributions $D_1, D_2 \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ with corresponding representation $D_k = f^k_{\mathbb{D}} + \sum_{t\in T^k} D^k_t$, $k = 1,2$, are equal if, and only if, $f^1 = f^2$, $\forall t \in T^1 \cap T^2 : D^1_t = D^2_t$, $\forall t \in T^1 \setminus T_2 : D^1_t = 0$ and $\forall t \in T^2 \setminus T_1 : D^2_t = 0$. Hence the definition of left and right sided evaluation and of the impulsive part are well defined. It is important to notice the condition that the set T in the definition of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ is *locally finite*, i.e. any intersection with some compact set is finite. If this condition is not fulfilled, it is, on the one hand, not true in general that the infinite sum of distribution with point support (as in the definition of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$) exists and, on the other hand, there are some locally infinite sums which define a distribution, but these distributions might have undesirable properties (see Remark 9).
An important motivation for the introductions of distributions as generalized functions is the property that every distribution has a derivative within the space of distributions. This property is preserved for the smaller space of piecewise-smooth distributions as the following proposition shows.

Proposition 2 (Derivative of piecewise-smooth distributions) *Let* $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ *and* $f_{\mathbb{D}} = D_{\text{reg}}$ *with* $f = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} f_i$ *for some locally finite set* $\{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}$ *and some smooth* $f_i \in \mathcal{C}^\infty$, $i \in \mathbb{Z}$. *Then*

$$D' = \left(\sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} f_i{}'\right)_{\mathbb{D}} + \sum_{i\in\mathbb{Z}} \Big(D(t_i+) - D(t_i-)\Big)\delta_{t_i} + D[\cdot]'. \tag{7}$$

In particular

$$D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \quad \Rightarrow \quad D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}.$$

Piecewise-smooth distributions also have antiderivatives which are piecewise-smooth distributions again, furthermore it is possible to make the antiderivative unique.

Proposition 3 (Unique distributional antiderivative) *For* $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ *and* $t_0 \in \mathbb{R}$ *there exists a unique distributional antiderivative*

$$H = \int_{t_0} D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$$

with $H' = D$ *and* $H(t_0-) = 0$.

It is possible to define a multiplication for piecewise-smooth distributions as stated in the next theorem. This multiplication is a generalization of the multiplication of functions but it is not commutative anymore.

Theorem 4 (Multiplication of piecewise-smooth distributions) *For* $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ *and* $t \in \mathbb{R}$ *let*

$$\begin{aligned} \delta_t D &:= D(t-)\delta_t, \quad \forall n \in \mathbb{N}:\ \delta_t^{(n+1)} D := \Big(\delta_t^{(n)} D\Big)' - \delta_t^{(n)} D', \\ D\delta_t &:= D(t+)\delta_t, \quad \forall n \in \mathbb{N}:\ D\delta_t^{(n+1)} := \Big(D\delta_t^{(n)}\Big)' - D'\delta_t^{(n)}. \end{aligned} \tag{8}$$

Let $F, G \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ *with representation* $F = f_{\mathbb{D}} + \sum_{t\in T_F} F[t]$ *and* $G = g_{\mathbb{D}} + \sum_{t\in T_G} G[t]$ *as in Definition 1. The product of* F *and* G *is defined by, using* (2) *and* (8),

$$FG := (fg)_{\mathbb{D}} + \sum_{t\in T_F} F[t]G_{\text{reg}} + \sum_{t\in T_G} F_{\text{reg}}G[t] = (fg)_{\mathbb{D}} + F[\cdot]G_{\text{reg}} + F_{\text{reg}}G[\cdot]. \tag{9}$$

The multiplication of piecewise-smooth distributions has the following properties, $F, G, H \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, $f, g \in \mathcal{C}^\infty_{\text{pw}}$:

(i) $FG \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$,
(ii) $f_{\mathbb{D}} g_{\mathbb{D}} = (fg)_{\mathbb{D}}$,
(iii) $F(GH) = (FG)H$, $(F+G)H = FH + GH$, $F(G+H) = FG + FH$,
(iv) $(FG)' = F'G + FG'$,

(v) $\operatorname{supp}(FG) \subseteq \operatorname{supp} F \cap \operatorname{supp} G$,

Remark 5 *Fuchssteiner [6] (see also [7]) studied the space of* almost bounded distributions*, which is very similar (but not equal) to the space of piecewise-smooth functions. He showed that the multiplication from Theorem 4 is the only one which fulfils the following conditions:*

(M1) $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ *is an associative differential algebra,*
(M2) $\forall f, g \in \mathcal{C}^\infty_{\mathrm{pw}}: \quad (fg)_{\mathbb{D}} = f_{\mathbb{D}} g_{\mathbb{D}}$,
(M3) $\forall t \in \mathbb{R}: \quad \mathbb{1}_{[t,\infty)_{\mathbb{D}}} \delta_t = \delta_t$.

In fact, Fuchssteiner gave a description of all multiplications fulfilling (M1) and (M2). These multiplications are parametrized by a set $M \subseteq \mathbb{R}$ *and fulfil*

$$\mathbb{1}_{[t,\infty)_{\mathbb{D}}} \delta_t = \begin{cases} \delta_t, & t \in M \\ 0, & t \in \mathbb{R} \backslash M \end{cases}$$

Since piecewise-smooth distributions are introduced in view of an applications to DAEs the condition (M3) is assumed, which then uniquely defines a multiplication on $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$*, namely the multiplication defined in Theorem 4. Hence this multiplication might be called* Fuchssteiner multiplication.

Remark 6 *From the recursive definition* (8) *an explicit representation can be derived easily,* $t \in \mathbb{R}$, $n \in \mathbb{N}$, $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$*:*

$$\begin{aligned} \delta_t^{(n)} D &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{(i)}(t-) \delta_t^{(n-i)}, \\ D \delta_t^{(n)} &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} D^{(i)}(t+) \delta_t^{(n-i)}. \end{aligned} \tag{10}$$

Remark 7 (Square of Dirac impulse) *It follows from the defintion of the multiplication that* $F[\cdot]G[\cdot] = 0$ *for all* $F, G \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ *and, in particular for* $\delta := \delta_0$,

$$\delta^2 = 0.$$

It is interesting to compare the different approaches in the literature with respect to the square of the Dirac impulse: In [23] it is claimed that it is impossible to define this square[1]*. A similar result is obtained in [22, Thm. 3.9], however, in the proof it is shown that the square of the Dirac impulse, if it exists, must be zero which contradicts the assumptions made in that paper. In [16] the equation* $\delta^2 - \frac{1}{\pi^2}\left(\frac{1}{x}\right)^2 = -\frac{1}{\pi^2}\frac{1}{x^2}$ *is established, where the left hand side is considered as a "single entity", this is motivated by quantum mechanics where* δ^2 *appears only in this context. The square of the Dirac impulse is well*

[1] [23, 3.IV]: "Im besonderen ist es nicht möglich, das Quadrat der δ-Funktion δ^2 zu bilden."

defined in [11], but only in a generalized space of distributions and it is shown that δ^2 is not a classical distribution. In [5] a commutative multiplication for a subspace of distributions is defined and there the square of the Dirac-impulse is zero.

The following definition introduces the restriction of piecewise-smooth distributions, which is a generalization of the restriction for functions defined by $f_M := \mathbb{1}_M f$ for some $f : \mathbb{R} \to \mathbb{R}$ and some $M \subseteq \mathbb{R}$. The distributional restriction is necessary to study *inconsistent initial values* for DAEs, because inconsistent initial values mean that the actual DAE is not valid in the past, hence it must be possible to formulate mathematically that the DAE (with distributional solutions) holds only on the time interval $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$ (see Definition 14).

Definition 8 *Distributional restriction Let $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ has the representation $D = f_{\mathbb{D}} + \sum_{t\in T} D_t$ as in Definition 1 and let $M \subseteq \mathbb{R}$ be a locally finite union of intervals (i.e. any compact set only intersects with finitely many intervals), then the restriction $D_M \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ of the piecewise-smooth distribution D to the set M is*

$$D_M := (f_M)_{\mathbb{D}} + \sum_{t\in T\cap M} D_t.$$

Clearly, in the above definition, the assumption that $M \subseteq \mathbb{R}$ is the locally finite union of intervals (LFUI) ensures that $f_M \in \mathcal{C}^\infty_{\mathrm{pw}}$. Furthermore, $T \cap M$ is a locally finite set (even for arbitrary sets $M \subseteq \mathbb{R}$), hence the above restriction is well defined. It is easy to see that the restriction from Definition 8 has the following properties:

(R1) The distributional restriction is a mapping

$$\{\ M \subseteq \mathbb{R} \mid M \text{ is a LFUI }\} \times \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}, \quad (M, D) \mapsto D_M$$

which is for each fixed $M \subseteq \mathbb{R}$ a projection, i.e. $D \mapsto D_M$ is linear and idempotent.

(R2) For $f \in \mathcal{C}^\infty_{\mathrm{pw}}$ and a LFUI $M \subseteq \mathbb{R}$ the distributional restriction fulfils

$$(f_M)_{\mathbb{D}} = (f_{\mathbb{D}})_M,$$

i.e. it is a generalization of restrictions of functions.

(R3) The restriction property for trivial cases is fulfilled, i.e. for all test functions $\varphi \in \mathcal{C}^\infty_0$, for all distributions $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ and and for all LFUIs $M \subseteq \mathbb{R}$ the following two implications hold:

$$\begin{aligned} \operatorname{supp}\varphi \subseteq M \quad &\Rightarrow \quad D_M(\varphi) = D(\varphi), \\ \operatorname{supp}\varphi \cap M = \emptyset \quad &\Rightarrow \quad D_M(\varphi) = 0. \end{aligned}$$

(R4) For any pairwise disjoint family of LFUIs $(M_i)_{i\in\mathbb{N}}$ with $M := \bigcup_{i\in\mathbb{N}} M_i$ a LFUI and any $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ the restriction fulfils

$$D_M = \sum_{i\in\mathbb{N}} D_{M_i},$$

in particular,

$$D_{M_1\cup M_2} = D_{M_1} + D_{M_2}.$$

Furthermore, for any disjoint LFUI sets $M_1, M_2 \subseteq \mathbb{R}$ the restriction fulfils

$$(D_{M_1})_{M_2} = 0.$$

Note that it is crucial that only piecewise-smooth distributions are considered as the following remark shows.

Remark 9 *For general distributions it is* not possible *to define a restriction with the properties (R1)-(R4). As an example consider the following (well defined!) distribution*

$$D = \sum_{n\in\mathbb{N}} d_n \delta_{d_n}, \quad d_n := \frac{(-1)^n}{n+1}, \; n \in \mathbb{N}.$$

The restriction to the interval $(0,\infty)$ *should then be*

$$D_{(0,\infty)} = \sum_{k\in\mathbb{N}} \frac{1}{2k} \delta_{\frac{1}{2k}},$$

but it is easy to see, that there exist test functions for which the infinite sum does not converge, hence the restriction is not defined.

2.3 Calculation rules for piecewise-smooth distributions

In this subsection some calculation rules for the restriction, multiplication and differentiation of piecewise-smooth distributions are given. These will be needed in later parts of this work and are also of general interest.

