

Pairs of Disjoint Dominating Sets and the Minimum Degree of Graphs

Christian Löwenstein and Dieter Rautenbach

Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany,
emails: {christian.loewenstein, dieter.rautenbach}@tu-ilmenau.de

Abstract. For a connected graph G of order n and minimum degree δ we prove the existence of two disjoint dominating sets D_1 and D_2 such that, if $\delta \geq 2$, then $|D_1 \cup D_2| \leq \frac{6}{7}n$ unless $G = C_4$, and, if $\delta \geq 5$, then $|D_1 \cup D_2| \leq 2^{\frac{1+\ln(\delta+1)}{\delta+1}}n$. While for the first estimate there are exactly six extremal graphs which are all of order 7, the second estimate is asymptotically best-possible.

Keywords. domination; domination number; domatic partition; domatic number; inverse domination; disjoint domination number

1 Introduction

We consider graphs $G = (V, E)$ with vertex set V and edge set E which are finite, simple and undirected.

Let $G = (V, E)$ be a graph and let $u \in V$ be a vertex. The neighbourhood $N_G(u)$ of u in G is the set $\{v \in V \mid uv \in E\}$ and the degree $d_G(u)$ of u in G is the number of edges incident with u . The minimum and maximum degree of a vertex in G are denoted by $\delta(G)$ and $\Delta(G)$. The closed neighbourhood $N_G[u]$ of $u \in G$ is the set $\{u\} \cup N_G(u)$. For some $i \in \mathbb{N}$ let $V_i = \{u \in V \mid d_G(u) = i\}$ and $V_{\geq i} = \{u \in V \mid d_G(u) \geq i\}$.

A set of vertices $D \subseteq V$ is said to *dominate* a vertex $u \in V$, if $N_G[u] \cap D \neq \emptyset$. D is a *dominating set* of G , if D dominates all vertices in V and the minimum cardinality of a dominating set of G is the domination number $\gamma(G)$ of G . Similarly, a pair (D_1, D_2) of disjoint sets of vertices $D_1, D_2 \subseteq V$ is said to dominate a vertex $u \in V$, if D_1 and D_2 dominate u . (D_1, D_2) is a *dominating pair*, if (D_1, D_2) dominates all vertices in V . The (total) cardinality of a pair (D_1, D_2) is $|D_1| + |D_2|$ and the minimum cardinality of a dominating pair is the *disjoint domination number* $\gamma\gamma(G)$ of G .

A path of length $l \geq 0$ in G is a sequence $P : u_0 u_1 u_2 \dots u_l$ of $l + 1$ distinct vertices of G such that $u_{i-1} u_i \in E$ for $1 \leq i \leq l$. A cycle of length $l \geq 3$ in G is a sequence $C : u_1 u_2 \dots u_l u_1$ such that $u_1, u_2, \dots, u_l \in V$ are l distinct vertices, $u_{i-1} u_i \in E$ for $2 \leq i \leq l$, and $u_1 u_l \in E$. A path of length $i + 1$ whose endvertices are of degree at least 3 and whose i internal vertices are all of degree 2 is called an *i -path*. A cycle of length $i + 1$ which contains i vertices of degree 2 and one vertex of degree at least 3 is called an *i -cycle*.

Domination is a classical and well-studied graph-theoretical notion [14, 15]. Among the most fundamental results on the domination number are upper bound for graphs which satisfy a minimum degree condition [1, 2, 4, 20–23].

The first such result is due to Ore [21] who observed that the complement of every minimal dominating set of a graph $G = (V, E)$ of minimum degree at least 1 is also a dominating set. This implies that every such graph has two disjoint dominating sets and hence

$$\gamma(G) \leq \frac{1}{2}|V|.$$

For graphs $G = (V, E)$ of minimum degree at least 2, Blank [4] and — independently — McCuaig and Shepherd [20] proved that

$$\gamma(G) \leq \frac{2}{5}|V|$$

unless G is one of the seven graphs H_1, H_2, \dots, H_7 in Figure 3.

Several authors studied so-called *domatic partitions* which are partitions of the vertex set of a graph into dominating sets. The maximum number of disjoint dominating sets into which a graph can be partitioned is known as the *domatic number* [6] (cf. Zelinka's contribution to [15]). Furthermore, graphs G having two disjoint minimum dominating sets [3] — i.e. graphs G with $\gamma\gamma(G) = 2\gamma(G)$ — and also the minimum intersection of pairs of minimum dominating sets [5, 9, 13] were considered.