Proposition 10 (Multiplication and restriction) *Let* $F, G \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ *and* $s, t \in \mathbb{R} \cup \{\pm\infty\}$ *with* $s \le t$*, then, for any* $\varepsilon > 0$*,*

$$\begin{aligned}
(FG)_{(s,t)} &= F_{(s,t)} G_{(s,t)}, \\
(FG)_{[s,t)} &= F_{[s,t)} G_{[s,t)} + F[s] G_{(s-\varepsilon,s)}, \\
(FG)_{(s,t]} &= F_{(s,t]} G_{(s,t]} + F_{(t,t+\varepsilon)} G[t], \\
(FG)_{[s,t]} &= F_{[s,t]} G_{[s,t]} + F_{(t,t+\varepsilon)} G[t] + F[s] G_{(s-\varepsilon,s)},
\end{aligned}$$

where $F[\pm\infty] = G[\pm\infty] = 0$.

Proposition 11 (Restrictions and derivatives) *For all* $-\infty \leq s \leq t \leq \infty$ *and* $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$,

$$\begin{aligned}
\left(D_{[s,t)}\right)' &= (D')_{[s,t)} + D(s-)\delta_s - D(t-)\delta_t,\\
\left(D_{(s,t)}\right)' &= (D')_{(s,t)} + D(s+)\delta_s - D(t-)\delta_t,\\
\left(D_{(s,t]}\right)' &= (D')_{(s,t]} + D(s+)\delta_s - D(t+)\delta_t,\\
\left(D_{[s,t]}\right)' &= (D')_{[s,t]} + D(s-)\delta_s - D(t+)\delta_t,
\end{aligned}$$

where $\delta_{\pm\infty} = 0$.

The last part of this subsection considers matrices with piecewise-smoothly distributional entries and under which condition these matrices are invertible and how the inverse looks like.

Definition 12 (Multiplication and invertibility of piecewise-smooth matrices) *For two matrices* $P \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times m}$, $Q \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m\times p}$, $n, m, p \in \mathbb{N}$, *with piecewise-smoothly distributional entries the matrix product is defined in the standard way, i.e., for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, p$,

$$(PQ)_{ij} = \sum_{k=1}^{m} P_{ik} Q_{kj},$$

where M_{ij} *denotes the* (i, j)*-entry of some matrix* M. *A square matrix* $M \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$, $n \in \mathbb{N}$, *is called* invertible (over $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$) *if, and only if, there exists a matrix* $M^{-1} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ *such that*

$$MM^{-1} = M^{-1}M = I,$$

where $I \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ *is the (distributional) identity matrix given by*

$$I_{ij} = \begin{cases} (\mathbb{1}_{\mathbb{R}})_{\mathbb{D}}, & i = j \\ 0, & i \neq j \end{cases}.$$

Note that no notational distinction between the matrices $I \in \mathbb{R}^{n\times n}$, $I \in (\mathcal{C}^\infty_{\mathrm{pw}})^{n\times n}$, *and* $I \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ *is made.*

Proposition 13 (Invertibility of piecewise-smooth matrices) *Consider a piecewise-smoothly distributional matrix* $M \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$, $n \in \mathbb{N}$, *with* M_{reg} *induced by* $M^{\mathrm{reg}} \in (\mathcal{C}^\infty_{\mathrm{pw}})^{n\times n}$, *i.e.* ${M^{\mathrm{reg}}}_{\mathbb{D}} = M_{\mathrm{reg}}$. *Then* M *is invertible if, and only if,* M^{reg} *is invertible over* $\mathcal{C}^\infty_{\mathrm{pw}}$, *i.e. there exists* $P \in (\mathcal{C}^\infty_{\mathrm{pw}})^{n\times n}$ *with* $M^{\mathrm{reg}}(t)P(t) = P(t)M^{\mathrm{reg}}(t) = I$ *for all* $t \in \mathbb{R}$.
If M *is invertible, then the inverse is given by*

$$M^{-1} = M_{\mathrm{reg}}^{-1} - M_{\mathrm{reg}}^{-1} M[\cdot] M_{\mathrm{reg}}^{-1}, \quad \text{where } M_{\mathrm{reg}}^{-1} := \left((M^{\mathrm{reg}})^{-1}\right)_{\mathbb{D}}.$$

Note that for a matrix $M^{\text{reg}} \in (\mathcal{C}^\infty_{\text{pw}})^{n\times n}$ the condition $\det M^{\text{reg}}(t) \neq 0$ for all $t \in \mathbb{R}$ is *not sufficient* for invertibility over $\mathcal{C}^\infty_{\text{pw}}$. Consider for example the 1×1 matrix M^{reg} given by $M^{\text{reg}}(t) = t$ on $(-\infty, 0)$ and $M^{\text{reg}}(t) = 1$ on $[0, \infty)$ whose determinant is non-zero everywhere, but the inverse is $(M^{\text{reg}})^{-1}(t) = 1/t$ on $(-\infty, 0)$ and $(M^{\text{reg}})^{-1}(t) = 1$ on $[0, \infty)$ which is not a piecewise-smooth function because $t \mapsto 1/t$ is not part of a globally smooth function as required by Definition 1.

3 Regularity of distributional DAEs

In this section, distributional DAEs of the form

$$E\dot{x} = Ax + f, \tag{11}$$

where $E, A \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{m\times n}$, $m, n \in \mathbb{N}$, $f \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$ and $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ are considered. Note that $\dot{x} := x'$ is just used for traditional reasons.

3.1 Definition of DAE-regularity

Definition 14 (Initial trajectory problem (ITP)) *Consider the distributional DAE* (11)*, let $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and $t_0 \in \mathbb{R}$, then $x \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is called a* solution of the initial trajectory problem (ITP) (11) *with initial trajectory x^0 and initial time t_0 if, and only if,*

$$\begin{aligned} x_{(-\infty,t_0)} &= x^0_{(-\infty,t_0)}, \\ (E\dot{x})_{[t_0,\infty)} &= (Ax + f)_{[t_0,\infty)}. \end{aligned}$$

Now ITPs are used to define regularity of the matrix pair (E, A) in (11), in short, (E, A) is called regular if, and only if, every ITP is uniquely solvable. Note that the notion "regularity" is already used for distributions, therefore in the following the notion "DAE-regularity" is used to distinguish it from the distributional regularity. However, if the context is clear just "regularity" will be used.

Definition 15 (DAE-regularity of (E, A)) *Consider the distributional DAE* (11)*. The matrix pair (E, A) is called* DAE-regular *if, and only if, for all inhomogeneities $f \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^m$, for all initial trajectories $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ and for all initial times $t_0 \in \mathbb{R}$ the corresponding ITP has a unique solution.*

Before formulating necessary and sufficient conditions for DAE-regularity, it is shown that regularity is invariant with respect to a certain system equivalence.

Proposition 16 (Regularity and system equivalence) *Let* $S \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m\times m}$ *and* $T \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ *both be invertible over* $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ *and let* (E, A) *from* (11) *be DAE-regular. Then* $\left(\widetilde{E}, \widetilde{A}\right) := (SET, SAT - SET')$ *is also DAE-regular. In fact,* $\widetilde{x}$ *is a solution of the ITP* (11) *with* $\left(\widetilde{E}, \widetilde{A}\right)$ *and inhomogeneity* $\widetilde{f} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^m$*, initial trajectory* $\widetilde{x}^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ *and initial time* $t_0 \in \mathbb{R}$ *if, and only if,* $x = T\widetilde{x}$ *is the solution of the ITP* (11) *with initial trajectory* $x^0 = T\widetilde{x}^0$*, initial time* t_0 *and inhomogeneity*

$$f := S^{-1}\widetilde{f}_{[t_0,\infty)} - S^{-1}[t_0]\left(\widetilde{A}\widetilde{x}^0 - \widetilde{E}\dot{\widetilde{x}}^0\right)_{(-\infty,t_0)}.$$

3.2 Necessary conditions for DAE-regularity

Theorem 17 ($n = m$) *Consider the distributional DAE* (11)*. If* (E, A) *is DAE-regular then* $n = m$*.*

This result is quite intuitive because, if $n > m$ then there are more variables than equations, so the system is underdetermined, hence uniqueness of solutions can not be expected. If $n < m$ then there are more equations than variables, hence the system is overdetermined and there exists inhomogeneities for which solutions do not exist. The next necessary conditions for regularity are more of technical nature.

Theorem 18 (Derivative and impulse array) *Consider the distributional DAE* (11) *with* (E, A) *DAE-regular.*

(i) Define the derivative array of order $p \in \mathbb{N}$ *as a block matrix*

$$\mathcal{M}^p \in \left((\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}\right)^{p+1\times p+2}$$

with, for $i = 1, \ldots, p+1$, $j = 1, \ldots, p+2$,

$$(\mathcal{M}^p)_{i,j} = \binom{i-1}{j-2}E^{(i-j+1)} - \binom{i-1}{j-1}A^{(i-j)},$$

with the convention that $\binom{0}{0} = 1$ *and* $\binom{n}{-k} = \binom{n}{n+k} = 0$ *for* $k > 0$, $n \in \mathbb{N}$*, i.e.*

$$\mathcal{M}^p = \begin{bmatrix} -A & E & & & & \\ -A' & E' - A & E & & & \\ -A'' & E'' - 2A' & 2E' - A & E & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ -A^{(p)} & E^{(p)} - pA^{(p-1)} & pE^{(p-1)} - \binom{p}{2}A^{(p-2)} & \binom{p}{2}E^{(p-2)} - \binom{p}{3}A^{(p-3)} & \cdots & E \end{bmatrix}$$

Then $M^p(t+)$ *and* $M^p(t-)$ *have full row rank for all* $p \in \mathbb{N}$ *and* $t \in \mathbb{N}$*.*

(ii) Define the impulse array of order (p,q), $p,q \in \mathbb{N}$, *as a block matrix*

$$\mathcal{N}^{p,q} \in \left((\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}\right)^{p+1\times q+1}$$

with, for $i = 1,\dots,p+1$, $j = 1,\dots,q+1$,

$$(\mathcal{N}^{p,q})_{i,j} = (-1)^{j-i}\left(\binom{j-1}{i-1}E^{(j-i)} + \binom{j-2}{i-1}A^{(j-i-1)}\right),$$

with the convention that $\binom{-1}{k} = 0$ *for all* $k \in \mathbb{N}$, *i.e.*

$$\mathcal{N}^{p,q} = \begin{bmatrix} E & -(E'+A) & E''+A' & \cdots & (-1)^q(E^{(q)}+A^{(q-1)}) \\ & -E & 2E'+A & \cdots & (-1)^{q-1}(qE^{(q-1)}+(q-1)A^{(q-2)}) \\ & & \ddots & & \vdots \\ & & & (-1)^pE \cdots & (-1)^{p-q}\left(\binom{q}{p}E^{q-p} + \binom{q-1}{p}A^{(q-p-1)}\right) \end{bmatrix}$$

Then for all $p \in \mathbb{N}$ *there exists* $q \in \mathbb{N}$ *such that* $\mathcal{N}^{p,q}(t+)$ *has full row rank for all* $t \in \mathbb{R}$.

Remark 19 *Applying Theorem 18 to the time-invariant case both conditions reduces to the simple condition that all matrices*

$$\begin{bmatrix} -A & E & & & & \\ & -A & E & & & \\ & & -A & E & & \\ & & & \ddots & \ddots & \\ & & & & -A & E \end{bmatrix}$$

have full row rank. Actually, this condition is equivalent to classical regularity of time-invariant DAEs [26].

3.3 Sufficient conditions for DAE-regularity

Theorem 20 (Concatination and additional impulses) *Consider a family of distributional DAEs* (11) *with the corresponding matrix pairs* (E_i, A_i), $i \in \mathbb{Z}$.

(i) If $(E_0,A_0),(E_1,A_1)$ *are DAE-regular, then*

$$(E,A) := \left(E_{0(-\infty,t_1)} + E_{1[t_1,\infty)}, A_{0(-\infty,t_1)} + A_{1[t_1,\infty)}\right)$$

is also DAE-regular for all $t_1 \in \mathbb{R}$.

(ii) *If (E_i, A_i), $i \in \mathbb{Z}$, is DAE-regular and $\{\ t_i \in \mathbb{R} \ \mid i \in \mathbb{Z}\ \}$ is a locally finite set, then*

$$(E, A) := \left(\sum_{i\in\mathbb{Z}} E_{i[t_i,t_{i+1})}, \sum_{i\in\mathbb{Z}} A_{i[t_i,t_{i+1})} \right)$$

is also DAE-regular.

(iii) *If (E_0, A_0) is DAE-regular, then*

$$(E, A) := (E_0 + E_1[t], A_0 + A_1[t])$$

is also DAE-regular for all $t \in \mathbb{R}$.

(iv) *If (E_0, A_0) is DAE-regular, then*

$$(E, A) := (E_0 + E_1[\cdot], A_0 + A_1[\cdot])$$

is also DAE-regular.

Corollary 21 *Consider the distributional DAE* (11)*, then (E, A) is DAE-regular if, and only if, $(E_{\text{reg}}, A_{\text{reg}})$ is DAE-regular.*

Remark 22 *The Corollary 21 does* not *state that the impulses in E and A have no influence on the solutions, in fact, the proof of the Theorem 20 reveals that the impulsive parts of E and A are preserved in an altered inhomogeneity. In general, the presence of Dirac impulses and its derivatives in E and A yield solutions which might depend also on the derivatives of the initial trajectory.*

Finally, some special distributional DAEs, namely distributional ordinary differential equations (ODEs) and pure distributional DAEs are studied and it turns out that these are DAE-regular.

Theorem 23 (Regularity of ODEs and pure DAEs) *Consider the distributional DAE* (11)*. If $(E, A) = (I, A)$ or $(E, A) = (N, I)$, where $N \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n\times n}$ is such that N_{reg} is a strictly lower triangular matrix, then (E, A) is DAE-regular.*

Remark 24 (Distributional ODEs) *The solution behaviour of a distributional ODE $\dot{x} = Ax + f$ differs significantly from the solution behaviour of a classical ODE. Firstly, the solution of the ITP can depend on derivatives of the initial trajectory, so the "dimension" of the solution space can be larger than the size of the system. Secondly, it can be shown that for the free homogeneouos distributional ODE $\dot{x} = Ax$, there exists, analogously as in the classical case, a fundamental solution $\Phi_{t_0} \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n\times n}$, with $t_0 \in \mathbb{R}$ such that $A_{(-\infty,t_0)}[\cdot] = 0$, i.e. every solution has the form $x = \Phi_{t_0} x_0$ and $x(t_0-) = x_0 \in \mathbb{R}^n$. However, different to the classical case, the fundamental solution need not to be an invertible matrix. As an example, consider the distributional ODE $\dot{x} = -\delta_0 x$ where all solutions are given by $x = \mathbb{1}_{(-\infty,0)_\mathbb{D}} x_0$ for $x_0 \in \mathbb{R}$.*

Finally the sufficient conditions can be summarized in the following way.

Corollary 25 (Generalized Weierstraß form) *Consider the distributional DAE* (11)*. If there exist invertible matrices* $S, T \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$*, a locally finite set* $\{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}$*, a family of matrices* $J_i \in (\mathcal{C}^\infty_{\mathrm{pw}})^{n_i \times n_i}$*,* $i \in \mathbb{Z}$*,* $0 \leq n_i \leq n$ *and a family of strictly lower triangular matrices* $N_i \in (\mathcal{C}^\infty_{\mathrm{pw}})^{(n-n_i)\times(n-n_i)}$*,* $i \in \mathbb{Z}$ *such that*

$$((SET)_{\mathrm{reg}}, (SAT - SET')_{\mathrm{reg}}) = \left(\sum_{i\in\mathbb{Z}} \begin{bmatrix} I & \\ & N_{i\mathbb{D}} \end{bmatrix}_{[t_i,t_{i+1})}, \sum_{i\in\mathbb{Z}} \begin{bmatrix} J_{i\mathbb{D}} & \\ & I \end{bmatrix}_{[t_i,t_{i+1})} \right),$$

then (E, A) *is DAE-regular.*

Remark 26 *For time-invariant DAEs, i.e.* $E, A \in \mathbb{R}^{n\times n}$*, the previous results show that DAE-regularity is identical to the classical regularity, defined by the condition* $\det(\lambda E - A) \in \mathbb{R}[\lambda]\backslash\{0\}$*.*

4 A simple electrical circuit example with distributional solutions

Consider the simple circuit shown in Figure 4. In the circuit, $C > 0$ is the

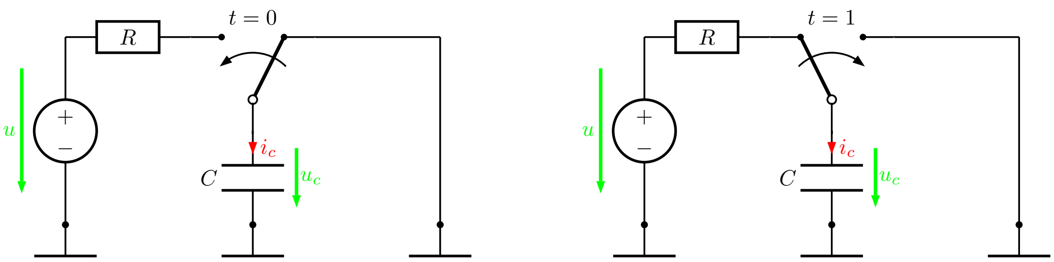

Fig. 1. A simple example circuit with a switch.

capacity of the capacitor, $R > 0$ is the resistance of the resistor and $u : \mathbb{R} \to \mathbb{R}$ is the input voltage. The state variables are i_c and u_c which are the current through the capacitor and the voltage over the capacitor, respectively. Before time $t = 0$, the switch is on the right side, i.e. the capacitor is bypassed. At $t = 0$, the switch moves to the left and the capacitor starts charging. After some time, the switch is moved back and the capacitor is bypassed again. The corresponding DAE reads as

$$E \begin{pmatrix} u_c' \\ i_c' \end{pmatrix} = A \begin{pmatrix} u_c \\ i_c \end{pmatrix} + f$$

where

$$E = \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix}, \quad A(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \begin{bmatrix} 0 & 1 \\ 1 & R \end{bmatrix}, & t \in [0,1) \end{cases}$$

and

$$f(t) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \begin{pmatrix} 0 \\ -u(t) \end{pmatrix}, & t \in [0,1). \end{cases}$$

Under the assumption that the input voltage satisfies $u \equiv 1$, the solution for the voltage u_c over the capacitor is given uniquely by

$$u_c(t) = \begin{cases} 0, & t \in \mathbb{R}\backslash[0,1), \\ 1 - e^{\frac{-t}{RC}}, & t \in [0,1). \end{cases}$$

Independently of the switch, the current i_c must fulfil the equation

$$Cu_c' = i_c.$$

Since the solution of u_c has a jump, there is no classical solution for i_c. However, if one allows for distributional solutions the equations are solvable and the (distributional) current i_c is given by

$$i_c = {i_c^{\text{reg}}}_{\mathbb{D}} + \underbrace{\left(e^{\frac{-1}{RC}} - 1\right)}_{=u_c(1+)-u_c(1-)} \delta_1,$$

where i_c^{reg} is the regular part of the distribution given by

$$i_c^{\text{reg}}(t) = \begin{cases} 0, & t \in \mathbb{R}\backslash[0,1), \\ \frac{1}{R}e^{\frac{-t}{RC}}, & t \in [0,1) \end{cases}$$

and δ_1 is the Dirac impulse at $t = 1$.
Another way to find a solution is by transforming the DAE (locally) into the so called Weierstraß normal form [24] (see also [13, Thm. 2.12]) via

$$S(t) = \begin{cases} \begin{bmatrix} 0 & 1 \\ \frac{1}{C} & 0 \end{bmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \begin{bmatrix} 1 & \frac{-1}{R} \\ 0 & \frac{1}{R} \end{bmatrix}, & t \in [0,1), \end{cases} \qquad T(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & C \end{bmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \begin{bmatrix} \frac{1}{C} & 0 \\ \frac{-1}{RC} & 1 \end{bmatrix}, & t \in [0,1). \end{cases}$$

The resulting DAE with $(\widetilde{E},\widetilde{A}) = (SET, SAT - SET')$ is then given by

$$\widetilde{E}(t) = \begin{cases} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & t \in [0,1) \end{cases}$$

and

$$\widetilde{A}^{\text{reg}}(t) = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \\ \begin{bmatrix} \frac{-1}{RC} & 0 \\ 0 & 1 \end{bmatrix}, & t \in [0,1), \end{cases} \qquad \widetilde{A} = \widetilde{A}^{\text{reg}}_{\mathbb{D}} + \begin{bmatrix} 0 & 0 \\ \frac{1-C}{C} & 0 \end{bmatrix} \delta_1.$$

Note that $\tilde{A}$ now contains a Dirac impulse. Actually, the transformed DAE is not much simpler, but in general a DAE in Weierstraß form is easier to solve, here the transformation is just done to illustrate that impulses can occur in the coefficient matrices. The unique solution of $\widetilde{E}\dot{z} = \widetilde{A}z + Sf$ is given by (again the input signal u is assumed to be constant and equal to one)

$$z^{\text{reg}}(t) = \begin{cases} \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & t \in \mathbb{R}\backslash[0,1), \\ \\ \begin{pmatrix} C\left(1 - e^{\frac{-t}{RC}}\right) \\ \frac{1}{R} \end{pmatrix}, & t \in [0,1), \end{cases} \qquad z = z^{\text{reg}}_{\mathbb{D}} + \begin{pmatrix} 0 \\ e^{\frac{-1}{RC}} - 1 \end{pmatrix} \delta_1.$$

Now it is easy to verify that Tz is equal to the solution found above.

5 Conclusion

To study time-varying DAEs of the form $E\dot{x} = Ax + f$ with jumps in the coefficients, the space of piecewise-smooth distributions was introduced as a solution space. With this space it is also possible to allow for distributional entries in the coefficient matrices. The well known concept of regularity for classical DAEs (i.e. DAEs with constant coefficients) was generalized, necessary and sufficient conditions were given for the regularity of matrix pairs (E, A) with piecewise-smoothly distributional entries. It seems that even for the classical time-varying case (i.e. E and A are smooth matrices) some of the conditions are new, in particular there is no general definition of regularity for time-varying DAEs. The presented framework is particularly suitable for

studying switched DAEs, for example Theorem 20(ii) states that switching between regular systems yields a new regular system, and in general, without the presented framework it seems difficult to study switched DAEs at all.
Although regular DAEs play an important role, there are cases were non-regular DAEs also arise in applications, for example rectangular descriptions of systems. It seems that the "behavioral approach", surveyed in [25], in combination with piecewise-smooth distributions as solutions will be a fruitful future research topic. Furthermore, for regular distributional DAEs, questions of control theory can be addressed and since the solution space as well as the control signal space are larger (they can include Dirac impulses) new methods and results are likely. Finally, the proposed distributional framework can be used to study reliability of linear networks, for example it is possible to study the situation that the failure of one component (which results in a new system description) induces impulsive solutions, which might destroy the system in reality.