Recently several authors initiated the study of the cardinalities of pairs of disjoint dominating sets in graphs. Kulli and Sigarkanti [19] introduce the *inverse domination number* which is the minimum cardinality of a dominating set whose complement contains a minimum dominating set (cf. [8, 11]).

Motivated by the inverse domination number, Hedetniemi et al. [17] defined and studied the disjoint domination number $\gamma\gamma(G)$ of a graph G . By Ore's observation,

$$\gamma\gamma(G) \leq |V|$$

for every graph $G = (V, E)$ without isolated vertices and Hedetniemi et al. characterized all extremal graphs for this bound. They also proved that it is NP-hard to determine $\gamma\gamma(G)$ even for chordal graphs G . In [17] they list 22 open problems in connection with the disjoint domination number, 9 of which were solved in [18].

It is a natural question why to devote special attention to the case of two disjoint dominating sets rather than k disjoint dominating sets for general k . The reason is that, by Ore's observation, the trivial necessary minimum degree condition is also sufficient for the existence of two disjoint dominating sets. For all fixed $k \geq 3$, it is NP-complete [12] to decide the existence of k disjoint dominating sets and no minimum degree condition is sufficient for the existence of three disjoint dominating sets. As a simple example attributed to Zelinka consider a bipartite graph with one partite set A containing $3\delta - 2$ vertices and a second partite set B containing $\binom{3\delta-2}{\delta}$ vertices each of which is adjacent to a different set of δ vertices from A . Clearly, this graph has minimum degree δ but does not contain three disjoint dominating sets.

Feige et al. [10] (cf. also [7]) proved that every graph G can be partitioned into

$$(1 - o(1)) \frac{\delta(G) + 1}{\ln \Delta(G)}$$

dominating sets where the $o(1)$ -term tends to 0 as $\Delta(G)$ tends to infinity. Considering the smallest k of these sets implies that every graph G has k disjoint dominating sets whose total cardinality is

$$(1 + o(1)) \frac{k \ln \Delta(G)}{\delta(G) + 1} |V|. \quad (1)$$

Our results in the present paper are

- a best-possible upper bound on the disjoint domination number of graphs of minimum degree at least 2 together with the characterization of the unique exceptional graph and the six extremal graphs (Theorem 6) and
- an asymptotically best-possible upper bound on the disjoint domination number of graphs of minimum degree at least 5 (Theorem 8).

The first result is inspired by McCuaig and Shepherd's [20] work and their seven exceptional graphs H_1, H_2, \dots, H_7 play an important role. The second result improves (1) for $k = 2$ and relies on a beautiful probabilistic argument used by Alon and Spencer [1] to prove the asymptotically best-possible bound

$$\gamma(G) \leq \frac{1 + \ln(\delta(G) + 1)}{\delta(G) + 1} |V|.$$

2 Graph of Minimum Degree at least 2

We first prove the desired bound for graphs which arise by suitably subdividing the edges of some multigraph which may contain multiple edges but no loops.

Theorem 1 *Let $G^* = (V^*, E^*)$ be a multigraph which may contain multiple edges but no loops such that every vertex is incident with at least 3 edges. Let $E_1^* \cup E_2^* \cup E_3^*$ be a partition of the edge set E^* of G^* .*

If the graph $G = (V, E)$ arises from G^ by subdividing every edge in E_i^* exactly i times for $1 \leq i \leq 3$, then G has a dominating pair (D_1, D_2) such that $V_{\geq 3} = V^* \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V|$.*

Proof: Let G^* and G be as in the statement of the result. We will prove the desired statement by explicitly describing the construction of a suitable dominating pair (D_1, D_2) for G . Initially, let $(D_1, D_2) = (\emptyset, \emptyset)$.

Note that the edges in E_i^* correspond exactly to the i -paths of G . Let $p_i = |E_i^*|$ for $1 \leq i \leq 3$. Furthermore, let $n_i = |V_i|$ and $n_{\geq i} = |V_{\geq i}|$ for $i \in \mathbb{N}$. Clearly, counting the vertices of G and the edges of G^* we obtain

$$|V| = n_{\geq 3} + p_1 + 2p_2 + 3p_3 \text{ and} \quad (2)$$

$$|E^*| = p_1 + p_2 + p_3 \geq \frac{3}{2}n_3 + 2n_{\geq 4}. \quad (3)$$

As a first step, we add all vertices in $V_{\geq 3} = V^*$ to either D_1 or D_2 .