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A Appendix: Proofs

A.1 Proof of Proposition 2

By (3) it is, for every $i \in \mathbb{Z}$,

$$\begin{aligned}
\Big(\mathbb{1}_{[t_i,t_{i+1})} f_i\Big)_{\mathbb{D}}{}' &= \Big(f_i \Big(\mathbb{1}_{[t_i,t_{i+1})}\Big)_{\mathbb{D}}\Big)' \\
&= f_i{}' \Big(\mathbb{1}_{[t_i,t_{i+1})}\Big)_{\mathbb{D}} + f_i \Big(\mathbb{1}_{[t_i,\infty)} - \mathbb{1}_{[t_{i+1},\infty)}\Big)_{\mathbb{D}}{}' \\
&= (\mathbb{1}_{[t_i,t_{i+1})} f_i{}')_{\mathbb{D}} + f_i \delta_{t_i} - f_i \delta_{t_{i+1}} \\
&= (\mathbb{1}_{[t_i,t_{i+1})} f_i{}')_{\mathbb{D}} + f_i(t_i) \delta_{t_i} - f_i(t_{i+1}) \delta_{t_{i+1}}.
\end{aligned}$$

Now (7) follows from $f_i(t_i) - f_{i-1}(t_i) = D(t_i+) - D(t_i-)$. Finally, (2) implies that $D[\cdot]'$ is again a locally finite sum of distributions with point support, hence $D' \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$. qed

A.2 Proof of Proposition 3

As already mentioned, every distribution $D \in \mathbb{D}$ has a distributional antiderivative and all antiderivatives only differ by a constant. It is first shown, that every distributional antiderivative H of a piecewise-smooth distribution $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is a piecewise-smooth distribution. Consider the representation $D = f_{\mathbb{D}} + \sum_{t\in T} D_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ as in Definition 1. Let $g : \mathbb{R} \to \mathbb{R}$ be a antiderivative of f, then $g \in \mathcal{C}^\infty_{\mathrm{pw}}$. For a fixed $t \in T$ and by (2), D_t can be written as

$$D_t = \sum_{i=0}^{n_t} a_t^i \delta_t^{(i)},$$

where $n_t \in \mathbb{N}$ and $a_t^0, \dots, a_t^{n_t} \in \mathbb{R}$. Clearly, one antiderivative of D_t is given by

$$a_t^0 \Big(\mathbb{1}_{[t,\infty)}\Big)_{\mathbb{D}} + \sum_{i=1}^{n_t} a_t^i \delta_t^{(i-1)}.$$

Now let

$$h = g + \sum_{t\in T} a_t^0 \mathbb{1}_{[t,\infty)} \in \mathcal{C}^\infty_{\mathrm{pw}}$$

and, for $t \in T$,

$$\widetilde{D}_t = \sum_{i=1}^{n_t} a_t^i \delta_t^{(i-1)},$$

then $H_1 = h_{\mathbb{D}} + \sum_{t\in T} \widetilde{D}_t \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ is a distributional antiderivative of D. Since all other antiderivatives only differ by a constant, all antiderivatives of D are piecewise-smooth distributions. Let

$$H = H_1 - H_1(t_0-)\mathbb{1}_{\mathbb{R}\mathbb{D}}$$

then H is the only distributional antiderivative with the property $H(t_0-) = 0$.

qed

A.3 Proof of Theorem 4

First observe that in (9) locally finiteness of T_F and T_G together with (5) ensures that indeed $\sum_{t\in T_F} F[t]g_{\mathbb{D}} = F[\cdot]g_{\mathbb{D}}$ and $\sum_{t\in T_G} f_{\mathbb{D}}G[t] = f_{\mathbb{D}}G[\cdot]$.

(i) Clearly, $fg \in \mathcal{C}^\infty_{\mathrm{pw}}$, hence it remains to show that $\sum_{t\in T_F} F[t]g_{\mathbb{D}}$ and $\sum_{t\in T_G} f_{\mathbb{D}}G[t]$ are piecewise-smooth distributions. From the definition it follows that

$$\begin{aligned} \forall t \in T_F: &\quad \operatorname{supp}(F[t]g_{\mathbb{D}}) \subseteq \{t\} \\ \forall t \in T_G: &\quad \operatorname{supp}(f_{\mathbb{D}}G[t]) \subseteq \{t\} \end{aligned}$$

which shows that $FG \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$.

(ii) This directly follows from the definition since $f_{\mathbb{D}}[\cdot] = 0 = g_{\mathbb{D}}[\cdot]$.

(iii) Simple calculations show the validity of these properties.

(iv) This part of the proof consists of four steps.

Step 1: It is shown that $(fg)_{\mathbb{D}}{}' = f_{\mathbb{D}}{}' g_{\mathbb{D}} + f_{\mathbb{D}} g_{\mathbb{D}}{}'$.

Let $f = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} f_i$ and $g = \sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} g_i$ with $f_i, g_i \in \mathcal{C}^\infty$, $i \in \mathbb{Z}$, and locally finite $\{ \ t_i \in \mathbb{R} \mid i \in \mathbb{Z} \ \}$. Note that both representations use the same intervals, but this is no restriction of generality. Then, by (7),

$$(fg)_{\mathbb{D}}{}' = \left(\sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} (f_i g_i)' \right)_{\mathbb{D}} + \sum_{i\in\mathbb{Z}} \Big(f(t_i+) g(t_i+) - f(t_i-) g(t_i-) \Big) \delta_{t_i},$$

and

$$\begin{aligned} f_{\mathbb{D}}{}' g_{\mathbb{D}} &= \left(\sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} f_i{}' g_i \right)_{\mathbb{D}} + \sum_{i\in\mathbb{Z}} \Big(f(t_i+) - f(t_i-) \Big) \delta_{t_i} g_{\mathbb{D}}, \\ f_{\mathbb{D}} g_{\mathbb{D}}{}' &= \left(\sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} f_i g_i{}' \right)_{\mathbb{D}} + \sum_{i\in\mathbb{Z}} f_{\mathbb{D}} \Big(g(t_i+) - g(t_i-) \Big) \delta_{t_i}. \end{aligned}$$

Since $(f_i g_i)' = f_i{}' g_i + f_i g_i{}'$, $\delta_{t_i} g_{\mathbb{D}} = g(t_i-) \delta_{t_i}$, and $f_{\mathbb{D}} \delta_{t_i} = f(t_i+) \delta_{t_i}$ for all $i \in \mathbb{Z}$, the assertion of Step 1 is shown.

Step 2: It is shown that $(F[t] g_{\mathbb{D}})' = F[t]' g_{\mathbb{D}} + F[t] g_{\mathbb{D}}{}'$ for all $t \in T_F$.

Since $F[t]$ is by (2) a finite sum of a Dirac impulse and its derivatives, it suffices to consider the case $F[t] = \delta_t^{(n)}$ for some $n \in \mathbb{N}$. Now the assertion follows directly from (8).

Step 3: It is shown that $(f_{\mathbb{D}} G[t])' = f_{\mathbb{D}}{}' G[t] + f G[t]'$ for all $t \in T_G$.

As in Step 2 it suffices to consider $G[t] = \delta_t^{(n)}$ for some $n \in \mathbb{N}$. Now the assertion follows again from (8).

Step 4: $(FG)' = F'G + FG'$ is shown.

Since T_F and T_G are locally finite it follows from Step 2 and 3 that

$$\begin{aligned} \Big(F[\cdot] g_{\mathbb{D}} \Big)' &= F[\cdot]' g_{\mathbb{D}} + F[\cdot] g_{\mathbb{D}}{}' \\ \Big(f_{\mathbb{D}} G[\cdot] \Big)' &= f_{\mathbb{D}}{}' G[\cdot] + f_{\mathbb{D}} G[\cdot]' \end{aligned}$$

Expanding the products yields

$$(FG)' = (fg)_{\mathbb{D}}{}' + (F[\cdot] g_{\mathbb{D}})' + (f_{\mathbb{D}} G[\cdot])'$$

and

$$\begin{aligned} F'G + FG' =& (f_{\mathbb{D}})' g_{\mathbb{D}} + (f_{\mathbb{D}})' G[\cdot] + F[\cdot]' g_{\mathbb{D}} + F[\cdot]' G[\cdot] \\ &+ f_{\mathbb{D}} g_{\mathbb{D}}{}' + f_{\mathbb{D}} G[\cdot] + F[\cdot] g_{\mathbb{D}}{}' + F[\cdot] G[\cdot]'. \end{aligned}$$

Hence

$$(FG)' - (F'G + FG') = (fg)_{\mathbb{D}}{}' - (f_{\mathbb{D}}{}' g_{\mathbb{D}} + f_{\mathbb{D}} g_{\mathbb{D}}{}') - F[\cdot]' G[\cdot] - F[\cdot] G[\cdot]'.$$

The regular parts of $F[\cdot]'$ and $G[\cdot]'$ are zero, hence $F[\cdot]'G[\cdot] = 0$ and $F[\cdot]G[\cdot]' = 0$. Using the equality from Step 1 now yields the assertion.

(v) This follows from

$$\begin{aligned}
\operatorname{supp} FG &= \operatorname{supp}\Big(f_{\mathbb{D}}g_{\mathbb{D}} + f_{\mathbb{D}}G[\cdot] + F[\cdot]g_{\mathbb{D}}\Big) \\
&\subseteq (\operatorname{supp} f_{\mathbb{D}} \cap \operatorname{supp} g_{\mathbb{D}}) \cup (\operatorname{supp} f_{\mathbb{D}} \cap \operatorname{supp} G[\cdot]) \\
&\quad \cup (\operatorname{supp} F[\cdot] \cap \operatorname{supp} g_{\mathbb{D}}) \cup (\operatorname{supp} F[\cdot] \cap \operatorname{supp} G[\cdot]) \\
&= (\operatorname{supp} f_{\mathbb{D}} \cup \operatorname{supp} F[\cdot]) \cap (\operatorname{supp} g_{\mathbb{D}} \cup \operatorname{supp} G[\cdot]) \\
&= \operatorname{supp} F \cap \operatorname{supp} G.
\end{aligned}$$

qed

A.4 Proof of Proposition 10

Let $M \subseteq \mathbb{R}$ be one of the four intervals with boundaries s and t, then by linearity of the restriction

$$(FG)_M = (F_{\text{reg}}G_{\text{reg}})_M + (F_{\text{reg}}G[\cdot])_M + (F[\cdot]G_{\text{reg}})_M.$$

First observe that $(F_{\text{reg}}G_{\text{reg}})_M = (F_{\text{reg}})_M(G_{\text{reg}})_M$. Furthermore,

$$(F_{\text{reg}}G[\cdot])_M = ((F_{\text{reg}})_M G[\cdot]_M\Big)_M + \Big((F_{\text{reg}})_{\mathbb{R}\setminus M} G[\cdot]_M\Big)_M + \Big(F_{\text{reg}}G[\cdot]_{\mathbb{R}\setminus M})\Big)_M,$$

where the term $\Big(F_{\text{reg}}G[\cdot]_{\mathbb{R}\setminus M})\Big)_M$ is zero, because $F_{\text{reg}}G[\cdot]_{\mathbb{R}\setminus M}$ is a distribution with zero regular part and whose support is a locally finite set contained in $\mathbb{R}\setminus M$, hence the restriction to M is zero by definition. Since the support of $(F_{\text{reg}})_M G[\cdot]_M$ is a locally finite set and is contained within M the outer restriction does not change it. Finally, the support of $(F_{\text{reg}})_{\mathbb{R}\setminus M}G[\cdot]_M$ is also a locally finite set and is contained in $\{s,t\}$, hence, if $s < t$,

$$(F_{\text{reg}}G[\cdot])_M = (F_{\text{reg}})_M G[\cdot]_M + (F_{\text{reg}})_{\mathbb{R}\setminus M}\Big(G[s] + G[t]\Big)_M$$

Analogously,

$$(F[\cdot]G_{\text{reg}})_M = F[\cdot]_M (G_{\text{reg}})_M + \Big(F[s] + F[t]\Big)_M (G_{\text{reg}})_{\mathbb{R}\setminus M}.$$

Now let $M = (s,t)$, then $\Big(G[s] + G[t]\Big)_M = 0 = \Big(F[s] + F[t]\Big)_M$, hence the assertion is shown in this case. For $M = [s,t)$ it is $\Big(G[s] + G[t]\Big)_M = G[s]$ and $\Big(F[s] + F[t]\Big)_M = F[s]$. From (10) it follows that the term $(F_{\text{reg}})_{\mathbb{R}\setminus M}G[s]$ depends only on the value $\Big((F_{\text{reg}})_{\mathbb{R}\setminus M}\Big)^{(i)}(s+)$, $i \in \mathbb{N}$, which is zero for all $i \in \mathbb{N}$, hence $(F_{\text{reg}})_{\mathbb{R}\setminus M}G[s] = 0$. Also from (10) it follows that $F[s](G_{\text{reg}})_{\mathbb{R}\setminus M} = F[s]G_{(s-\varepsilon,s)}$ for any $\varepsilon > 0$. This shows the assertion for $M = [s,t)$. Analogous

arguments show the validity of the assertions for $M = (s,t]$ and $M = [s,t]$.
If $s = t$, then

$$\begin{aligned}(FG)_{[s,t]} &= (FG)[s] = (F_{\text{reg}}G[\cdot])[s] + (F[\cdot]G_{\text{reg}})[s] \\ &= F_{\text{reg}}G[s] + F[s]G_{\text{reg}} \\ &= \underbrace{F[s]G[s]}_{=0} + F_{\text{reg}(s,s+\varepsilon)}G[s] + F[s]G_{\text{reg}(s-\varepsilon,s)} \\ &= F_{[s,t]}G_{[s,t]} + F_{(t,t+\varepsilon)}G[s] + F[s]G_{(s-\varepsilon,s)}\end{aligned}$$

qed

A.5 Proof of Proposition 11

Let $D_{\text{reg}} = \left(\sum_{i\in\mathbb{Z}} \mathbb{1}_{[t_i,t_{i+1})} f_i\right)_{\mathbb{D}}$ for some locally finite set $\{\, t_i \in \mathbb{R} \mid i \in \mathbb{Z} \,\}$ and $f_i \in \mathcal{C}^\infty$, $i \in \mathbb{Z}$. Assume, without restriction, that $s, t \in \{\, t_i \mid i \in \mathbb{Z} \,\}$. From (7) follows

$$\begin{aligned}(D_{[s,t)})' - (D')_{[s,t)} &= \sum_{i\in\mathbb{Z}} \Big(D_{[s,t)}(t_i+) - D_{[s,t)}(t_i-)\Big)\,\delta_{t_i} \\ &\quad - \left(\sum_{i\in\mathbb{Z}} \Big(D(t_i+) - D(t_i-)\Big)\delta_{t_i}\right)_{[s,t)} \\ &= \Big(D_{[s,t)}(s+) - D_{[s,t)}(s-)\Big)\,\delta_s + \Big(D_{[s,t)}(t+) - D_{[s,t)}(t-)\Big)\,\delta_t \\ &\quad - \Big(D(s+) - D(s-)\Big)\delta_s \\ &= -D(t-)\delta_t + D(s-)\delta_s.\end{aligned}$$

This shows the first formula. Since $D[\tau]' = D'[\tau] - \Big(D(\tau+) - D(\tau-)\Big)\delta_\tau$ for all $\tau \in \mathbb{R}$ the other three formulae follow easily. qed

A.6 Proof of Proposition 13

If M^{reg} is invertible over $\mathcal{C}^\infty_{\text{pw}}$ then

$$\begin{aligned}MM^{-1} &= (M_{\text{reg}} + M[\cdot])\left(M_{\text{reg}}^{-1} - M_{\text{reg}}^{-1}M[\cdot]M_{\text{reg}}^{-1}\right) \\ &= \underbrace{M_{\text{reg}}M_{\text{reg}}^{-1}}_{=I} \underbrace{- M_{\text{reg}}M_{\text{reg}}^{-1}M[\cdot]M_{\text{reg}}^{-1} + M[\cdot]M_{\text{reg}}^{-1}}_{=0} - \underbrace{M[\cdot]M_{\text{reg}}^{-1}M[\cdot]M_{\text{reg}}^{-1}}_{=0},\end{aligned}$$

where the last zero follows from the fact the product of two piecewise-smooth distributions with zero regular part is zero. An analogous calculation shows $M^{-1}M = I$. Hence sufficiency is shown.

Now assume that M is invertible over $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, i.e. there exists a matrix $M^{-1} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{n\times n}$ such that $MM^{-1} = I$. Let $M^{-1} = (M^{-1})_{\mathrm{reg}} + M^{-1}[\cdot]$, then

$$\begin{aligned} I = MM^{-1} &= (M_{\mathrm{reg}} + M[\cdot])((M^{-1})_{\mathrm{reg}} + M^{-1}[\cdot]) \\ &= M_{\mathrm{reg}}(M^{-1})_{\mathrm{reg}} + \underbrace{M_{\mathrm{reg}}M^{-1}[\cdot] + M[\cdot](M^{-1})_{\mathrm{reg}}}_{=:H}. \end{aligned}$$

Since $H[\cdot] = H$ and $I[\cdot] = 0$, it follows that H must be zero. This implies

$$I = M_{\mathrm{reg}}(M^{-1})_{\mathrm{reg}} = (M^{\mathrm{reg}}(M^{-1})^{\mathrm{reg}})_{\mathbb{D}}$$

where $(M^{-1})^{\mathrm{reg}} \in (\mathcal{C}^\infty_{\mathrm{pw}})^{n\times n}$ is such that $(M^{-1})_{\mathrm{reg}} = {(M^{-1})^{\mathrm{reg}}}_{\mathbb{D}}$. Hence M^{reg} is invertible over $\mathcal{C}^\infty_{\mathrm{pw}}$ with inverse $(M^{-1})^{\mathrm{reg}}$. Finally, from $H = 0$ and the invertibility of M_{reg} it follows that

$$M^{-1}[\cdot] = -(M_{\mathrm{reg}})^{-1}M[\cdot](M^{-1})_{\mathrm{reg}} = -M_{\mathrm{reg}}^{-1}M[\cdot]M_{\mathrm{reg}}^{-1},$$

hence M^{-1} is unique. qed

A.7 Proof of Proposition 16

It will be shown that every ITP

$$\widetilde{E}\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{f}, \quad \widetilde{x}_{(-\infty,t_0)} = \widetilde{x}^0_{(-\infty,t_0)},$$

with $t_0 \in \mathbb{R}$, $\widetilde{x}^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$, $\widetilde{f} \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$, has a unique solution.
Step 1: Existence of a solution.
Let x be the solution of the ITP

$$E\dot{x} = Ax + f, \quad x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)},$$

where

$$f = S^{-1}\widetilde{f}_{[t_0,\infty)} - S^{-1}[t_0]\left(\widetilde{A}\widetilde{x}^0 - \widetilde{E}\dot{\widetilde{x}}^0\right)_{(-\infty,t_0)}$$

and $x^0 = T\widetilde{x}^0$. It will be shown that $\widetilde{x} := T^{-1}x$ is the desired solution. First observe that, by Proposition 10,

$$\widetilde{x}_{(-\infty,t_0)} = (T^{-1}x)_{(-\infty,t_0)} = T^{-1}_{(-\infty,t_0)}x^0_{(-\infty,t_0)} = T^{-1}_{(-\infty,t_0)}(T\widetilde{x}^0)_{(-\infty,t_0)} = \widetilde{x}^0_{(-\infty,0)}.$$

Hence it remains to show that

$$(\widetilde{E}\dot{\widetilde{x}})_{[t_0,\infty)} = (\widetilde{A}\widetilde{x})_{[t_0,\infty)} + \widetilde{f}_{[t_0,\infty)},$$

which is equivalent to

$$S^{-1}(\widetilde{E}\dot{\widetilde{x}})_{[t_0,\infty)} = S^{-1}(\widetilde{A}\widetilde{x})_{[t_0,\infty)} + S^{-1}\widetilde{f}_{[t_0,\infty)}.$$

Note that from Proposition 10 and Property (R4) it follows that, for any $M \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^{m\times h}$ and $h = 1$ or $h = n$,

$$\begin{aligned} S^{-1}M_{[t_0,\infty)} &= (S^{-1}M_{[t_0,\infty)})_{(-\infty,t_0)} + (S^{-1}M_{[t_0,\infty)})_{[t_0,\infty)} \\ &= 0 + S^{-1}_{[t_0,\infty)}M_{[t_0,\infty)} \\ &= (S^{-1}M)_{[t_0,\infty)} - S^{-1}[t_0]M_{(-\infty,t_0)}. \end{aligned}$$

Hence $\widetilde{x}$ must fulfil

$$\begin{aligned} &(S^{-1}\widetilde{E}\dot{\widetilde{x}})_{[t_0,\infty)} - S^{-1}[t_0](\widetilde{E}\dot{\widetilde{x}})_{(-\infty,t_0)} \\ &\quad = (S^{-1}\widetilde{A}\widetilde{x})_{[t_0,\infty)} - S^{-1}[t_0](\widetilde{A}\dot{\widetilde{x}})_{(-\infty,t_0)} + (S^{-1}\widetilde{f})_{[t_0,\infty)} - S^{-1}[t_0]\widetilde{f}_{(-\infty,t_0)}. \end{aligned}$$

From $0 = (T^{-1}T)' = (T^{-1})'T + T^{-1}T'$ it follows, that

$$(T^{-1})' = -T^{-1}T'T^{-1},$$

hence

$$S^{-1}\widetilde{E}\dot{\widetilde{x}} = S^{-1}SET(T^{-1}x)' = E\dot{x} - ET'T^{-1}x$$

and

$$S^{-1}\widetilde{A}\widetilde{x} = S^{-1}(SAT - SET')T^{-1}x = Ax - ET'T^{-1}x.$$

Since, by assumption, $(E\dot{x})_{[t_0,\infty)} = (Ax)_{[t_0,\infty)} + f_{[t_0,\infty)}$, it remains to show that

$$f_{[t_0,\infty)} = S^{-1}[t_0](\widetilde{E}\dot{\widetilde{x}})_{(-\infty,t_0)} - S^{-1}[t_0](\widetilde{A}\widetilde{x})_{(-\infty,t_0)} + (S^{-1}\widetilde{f})_{[t_0,\infty)} - S^{-1}[t_0]\widetilde{f}_{(-\infty,t_0)}.$$

Together with Proposition 10 and Proposition 11 this follows from

$$\begin{aligned} (\widetilde{E}\dot{\widetilde{x}})_{(-\infty,t_0)} &= \widetilde{E}_{(\infty,t_0)}\dot{\widetilde{x}}_{(\infty,t_0)} \\ &= \widetilde{E}_{(\infty,t_0)}\left(\left(\widetilde{x}_{(-\infty,t_0)}\right)' + \widetilde{x}(t_0-)\delta_{t_0}\right) \\ &= \widetilde{E}_{(\infty,t_0)}\left(\left(\widetilde{x}^0_{(-\infty,t_0)}\right)' + \widetilde{x}^0(t_0-)\delta_{t_0}\right) \\ &= \widetilde{E}_{(\infty,t_0)}\dot{\widetilde{x}}^0_{(\infty,t_0)} = (\widetilde{E}\dot{\widetilde{x}}^0)_{(-\infty,t_0)}, \\ (\widetilde{A}\widetilde{x})_{(-\infty,t_0)} &= (\widetilde{A}\widetilde{x}^0)_{(-\infty,t_0)}, \\ (S^{-1}\widetilde{f})_{[t_0,\infty)} - S^{-1}[t_0]\widetilde{f}_{(-\infty,t_0)} &= S^{-1}_{[t_0,\infty)}\widetilde{f}_{[t_0,\infty)} = S^{-1}\widetilde{f}_{[t_0,\infty)}, \end{aligned}$$

and the definition of f.