If $u, v \in V_{\geq 3}$ are the endvertices of an i -path P , then we call P *good*, if either $i \in \{1, 3\}$ and u and v do not both lie in one of the two sets D_1 and D_2 , or $i = 2$ and u and v both lie in one of the two sets D_1 and D_2 , i.e.

$$\begin{aligned} \text{either } & i \in \{1, 3\} \quad \text{and } |\{u, v\} \cap D_1| = |\{u, v\} \cap D_2| = 1, \\ \text{or } & i = 2 \quad \text{and } \{|\{u, v\} \cap D_1|, |\{u, v\} \cap D_2|\} = \{0, 2\}. \end{aligned}$$

We call i -paths *bad*, if they are not good and denote the number of bad i -paths by b_i for $1 \leq i \leq 3$.

We assume that the vertices in $V_{\geq 3} = V^*$ are added to either D_1 or D_2 in such a way that the total number of bad i -paths is as small as possible, i.e.

$$(b_1 + b_2 + b_3) \rightarrow \min. \quad (4)$$

Next, for every good i -path, we add $i - 1$ of the internal vertices to either D_1 or D_2 and for every bad i -path, we add all i internal vertices to either D_1 or D_2 in such a way that (D_1, D_2) dominates all vertices of degree 2 and as many vertices of degree at least 3 as possible, i.e. if \dot{V}_i and $\dot{V}_{\geq i}$ denote the sets of vertices in V_i and $V_{\geq i}$ which are not — yet — dominated by (D_1, D_2) , $\dot{n}_i = |\dot{V}_i|$, and $\dot{n}_{\geq i} = |\dot{V}_{\geq i}|$, then

$$\dot{n}_{\geq 3} \rightarrow \min. \quad (5)$$

Clearly, we may assume that the internal vertices of all i -paths are added to either D_1 or D_2 as indicated in Figure 1 where all vertices within squares belong to one of the two sets D_1 or D_2 and all vertices within cycles belong to the other set.

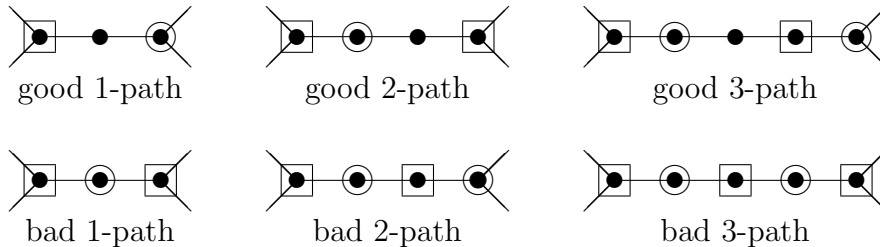


Figure 1

Let \ddot{V}_j and $\ddot{V}_{\geq j}$ denote the set of vertices in V_j and $V_{\geq j}$ which do not belong to a bad i -path or a good 3-path. Let $\ddot{n}_j = |\ddot{V}_j|$ and $\ddot{n}_{\geq j} = |\ddot{V}_{\geq j}|$. Since all vertices in $V_{\geq 3}$ which lie on a bad i -path or a good 3-path are already dominated by (D_1, D_2) , we have

$$\dot{n}_3 \leq n_3 \quad (6)$$

and

$$\dot{n}_{\geq 3} \leq \ddot{n}_{\geq 3}. \quad (7)$$

Claim 1

$$(b_1 + b_2 + b_3) \leq \frac{1}{2}(p_1 + p_2 + p_3) - \frac{1}{4}n_3 - \ddot{n}_{\geq 4} - \frac{1}{2}\ddot{n}_3 \quad (8)$$

Proof of Claim 1: It follows by the handshaking lemma that

$$2(p_1 + p_2 + p_3) = \sum_{i \geq 3} i n_i.$$

Furthermore, by (4), every vertex in $V_{\geq 3}$ belongs at least to as many good i -paths than bad i -paths. Therefore, another application of the handshaking lemma yields

$$\begin{aligned} 2 \left(\sum_{i=1}^3 p_i - \sum_{i=1}^3 b_i \right) &\geq \sum_{i \geq 3} i \ddot{n}_i + \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor (n_i - \ddot{n}_i) \\ &= \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i + \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor n_i. \end{aligned}$$