Step 2: Uniqueness of a solution.

Let $\widetilde{x}_1$ and $\widetilde{x}_2$ be two solutions of the ITP

$$\widetilde{E}\dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{f}, \quad \widetilde{x}_{(-\infty,t_0)} = \widetilde{x}^0_{(-\infty,t_0)}$$

for some $t_0 \in \mathbb{R}$, $\widetilde{x}^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$, $\widetilde{f} \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$. Then $\widetilde{z} := \widetilde{x}_1 - \widetilde{x}_2$ is a solution of the ITP

$$\widetilde{E}\dot{\widetilde{z}} = \widetilde{A}\widetilde{z}, \quad \widetilde{z}_{(-\infty,t_0)} = 0.$$

It will be shown that $z = T\widetilde{z}$ is a solution of the ITP

$$E\dot{z} = Az, \quad z_{(-\infty,t_0)} = 0,$$

it then follows from the DAE-regularity of (E, A) that $z = 0$, hence $\widetilde{z} = 0$ and the uniqueness of solutions is shown.
Clearly, $z_{(-\infty,t_0)} = 0$, hence it remains to show that $(E\dot{z})_{[t_0,\infty)} = (Az)_{[t_0,\infty)}$. It is, by Proposition 10,

$$\begin{aligned} 0 &= (\widetilde{E}\dot{\widetilde{z}})_{[t_0,\infty)} - (\widetilde{A}\widetilde{z})_{[t_0,\infty)} \\ &= S_{[t_0,\infty)}(E\dot{z} - Az)_{[t_0,\infty)} + S[t_0](E\dot{z} - Az)_{(-\infty,t_0)} \\ &= S(E\dot{z} - Az)_{[t_0,\infty)} + 0, \end{aligned}$$

hence $(E\dot{z})_{[t_0,\infty)} = (Az)_{[t_0,\infty)}$. qed

A.8 Proof of Theorem 17

Step 1: $m \leq n$
Seeking a contradiction assume $m > n$. Let $E = {E^{\text{reg}}}_{\mathbb{D}} + E[\cdot]$, where $E^{\text{reg}} \in (\mathcal{C}^\infty_{\text{pw}})^{m\times n}$. Let $r_E : \mathbb{R} \to \mathbb{N}, t \mapsto r_E(t) := \operatorname{rk} E^{\text{reg}}(t)$, then there exists an open Interval $J \subseteq \mathbb{R}$ such that r_E is constant on J (see e.g. [13, Thm. 3.25]) and $E^{\text{reg}}\big|_J$ is smooth. Let $r := r(t)$ for some $t \in J$, then $r \leq n < m$. In particular, there exists an invertible $S \in (\mathcal{C}^\infty(J \to \mathbb{R}))^{m\times m}$ such that

$$SE^{\text{reg}}\Big|_J = \begin{bmatrix} \widetilde{E} \\ 0_{(m-n-r)\times n} \\ 0_{(m-n)\times n} \end{bmatrix} =: \begin{bmatrix} \widehat{E} \\ 0_{(m-n)\times n} \end{bmatrix}$$

for some $\widetilde{E} \in (\mathcal{C}^\infty(J \to \mathbb{R}))^{r\times n}$ (Doležal's Theorem, [4]) and corresponding $\widehat{E} \in (\mathcal{C}^\infty(J \to \mathbb{R}))^{n\times n}$. Without restriction, it can be assumed that $\inf_{t\in J} \det S(t) > 0$ and $E_J[\cdot] = A_J[\cdot] = 0$ (if these conditions are not fulfilled a reduction of the size of the open interval J yields these properties). Hence it is possible to extend the matrix function S to the whole time interval $\mathbb{R}$ such that $S \in (\mathcal{C}^\infty)^{m\times m}$ and $S^{-1} \in (\mathcal{C}^\infty)^{m\times m}$ exists and

$$SE = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \quad SA = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},$$

where $E_1, A_1 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n\times n}$ and $(E_2)_J = 0$. By Proposition 16 the pair (SE, SA) is still DAE-regular, in particular the DAE must have a local solution on the interval J for all inhomogeneities. Let $A_2 = {A_2^{\text{reg}}}_{\mathbb{D}} + A_2[\cdot]$ for some

$A_2^{\text{reg}} \in (\mathcal{C}_{\text{pw}}^\infty)^{(m-n)\times(m-n)}$. It now follows that $A_2^{\text{reg}}\big|_J$ must have full row rank, because otherwise there would exist $t \in J$ and an invertible matrix $M \in \mathbb{R}^{m\times m}$ such that the last row of $MSA(t+)$ and of $MSE(t+)$ is zero, hence for every inhomogeneity f with $f(t+) \neq 0$ any ITP with $t_0 \leq t$ would not have a solution. Firstly, this implies $m \leq 2n$. Secondly, by Doležal's Theorem there exists a matrix function $T \in \mathcal{C}^\infty(J \to \mathbb{R})^{n\times n}$ such that

$$A_2 T = [0\ I] \text{ on } J$$

and it is possible to extend T on the whole axis, such that $T \in (\mathcal{C}^\infty)^{n\times n}$ with $T^{-1} \in (\mathcal{C}^\infty)^{n\times n}$ (possible by reducing the size of J). Now let

$$(\widetilde{E}, \widetilde{A}) := (SET, SAT - SET') = \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right),$$

then $(\widetilde{E}, \widetilde{A})$ is DAE-regular by Proposition 16 and

$$E_{21J} = 0,\ E_{22J} = 0,\ A_{21J} = 0,\ A_{22J} = I_J,$$

furthermore, the size of E_{11} is $n \times (2n - m)$. Since $m > n$ was assumed it follows that E_{11} has a strictly smaller size than E and has more rows than columns. On the interval J the system $(\widetilde{E}, \widetilde{A})$ reads as

$$\begin{aligned} E_{11}\dot{z}_1 + E_{12}\dot{z}_2 &= A_{11}z_1 + A_12z_2 + f_1, \\ 0 &= z_2 + f_2, \end{aligned}$$

where $f_1 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$, $f_2 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{m-n}$, $z_1 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{2n-m}$, $z_2 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{m-n}$. If $2m = n$, then the above equation is equivalent to

$$f_1 = A_{12}f_2 - E_{12}f_2' \quad \text{on } J,$$

hence it is only solvable on the interval J if the inhomogeneity is not chosen arbitrarily. Hence $2m > n$. In this case, substituting z_2 in the above equation by f_2 yields that the system (E, A) has a local solution on the interval J if, and only if, the system (E_{11}, A_{11}) has a local solution on the interval J. Let $(E^0, A^0) := (E, A)$ and $(E^1, A^1) := (E_{11}, A_{11})$ with size $m^0 \times n^0 := m \times n$ and $m^1 \times n^1 := n^0 \times (2n^0 - m^0)$. It is now possible to repeat the above arguments to get a sequence of matrix-pairs (E^i, A^i), $i \in \mathbb{N}$, with strictly decreasing size $m^i \times n^i$ such that $0 \leq m^i < m^{i-1}$ and $0 \leq n^i < m^{i-1}$. Clearly, this is a contradiction.

Step 2: $n \leq m$

Seeking a contradiction assume $n > m$. With analogous arguments as in the first step it is possible to find an open interval $J \subseteq \mathbb{R}$ and invertible matrices

$S \in (\mathcal{C}^\infty)^{n\times n}$, $T \in (\mathcal{C}^\infty)^{m\times m}$ such that

$$(\widetilde{E}, \widetilde{A}) := (SET, SAT - SET') = \left(\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right)$$

is DAE-regular and

$$E_{12J} = 0,\ E_{22J} = 0,\ A_{12J} = 0,\ A_{22J} = I_J.$$

To get this result it was used that $S^{-1}\left[\begin{smallmatrix} A_{12} \\ A_{22} \end{smallmatrix}\right]$ must have full column rank on J, because otherwise there would exist a component of the solution vector x which does not "appear" in the DAE on J and hence the ITP could not have a unique solution. The size of E_{11} is $(2m-n)\times n$, from which follows that $2m \geq n$. The system $(\widetilde{E}, \widetilde{A})$ restricted to J reads as

$$\begin{aligned} E_{11}\dot{z}_1 &= A_{11}z_1 + f_1 \\ E_{21}\dot{z}_1 &= A_{21}z_1 + z_2 + f_2. \end{aligned}$$

If $2m = n$ then only the equation

$$z_2 = A_{21}z_1 - E_{21}\dot{z}_1 + f_2$$

remains, which is clearly not uniquely solvable (with a given initial trajectory) on the interval J, because z_1 can be altered on J and together with the corresponding z_2 it is still a solution of the same ITP. Hence $2m > n$. Similar as in the first step it is now again possible to construct a sequence of systems (E^i, A^i), $i \in \mathbb{N}$, with strictly decreasing size $m^i \times n^i$ such that $m^i > n^i$ and $0 < m^i < m^{i-1}$. This is a contradiction. qed

A.9 Proof of Theorem 18

(i) Taking successively the derivative of the equation $E\dot{x} = Ax + f$ yields

$$\begin{aligned} E\dot{x} - Ax &= f \\ E\ddot{x} + (E' - A)\dot{x} - A'x &= f' \\ E\dddot{x} + (2E' - A)\ddot{x} + (E'' - 2A')\dot{x} - A''x &= f'' \\ &\vdots \end{aligned}$$

and it follows inductively that, for all $p \in \mathbb{N}$,

$$\mathcal{M}^p \begin{pmatrix} x \\ \dot{x} \\ \vdots \\ x^{(p)} \\ x^{(p+1)} \end{pmatrix} = \begin{pmatrix} f \\ f' \\ \vdots \\ f^{(p)} \end{pmatrix}$$

and, in particular, for all $t \in \mathbb{R}$,

$$\mathcal{M}^p(t\pm) \begin{pmatrix} x(t\pm) \\ \dot{x}(t\pm) \\ \vdots \\ x^{(p)}(t\pm) \\ x^{(p+1)}(t\pm) \end{pmatrix} = \begin{pmatrix} f(t\pm) \\ f'(t\pm) \\ \vdots \\ f^{(p)}(t\pm) \end{pmatrix}$$

Since (E, A) is assumed to be DAE-regular there exists a solution for any given right-hand side, hence $\mathcal{M}^p(t+)$ and $\mathcal{M}^p(t-)$ must both have full row rank.

(ii) For a fixed $t_0 \in \mathbb{R}$ consider the impulsive part of the DAE (11) at t_0:

$$(E\dot{x})[t_0] = (Ax + f)[t_0]$$

or, equivalently,

$$E_{(t_0,\infty)}\dot{x}[t_0] - A_{(t_0,\infty)}x[t_0] = A[t_0]x_{(-\infty,t_0)} - E[t_0]\dot{x}_{(-\infty,t_0)} + f[t_0] =: \widetilde{f}[t_0].$$

The right-hand side can be assumed to be arbitrary, and since (E, A) is DAE-regular it follows that the operator

$$\begin{aligned} (E \tfrac{\mathrm{d}}{\mathrm{dt}} - A)_{t_0} : (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n &\to \{\ D_{t_0} \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \ \mid \ \operatorname{supp} D_{t_0} \subseteq \{t_0\}\ \}, \\ x &\mapsto E_{(t_0,\infty)}\dot{x}[t_0] - A_{(t_0,\infty)}x[t_0] \end{aligned}$$

must be surjective. Assume

$$x[t_0] = \sum_{i=0}^{p} x_i \delta_{t_0}^{(i)}$$

for some $p \in \mathbb{N}$ and $x_0, x_1, \ldots, x_p \in \mathbb{R}^n$, then

$$\dot{x}[t_0] = \sum_{i=0}^{p+1} x_{i-1} \delta_{t_0}^{(i)},$$

where $x_{-1} := x(t+) - x(t-)$. Using (10), one gets

$$\begin{aligned}
E_{(t_0,\infty)}\dot{x}[t_0] - A_{(t_0,\infty)}x[t_0] &= \sum_{j=0}^{p+1}\sum_{i=0}^{j}(-1)^i\tbinom{j}{i}E^{(i)}(t_0+)x_{j-1}\delta_{t_0}^{(j-i)} \\
&\quad - \sum_{j=0}^{p}\sum_{i=0}^{j}(-1)^i\tbinom{j}{i}A^{(i)}(t_0+)x_j\delta_{t_0}^{(j-i)} \\
&= \sum_{i=0}^{p+1}\sum_{j=i-1}^{p}(-1)^{j-i+1}\tbinom{j+1}{i}E^{(j-i+1)}(t_0+)x_j\delta_{t_0}^{(i)} \\
&\quad - \sum_{i=0}^{p}\sum_{j=i}^{p}(-1)^{j-i}\tbinom{j}{i}A^{(j-i)}(t_0+)x_j\delta_{t_0}^{(i)} \\
&= \sum_{i=0}^{p+1}a_i\delta_{t_0}^{(i)} \overset{!}{=} \tilde{f}[t_0]
\end{aligned}$$

where

$$\mathbb{R}^{(p+2)n} \ni \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{p+1} \end{pmatrix} := \mathcal{N}^{p+1,p+1}(t_0+)\begin{pmatrix} x_{-1} \\ x_0 \\ \vdots \\ x_p \end{pmatrix}.$$

Note that, in particular, for $i = 0, 1, \ldots, p+1$

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_i \end{pmatrix} = \mathcal{N}^{i,p+1}(t_0+)\begin{pmatrix} x_{-1} \\ x_0 \\ \vdots \\ x_p \end{pmatrix}.$$

Since $a_0, a_1, \ldots,$ are given by $\tilde{f}[t_0]$ they can be arbitrary. Hence there must exists $q_0 \in \mathbb{N}$ such that $\mathcal{N}^{0,q_0}$ has full row rank, otherwise not all values for a_0 can be "produced". In general, for every $i \in \mathbb{N}$ there must exists $q_i \in \mathbb{N}$ such that $\mathcal{N}^{i,q_i}$ has full row rank to guarantee that every vector $(a_0^\top, a_1^\top, \ldots, a_i^\top)^\top$ can be obtained. This proves the theorem.

qed

A.10 *Proof of Theorem 20*

(i) If $t_0 \geq t_1$, then the ITP for (E, A) is identical to the ITP for (E_1, A_1), hence only $t_0 < t_1$ needs to be considered. For $x^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ and $f \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ let x^1 be the unique solution of the ITP (E_0, A_0), $x^1_{(-\infty,t_0)} =$

$x^0_{(-\infty,t_0)}$ with inhomogeneity f and let x be the unique solution of the ITP (E_1, A_1), $x_{(-\infty,t_1)} = x^1_{(-\infty,t_1)}$ with inhomogeneity f.
It will be shown that x is also the unique solution of the ITP (E, A), $x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$. First observe that $x_{(-\infty,t_0)} = x^1_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$ because $t_0 < t_1$. Secondly, the following equivalences hold (using Proposition 10)

$$\begin{aligned} & (E\dot{x})_{[t_0,\infty)} = (Ax + f)_{[t_0,\infty)} \\ \Leftrightarrow\ & (E\dot{x})_{[t_0,t_1)} = (Ax + f)_{[t_0,t_1)} \quad \wedge \quad (E\dot{x})_{[t_1,\infty)} = (Ax + f)_{[t_1,\infty)} \\ \Leftrightarrow\ & (E_0\dot{x}^1)_{[t_0,t_1)} = (A_0x^1 + f)_{[t_0,t_1)} \quad \wedge \quad (E_1\dot{x})_{[t_1,\infty)} = (A_1x + f)_{[t_1,\infty)}. \end{aligned}$$

The last expression is true by the definition of x^1 and x, hence x is a solution of the ITP.
It remains to show that x is unique. Assume there is another solution $\widetilde{x}$. Since, by definition, $\widetilde{x}_{(-\infty,t_0)} = x^0_{(-\infty,t_0)} = x_{(-\infty,t_0)}$, it remains to show that $\widetilde{x}_{[t_0,t_1)} = x_{[t_0,t_1)}$ and $\widetilde{x}_{[t_1,\infty)} = x_{[t_1,\infty)}$. Let z and $\widetilde{z}$ be the solutions of the ITP (E_0, A_0), $z_{(-\infty,t_1)} = x_{(-\infty,t_1)}$ and $\widetilde{z}_{(-\infty,t_1)} = \widetilde{x}_{(-\infty,t_1)}$, resp., then

$$(E_0\dot{z})_{[t_0,t_1)} = (E_0\dot{x})_{[t_0,t_1)} = (A_0x + f)_{[t_0,t_1)} = (A_0z + f)_{[t_0,t_1)}$$

and

$$(E_0\dot{\widetilde{z}})_{[t_0,t_1)} = (E_0\dot{\widetilde{x}})_{[t_0,t_1)} = (A_0\widetilde{x} + f)_{[t_0,t_1)} = (A_0\widetilde{z} + f)_{[t_0,t_1)}.$$

Hence z and $\widetilde{z}$ are also solutions of the ITP (E_0, A_0), $z_{(-\infty,t_0)} = x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$ and $\widetilde{z}_{(-\infty,t_0)} = \widetilde{x}_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$, resp. Since (E_0, A_0) is DAE-regular it follows that $z = \widetilde{z}$ and therefore $x_{[t_0,t_1)} = \widetilde{x}_{[t_0,t_1)}$. Finally, observe that x and $\widetilde{x}$ are solutions of the ITP (E_1, A_1), $x_{(-\infty,t_1)} = x_{(-\infty,t_1)}$ and $\widetilde{x}_{(-\infty,t_1)} = \widetilde{x}_{(-\infty,t_1)}$, resp. Since (E_1, A_1) is DAE-regular and $x_{(-\infty,t_1)} = \widetilde{x}_{(-\infty,t_1)}$, it follows that $x = \widetilde{x}$.

(ii) Consider the ITP (E, A), $x_{(-\infty,\tau_0)} = \xi^0_{(-\infty,\tau_0)}$ for some initial trajectory ξ^0 and $\tau_0 \in \mathbb{R}$. Without restriction of generality it may be assumed that $t_0 \leq \tau_0 < t_1$ (just by changing the indices). Let x^0 be the solution of the ITP (E_0, A_0), $x^0_{(-\infty,\tau_0)} = \xi^0_{(-\infty,\tau_0)}$ and, for $i \in \mathbb{N}$, let x^{i+1} be the solution of the ITP (E_{i+1}, A_{i+1}), $x^{i+1}_{(-\infty,t_{i+1})} = x^i_{(-\infty,t_{i+1})}$. Then $x = \lim_{i\to\infty} x^i$ is a well defined distribution and it follows by inductively repeating the same arguments as in (i) that x is the unique solution of the ITP (E, A), $x_{(-\infty,\tau_0)} = \xi^0_{(-\infty,\tau_0)}$. Hence (E, A) is DAE-regular.

(iii) Consider the ITP $(E_0+E_1[t], A_0+A_1[t])$, $x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$ for some $x^0 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$, $t_0 \in \mathbb{R}$ and with a inhomogeneity $f \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$. Clearly, if $t_0 > t$ this ITP is identical to the ITP (E_0, A_0) with the same initial trajectory and inhomogeneity. Hence it remains to consider $t_0 \leq t$. Let $\hat{x}$ be the solution of the ITP (E_0, A_0), $\hat{x}_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$ with inhomogeneity f and let x be the solution of the ITP (E_0, A_0), $x_{(-\infty,t)} = \hat{x}_{(-\infty,t)}$ with inhomogeneity $\hat{f} := f + A_1[t]\hat{x} - E_1[t]\dot{\hat{x}}$. It will be shown that x is the unique solution of the ITP (E, A), $x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$ with inhomogeneity f. First observe that $x_{(-\infty,t_0)} = \hat{x}_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$. Secondly,

$$\begin{aligned}\Big((E_0 + E_1[t])\dot{x}\Big)_{[t_0,t)} &= (E_0\dot{x})_{[t_0,t)} = (E_0\dot{\hat{x}})_{[t_0,t)} = (A_0\hat{x} + f)_{[t_0,t)} \\ &= \Big((A_0 + A_1[t])x + f\Big)_{[t_0,t)}\end{aligned}$$

and, because $x_{(-\infty,t)} = \hat{x}_{(-\infty,t)}$,

$$\begin{aligned}\Big((E_0 + E_1[t])\dot{x}\Big)_{[t,\infty)} &= (E_0\dot{x})_{[t,\infty)} + E_1[t]\dot{x} \\ &= (A_0x + \hat{f})_{[t,\infty)} + E_1[t]\dot{\hat{x}} \\ &= (A_0x + f + A_1[t]\hat{x})_{[t,\infty)} \\ &= \Big((A_0 + A_1[t])x + f\Big)_{[t,\infty)}.\end{aligned}$$

Hence it remains to show uniqueness of the solution x. Therefore, let $\widetilde{x}$ also be a solution of the ITP (E, A), $\widetilde{x}_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}$ with inhomogeneity f. With the same arguments as above it follows that $\widetilde{x}_{(-\infty,t)} = x_{(-\infty,t)} = \hat{x}_{(-\infty,t)}$. Now

$$\begin{aligned}&\Big((E_0 + E_1[t])\dot{x}\Big)_{[t,\infty)} = \Big((A_0 + A_1[t])x + f\Big)_{[t,\infty)} \\ \Leftrightarrow\quad &(E_0\dot{x})_{[t,\infty)} = (A_0x + \hat{f})_{[t,\infty)}\end{aligned}$$

and the same for $\widetilde{x}$, hence x and $\widetilde{x}$ are both solutions of the ITP (E_0, A_0), $x_{(-\infty,t)} = \hat{x}_{(-\infty,t)}$ with inhomogeneity $\hat{f}$. Because (E_0, A_0) is DAE-regular it follows that $x = \widetilde{x}$.