Combining these two observations, we obtain

$$\begin{aligned} 2(b_1 + b_2 + b_3) &\leq 2(p_1 + p_2 + p_3) - \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i - \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor n_i \\ &= (p_1 + p_2 + p_3) + \sum_{i \geq 3} \frac{i}{2} n_i - \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i - \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor n_i \\ &\leq (p_1 + p_2 + p_3) - \frac{1}{2}n_3 - 2\ddot{n}_{\geq 4} - \ddot{n}_3 \end{aligned}$$

which is equivalent to the statement of the claim. \square

For the purpose of the present proof we will consider a suitable directed graph

$$\vec{G}^*$$

with vertex set $V^* = V_{\geq 3}$ which contains a directed edge (u, v) from u to v for every good 2-path $P : uxyv$ in G such that $y \in D_1 \cup D_2$, i.e. a directed edge “ (u, v) ” indicates that v is already properly dominated by the vertices on P . (Note that \vec{G}^* can contain multiple directed edges.)

For a vertex $u \in V_{\geq 3}$ let

$$T_u$$

denote the set of vertices $v \in V_{\geq 3}$ such that \vec{G}^* contains a directed path from u to v .

Claim 2 *If $v \in T_u$ for some $u \in \dot{V}_{\geq 3}$, then v is not contained in a bad i -path or a good 3-path in G and v is not the endvertex of two directed edges in \vec{G}^* .*

Proof of Claim 2: For contradiction, we assume that vertices u and v as stated in the claim exist.

Let $P : u_0 u_1 \dots u_l$ be a directed path in \vec{G}^* from $u = u_0$ to $v = u_l$. By definition, every directed edge (u_{r-1}, u_r) for some $1 \leq r \leq l$ corresponds to a good 2-path $P_r : u_{r-1} x_r y_r u_r$ with $y_r \in D_s$ for some fixed $s \in \{1, 2\}$. If we replace the vertex y_r in D_s with x_r for $1 \leq r \leq l$, then, by the assumption, all vertices which were dominated by (D_1, D_2) — in particular v — are still dominated by the new pair and the total number of bad i -path remains unchanged. Since u is dominated by the new pair, $\dot{n}_{\geq 3}$ is reduced by 1, which is a contradiction to (5). \square

By Claim 2, the sets T_u for $u \in \dot{V}_{\geq 3}$ induce disjoint rooted trees \vec{T}_u within \vec{G}^* with root u . Furthermore, again by Claim 2, every leaf of \vec{T}_u which is different from u is the endvertex of at least two good 1-paths. Clearly, the sum of the number of good 1-paths which contain u and the number of leaves in \vec{T}_u is at least $d_G(u) \geq 3$. Therefore, we can associate 3 good 1-paths to every vertex in $\dot{V}_{\geq 3}$ such that every good 1-path is associated at most twice to vertices in $\dot{V}_{\geq 3}$. By double counting, we obtain

$$\dot{n}_{\geq 3} \leq \frac{2}{3}(p_1 - b_1) \leq \frac{2}{3}p_1. \quad (9)$$

We now turn (D_1, D_2) into a dominating pair of G by adding at most $\dot{n}_{\geq 3}$ vertices to the two sets and possibly moving some vertices from D_s to D_{3-s} , if all their neighbours belong to D_s .

We are ready to estimate the cardinality of (D_1, D_2) .

$$\begin{aligned} |D_1 \cup D_2| &\leq n_{\geq 3} + b_1 + p_2 + b_2 + 2p_3 + b_3 + \dot{n}_{\geq 3} \\ &\stackrel{(8)}{\leq} n_{\geq 3} + \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 - \frac{1}{4}n_3 - \ddot{n}_{\geq 4} - \frac{1}{2}\ddot{n}_3 + \dot{n}_{\geq 3} \\ &= \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 + (\dot{n}_{\geq 4} - \ddot{n}_{\geq 4}) + \frac{1}{2}(\dot{n}_3 - \ddot{n}_3) \\ &\stackrel{(7)}{\leq} \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 \\ &\stackrel{(9)}{\leq} \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 + \left(\frac{1}{4}p_1 - \frac{3}{8}\dot{n}_3\right) \\ &\stackrel{(6)}{\leq} \frac{3}{4}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{7}{8}n_3 + n_{\geq 4} \\ &\stackrel{(3)}{\leq} \frac{3}{4}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{7}{8}n_3 + n_{\geq 4} + \left(\frac{1}{14}(p_1 + p_2 + p_3) - \frac{3}{28}n_3 - \frac{1}{7}n_{\geq 4}\right) \\ &= \frac{23}{28}p_1 + 2 \cdot \frac{11}{14}p_2 + 3 \cdot \frac{6}{7}p_3 + \frac{43}{56}n_3 + \frac{6}{7}n_{\geq 4} \\ &\stackrel{(2)}{\leq} \frac{6}{7}|V|, \end{aligned}$$

where equality is only possible if $p_1 = p_2 = n_3 = 0$, i.e. every vertex in G belongs to a 3-path and no vertex has degree exact 3.

In this case

$$|V| = 3p_3 + n_{\geq 4}, \quad (10)$$

$$p_3 \geq 2n_{\geq 4} \quad (11)$$

and we construct a dominating pair (D_1, D_2) for G in the following way: First we add all vertices in $V_{\geq 4}$ to either D_1 or D_2 in such a way that the number of bad 3-paths is minimum as in (4). Clearly, every vertex in $V_{\geq 4}$ belongs to a good 3-path. Therefore, we can turn (D_1, D_2) to a dominating pair of G by adding exactly two internal vertices of every 3-path to either D_1 or D_2 as indicated in Figure 2.

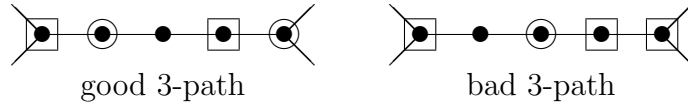


Figure 2

Now

$$\begin{aligned} |D_1 \cup D_2| &\leq n_{\geq 4} + 2p_3 \\ &\stackrel{(11)}{\leq} n_{\geq 4} + 2p_3 + \left(\frac{1}{7}p_3 - \frac{2}{7}n_{\geq 4} \right) \\ &= \frac{5}{7}n_{\geq 3} + \frac{15}{7}p_3 \\ &\stackrel{(10)}{\leq} \frac{5}{7}|V| \\ &< \frac{6}{7}|V|, \end{aligned}$$

and the proof is complete. \square

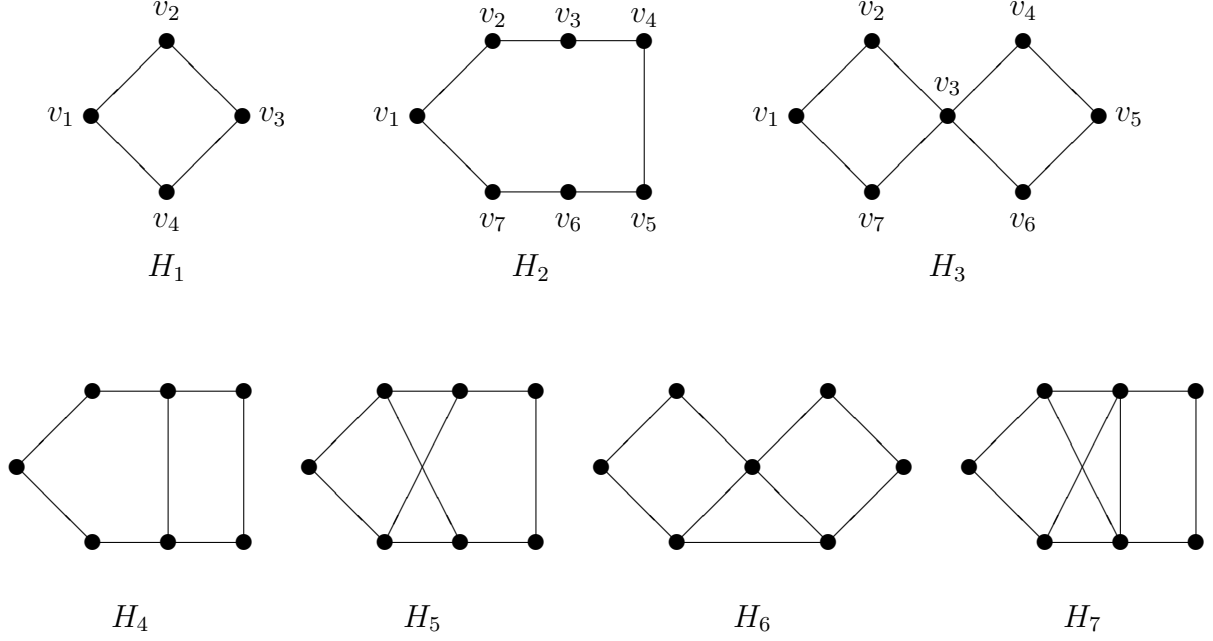


Figure 3

Lemma 2 (i) $\gamma\gamma(H_1) = 4$, $\gamma\gamma(H_2) = \dots = \gamma\gamma(H_7) = 6$.