(iv) Let $T = \{\, t_i \in \mathbb{R} \mid i \in \mathbb{Z} \,\}$ be a locally finite set such that $E_1[\cdot] = \sum_{i\in\mathbb{Z}} E_1[t_i]$ and $A_1[\cdot] = \sum_{i\in\mathbb{Z}} A_1[t_i]$. Furthermore, let $\widetilde{E}_0 = E_0$, $\widetilde{A}_0 = A_0$ and, for $k \in \mathbb{N}$, $\widetilde{E}_{k+1} = \widetilde{E}_k + E_1[t_k]$, $\widetilde{E}_{-k-1} = \widetilde{E}_{-k} + E_1[t_{-k}]$, $\widetilde{A}_{k+1} = \widetilde{A}_k + A_1[t_k]$, $\widetilde{A}_{-k-1} = \widetilde{A}_{-k} + A_1[t_{-k}]$. Then it follows inductively from (iii) that $(\widetilde{E}_i, \widetilde{A}_i)$ is DAE-regular for all $i \in \mathbb{Z}$. Finally,

$$(E_0 + E_1[\cdot], A_0 + A_1[\cdot]) = \left(\sum_{i\in\mathbb{Z}} \widetilde{E}_{i[t_i,t_{i+1})}, \sum_{i\in\mathbb{Z}} \widetilde{A}_{i[t_i,t_{i+1})}\right)$$

and regularity follows from (ii).

qed

A.11 Proof of Theorem 23

By Theorem 20 it suffices to consider the impulse free case, i.e. $E[\cdot] = 0$ and $A[\cdot] = 0$.

The distributional ODE case, i.e. $(E, A) = (I, A)$.
Let $A^{\mathrm{reg}} \in (\mathcal{C}^\infty_{\mathrm{pw}})^{n\times n}$ such that $A^{\mathrm{reg}}_{\mathbb{D}} = A_{\mathrm{reg}} = A$ and consider the standard homogeneous ODE $\dot{x} = A^{\mathrm{reg}}x$. Let $\phi(\cdot, t_0) : \mathbb{R} \to \mathbb{R}^{n\times n}$, $t_0 \in \mathbb{R}$, be its fundamental solution, i.e. $\phi(\cdot, t_0)$ is absolutely continuous, $\phi(\cdot, t_0)' = A^{\mathrm{reg}}\phi(\cdot, t_0)$ and $\phi(t_0, t_0) = I$ [20, C.4].
It will be shown first, that $\phi(\cdot, t_0)$ is piecewise-smooth as in Definition 1. Let $T = \{ \tau_i \in \mathbb{R} \mid i \in \mathbb{Z} \}$ be a locally finite set such that $A^{\mathrm{reg}} = \sum_{i\in\mathbb{Z}} A_{i[\tau_i,\tau_{i+1})}$ for some family of smooth matrices $(A_i)_{i\in\mathbb{Z}}$. For $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ let $\phi_i(\cdot, t)$ be the fundamental solutions of $\dot{x} = A_i x$. Then $\phi_i(\cdot, t)$ is smooth for all $i \in \mathbb{Z}$ because each A_i is smooth. Since the ODEs $\dot{x} = A^{\mathrm{reg}}x$ and $\dot{x} = A_i x$ are identical on the interval $[\tau_i, \tau_{i+1})$ the fundamental solution restricted to this interval are also identical if the initial time fulfils $t \in [\tau_i, \tau_{i+1})$, hence $\phi(s, t) = \phi_i(s, t)$ for all $s, t \in [t_i, t_{i+1})$. For a fixed $t_0 \in \mathbb{R}$ this yields $\phi(t, t_0) = \phi_i(t, t_i)\phi(t_i, t_0)$ where $i \in \mathbb{Z}$ is chosen such that $t \in [t_i, t_{i+1})$. Now it follows that

$$\phi(\cdot, t_0) = \sum_{i\in\mathbb{Z}} \Big(\phi_i(\cdot, t_i)\phi(t_i, t_0)\Big)_{[t_i,t_{i+1})},$$

which shows that $\phi(\cdot, t_0)$ is piecewise-smooth. Note furthermore, that each $\phi_i(\cdot, t_i)$ is invertible with $\phi_i(\cdot, t_i)^{-1} \in (\mathcal{C}^\infty)^{n\times n}$, this shows that the inverse $\phi(\cdot, t_0)^{-1}$ is also piecewise-smooth.
It will be shown now that the ITP

$$\begin{aligned} x_{(-\infty,t_0)} &= x^0_{(-\infty,t_0)}, \\ \dot{x}_{[t_0,\infty)} &= (Ax + f)_{[t_0,\infty)}, \end{aligned}$$

where $x^0, f \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ and $t_0 \in \mathbb{R}$, has the unique solution

$$x = x^0_{(-\infty,t_0)} + \left(\phi(\cdot, t_0)_{\mathbb{D}} x^0(t_0-) + \phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)^{-1}_{\mathbb{D}} f\right)_{[t_0,\infty)}.$$

It must first be shown, that $\dot{x}_{[t_0,\infty)} = (Ax)_{[t_0,\infty)}$. Proposition 10, Proposition 11 and $\phi(\cdot, t_0)'_{\mathbb{D}} = A\phi(\cdot, t_0)_{\mathbb{D}}$ yield

$$\begin{aligned} \dot{x}_{[t_0,\infty)} &= -x^0(t_0-)\delta_{t_0} + \left(\phi(\cdot, t_0)'_{\mathbb{D}} x^0(t_0-) + \left(\phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)^{-1}_{\mathbb{D}} f\right)'\right)_{[t_0,\infty)} \\ &\quad + \left(\underbrace{\phi(\cdot, t_0)_{\mathbb{D}}(t_0-)}_{=I} x^0(t_0-) + \underbrace{\left(\phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)^{-1}_{\mathbb{D}} f\right)(t_0-)}_{=0}\right)\delta_{t_0} \\ &= \left(A\phi(\cdot, t_0)_{\mathbb{D}} x^0(t_0-) + A\phi(\cdot, t_0)_{\mathbb{D}} \int_{t_0} \phi(\cdot, t_0)^{-1}_{\mathbb{D}} f + f\right)_{[t_0,\infty)} \end{aligned}$$

and, since $A[t_0]=0$,

$$(Ax+f)_{[t_0,\infty)} = \underbrace{\left(Ax^0_{(-\infty,t_0)}\right)_{[t_0,\infty)}}_{=0} + \left(A\phi(\cdot,t_0)_{\mathbb{D}}x^0(t_0-) + A\phi(\cdot,t_0)_{\mathbb{D}}\int_{t_0}\phi(\cdot,t_0)_{\mathbb{D}}^{-1}f + f\right)_{[t_0,\infty)},$$

which shows that x is a solution of the ITP.
It remains to show that the proposed solution is unique. Assume that $x_1, x_2 \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ are solutions of the same ITP, then $e = x_1 - x_2$ fulfils $e_{(-\infty,t_0)} = 0$ and

$$\dot{e}_{[t_0,\infty)} = (Ae)_{[t_0,\infty)}.$$

Note that $\dot{e}_{(-\infty,t_0)} = 0 = (Ae)_{(-\infty,t_0)}$, hence e is a solution of $\dot{e} = Ae$ with $e(t_0-) = 0$ and it must be shown that $e = 0$ is the only solution of $\dot{e} = Ae$ with $e(t_0-) = 0$. Let $e \in (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n$ be any solution of $\dot{e} = Ae$ with $e(t_0-) = 0$ and let $\eta := \phi(\cdot,t_0)_{\mathbb{D}}^{-1}e$. Then, since $e = \phi(\cdot,t_0)_{\mathbb{D}}\eta$,

$$\dot{e} = \underbrace{A\phi(\cdot,t_0)_{\mathbb{D}}\eta}_{=Ae=\dot{e}} + \phi(\cdot,t_0)_{\mathbb{D}}\dot{\eta},$$

hence

$$\dot{\eta} = 0.$$

This implies that η, as an distributional antiderivative of zero, is a constant distribution and since $\eta(t_0-) = e(t_0-) = 0$ it follows that $\eta = 0$. This shows that $e = 0$ and it is shown that x as given above is the only solution of the ITP.
The pure distributional DAE case, i.e. $(E,A) = (N,I)$ *with* N_{reg} *a strictly lower triangular matrix.*
First observe that $(N\dot{x})_{[t_0,\infty)} = N_{[t_0,\infty)}\dot{x}$, so the ITP

$$\begin{aligned} x_{(-\infty,t_0)} &= x^0_{(-\infty,t_0)}, \\ (N\dot{x})_{[t_0,\infty)} &= (x+f)_{[t_0,\infty)}, \end{aligned}$$

can equivalently reformulated into a distributional DAE without explicit initial condition:

$$N_{\mathrm{itp}}\dot{x} = x + f_{\mathrm{itp}},$$

where $N_{\mathrm{itp}} = N_{[t_0,\infty)}$ and $f_{\mathrm{itp}} = -x^0_{(-\infty,t_0)} + f_{[t_0,\infty)}$. The matrix N_{itp} is still a strictly lower triangular matrix. Consider the operator

$$N_{\mathrm{itp}}\tfrac{\mathrm{d}}{\mathrm{dt}} : (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty})^n, \quad x \mapsto N_{\mathrm{itp}}\dot{x},$$

and its powers

$$(N_{\mathrm{itp}}\tfrac{\mathrm{d}}{\mathrm{dt}})^0 := x \mapsto x, \quad \forall i \in \mathbb{N}: \ (N_{\mathrm{itp}}\tfrac{\mathrm{d}}{\mathrm{dt}})^{i+1} := x \mapsto (N_{\mathrm{itp}}\tfrac{\mathrm{d}}{\mathrm{dt}})\left((N_{\mathrm{itp}}\tfrac{\mathrm{d}}{\mathrm{dt}})^i(x)\right).$$

Then, since N_{itp} is a strictly lower triangular matrix, the operator $N_{\text{itp}} \frac{\mathrm{d}}{\mathrm{dt}}$ is nilpotent, i.e. there exists $\nu \in \mathbb{N}$ such that $(N_{\text{itp}} \frac{\mathrm{d}}{\mathrm{dt}})^\nu$ is the zero operator. From this it follows that the operator

$$(N_{\text{itp}} \tfrac{\mathrm{d}}{\mathrm{dt}} - I) : (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n \to (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n, \quad x \mapsto N_{\text{itp}}\dot{x} - x$$

is bijective with inverse

$$(N_{\text{itp}} \tfrac{\mathrm{d}}{\mathrm{dt}} - I)^{-1} = -\sum_{i=0}^{\nu-1} (N_{\text{itp}} \tfrac{\mathrm{d}}{\mathrm{dt}})^i.$$

Hence the unique solution of the ITP is given by

$$x = -\sum_{i=0}^{\nu-1} \left(N_{[t_0,\infty)} \tfrac{\mathrm{d}}{\mathrm{dt}} \right)^i \left(-x^0_{(-\infty,t_0)} + f_{[t_0,\infty)} \right).$$

qed

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