(ii) If $G = (V, E) \in \{H_1, H_2, H_3\}$ and $v \in V$, then G has a minimum dominating pair (D_1, D_2) such that $v \in D_1$.

(iii) If $G = (V, E) \in \{H_1, H_2, H_3\}$ and $v \in V$, then there is a pair (D_1, D_2) of disjoint sets of vertices of G such that $|D_1 \cup D_2| = \gamma\gamma(G) - 1$, $v \in D_1$, D_1 is a dominating set, and $V \setminus \{v\} \subseteq N_G[D_2]$.

(iv) If G arises from a path $P : v_1v_2 \dots v_rv_{r+1} \dots v_{r+s}$ by adding the edge v_1v_r such that $r \in \{3, 4, 5\}$ and $s \in \{1, 3, 4, 5\}$, then G has a minimum dominating pair (D_1, D_2) with $v_{r+s} \in D_1$, $v_{r+s-1} \in D_2$, and $v_r \subseteq D_1 \cup D_2$. Furthermore, $\gamma\gamma(G) \leq \frac{6}{7}|V|$ with equality if and only if $(r, s) = (4, 3)$.

Proof: Since (i) is easily verified, we proceed to (ii).

Clearly, $(\{v_1, v_3\}, \{v_2, v_4\})$ is a dominating pair of H_1 , $(\{v_1, v_3, v_6\}, \{v_2, v_4, v_7\})$ is a dominating pair of H_2 , and $(\{v_1, v_5, v_6\}, \{v_3, v_4, v_7\})$ is a dominating pair of H_3 . By symmetry - considering suitable automorphisms of the graphs, (ii) follows.

If $G = H_1$, then let $(D_1, D_2) = (\{v_1, v_2\}, \{v_3\})$, and, if $G = H_2$, then let $(D_1, D_2) = (\{v_1, v_4, v_5\}, \{v_3, v_6\})$. In both cases $v_1 \in D_1$, D_1 is dominating, and $V \setminus \{v_1\} \subseteq N_G[D_2]$ which, by symmetry, implies (iii) for $G \in \{H_1, H_2\}$.

If $G = H_3$ and $(D_1, D_2) = (\{v_1, v_4, v_6\}, \{v_3, v_5\})$, then $v_1 \in D_1$, D_1 is dominating and $V \setminus \{v_1\} \subseteq N_G[D_2]$. If $G = H_3$ and $(D_1, D_2) = (\{v_2, v_3, v_6\}, \{v_5, v_7\})$, then $v_2 \in D_1$, D_1 is dominating and $V \setminus \{v_2\} \subseteq N_G[D_2]$. Finally, if $G = H_3$ and $(D_1, D_2) = (\{v_3, v_6, v_7\}, \{v_1, v_5\})$, then $v_3 \in D_1$, D_1 is dominating and $V \setminus \{v_3\} \subseteq N_G[D_2]$. By symmetry, the above observations imply (iii) for $G = H_3$.

Now let G be as in (iv). It is easy to verify that the Table 1 defines suitable minimum dominating pairs for G which completes the proof. \square

r	s	D_1	D_2
3	1	$\{v_2, v_4\}$	$\{v_3\}$
3	3	$\{v_3, v_6\}$	$\{v_2, v_5\}$
3	4	$\{v_2, v_4, v_7\}$	$\{v_3, v_6\}$
3	5	$\{v_3, v_5, v_8\}$	$\{v_2, v_4, v_7\}$
4	1	$\{v_2, v_5\}$	$\{v_3, v_4\}$
4	3	$\{v_2, v_4, v_7\}$	$\{v_1, v_3, v_6\}$
4	4	$\{v_2, v_5, v_8\}$	$\{v_3, v_4, v_7\}$
4	5	$\{v_3, v_4, v_6, v_9\}$	$\{v_2, v_5, v_8\}$
5	1	$\{v_2, v_4, v_6\}$	$\{v_3, v_5\}$
5	3	$\{v_3, v_5, v_8\}$	$\{v_2, v_4, v_7\}$
5	4	$\{v_2, v_4, v_6, v_9\}$	$\{v_3, v_5, v_8\}$
5	5	$\{v_3, v_5, v_7, v_{10}\}$	$\{v_2, v_4, v_6, v_9\}$

Table 1

Lemma 3 *If $G = (V, E)$ is a graph such that*

- (i) $\delta(G) \geq 2$,
- (ii) G is connected,
- (iii) $V_{\geq 3}$ is independent, and
- (iv) $G \notin \{H_1, H_2, H_3\}$,

then G has a dominating pair (D_1, D_2) with $V_{\geq 3} \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V|$.

Proof: For contradiction, we assume that $G = (V, E)$ is a counterexample of minimum order. It is easy to check that $|V| \geq 5$.

Claim 1 *There is no path $P : v_1v_2v_3v_4v_5$ in G such that the vertices v_1, v_2, v_3 , and v_4 are of degree 2 and $v_1v_5 \notin E$.*

Proof of Claim 1: For contradiction, we assume that a path P as described in the claim exists. The graph

$$\begin{aligned} G' &= G[V \setminus \{v_2, v_3, v_4\}] + v_1v_5 \\ &= (V \setminus \{v_2, v_3, v_4\}, (E \setminus \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}) \cup \{v_1v_5\}) \end{aligned}$$

satisfies (i)-(iii) of the hypothesis.

If $G' \in \{H_1, H_2, H_3\}$, then G is either H_2 , or a cycle of length 10 or arises from H_3 by subdividing one edge three times. In all three cases the desired result follows easily. Hence, we may assume that $G' \notin \{H_1, H_2, H_3\}$.

By the choice of G , this implies the existence of a dominating pair (D'_1, D'_2) of G' with $V_{\geq 3} = V'_{\geq 3} \subseteq D'_1 \cup D'_2$ and $|D'_1 \cup D'_2| < \frac{6}{7}(|V| - 3)$. Since $d_{G'}(v_1) = 2$, either v_1 or v_5 belong to $D'_1 \cup D'_2$.

If $v_1 \notin D'_1 \cup D'_2$ and $v_5 \in D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_3\}, D'_2 \cup \{v_2\})$, if $v_1 \in D'_1$ and $v_5 \in D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_4\}, D'_2 \cup \{v_2\})$, and if $v_1 \in D'_1$ and $v_5 \notin D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_4\}, D'_2 \cup \{v_3\})$. In all three cases (D_1, D_2) is a dominating pair of G with

$$|D_1 \cup D_2| = |D'_1 \cup D'_2| + 2 < \frac{6}{7}(|V| - 3) + 2 < \frac{6}{7}|V|$$

which is a contradiction. By symmetry, this completes the proof. \square

Claim 2 *There is no cycle $C : v_1v_2v_3v_4v_1$ in G such that $d_G(v_1) + d_G(v_3) \geq 7$, $d_G(v_2) = d_G(v_4) = 2$ and $G[V \setminus \{v_2, v_4\}]$ has two components with vertex sets $\{v_1\} \cup U_1$ and $\{v_3\} \cup U_3$ such that $v_1 \notin U_1$ and $v_3 \notin U_3$. (Note that one of the two sets U_1 and U_3 may be empty.)*

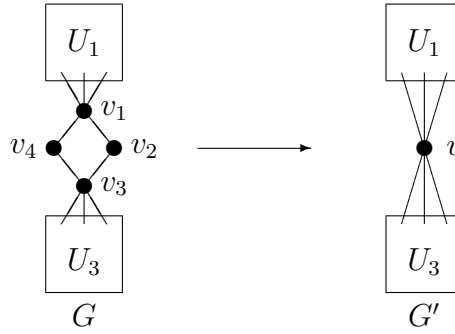


Figure 4

Proof of Claim 2: For contradiction, we assume that a cycle C as described in the claim exists. The graph G' which arises by contracting the cycle C to a single vertex v satisfies (i)-(iii) of the hypothesis. Since $d_{G'}(v) \geq 3$, the graph G' is different from H_1 . Therefore, by Lemma 2 (i) and the choice of G , G' has a dominating pair (D'_1, D'_2) such that $v \in D'_1$ and $|D'_1 \cup D'_2| \leq \frac{6}{7}(|V| - 3)$. By symmetry, we may assume that v has a neighbour v' in $D'_2 \cap V_1$. Now (D_1, D_2) with

$$\begin{aligned} D_1 &= \{v_1, v_2\} \cup (D'_1 \cap U_1) \cup (D'_2 \cap U_3) \text{ and} \\ D_2 &= \{v_3\} \cup (D'_2 \cap U_1) \cup (D'_1 \cap U_3) \end{aligned}$$

is a dominating pair of G with

$$|D_1 \cup D_2| = |(D'_1 \setminus \{v\}) \cup D'_2| + 3 \leq \left(\frac{6}{7}(|V| - 3) - 1 \right) + 2 < \frac{6}{7}|V|,$$

which is a contradiction. \square

Claim 3 *There are no six vertices $v_1, v_2, v_3, v_4, v_5, v_6 \in V$ such that*

$$v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_2v_6, v_4v_6 \in E,$$

$v_1, v_3, v_5,$ and v_6 are of degree 2, v_2 and v_4 are of degree 3, $G[V \setminus \{v_2\}]$ is not connected.

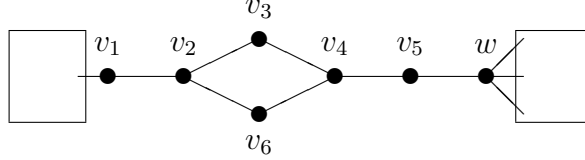


Figure 5

Proof of Claim 3: For contradiction, we assume that six vertices v_1, v_2, \dots, v_6 as described in the claim exist. Let w be the neighbour of v_5 different from v_4 . The graph

$$G' = G[V \setminus \{v_2, v_3, v_4, v_5, v_6\}] + v_1w$$

satisfies (i)-(iii) of the hypothesis.

Since the edge v_1w is a bridge of G' , $G' \notin \{H_1, H_2, H_3\}$. By the choice of G , this implies the existence of a dominating pair (D'_1, D'_2) of G' with $V_{\geq 3} \setminus \{v_2, v_4\} \subseteq D'_1 \cup D'_2$ and $|D'_1 \cup D'_2| < \frac{6}{7}(|V| - 5)$. Since $d_{G'}(v_1) = 2$, either $v_1 \in D'_1 \cup D'_2$ or $w \in D'_1 \cup D'_2$.

If $v_1 \notin D'_1 \cup D'_2$ and $w \in D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_4, v_6\}, D'_2 \cup \{v_2, v_3\})$, if $v_1 \in D'_1$ and $w \notin D'_1 \cup D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_2, v_5\}, D'_2 \cup \{v_3, v_4\})$, if $v_1 \in D'_1$ and $w \in D'_1$, then let $(D_1, D_2) = (D'_1 \cup \{v_4\}, D'_2 \cup \{v_2, v_5\})$, and if $v_1 \in D'_1$ and $w \in D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_4, v_5\}, D'_2 \cup \{v_2, v_3\})$. In all four cases (D_1, D_2) is a dominating pair of G with

$$|D_1 \cup D_2| \leq |D'_1 \cup D'_2| + 4 \leq \frac{6}{7}(|V| - 5) + 4 < \frac{6}{7}|V|$$

which is a contradiction. By symmetry, this completes the proof. \square

By Claim 1, for every i -path in G we have $i \in \{1, 2, 3\}$ and for every i -cycle in G we have $i \in \{2, 3, 4\}$.

If G has no i -cycle, then the desired result follows from Theorem 1. Hence, we may assume that

$$C : v_1v_2 \dots v_rv_1$$

with $r \in \{3, 4, 5\}$ is an $(r - 1)$ -cycle and $d_G(v_r) \geq 3$. If $d_G(v_r) = 3$, then there is an $(s - 1)$ -path

$$P : v_rv_{r+1} \dots v_{r+s}$$

in G with $s \in \{2, 3, 4\}$, $v_{r+1} \notin \{v_1, v_{r-1}\}$, and $d_G(v_{r+s}) \geq 3$. If $d_G(v_r) \geq 4$, then let $s = 0$, i.e. $s \in \{0, 2, 3, 4\}$.

Claim 4 *$d_G(v_r) \leq 4$ and, if $d_G(v_r) = 3$, then $d_G(v_{r+s}) = 3$.*

Proof of Claim 4: For contradiction, we assume that $d_G(v_r) \geq 5$ or that $d_G(v_r) = 3$ and $d_G(v_{r+s}) \geq 4$. The graph $G' = G[V \setminus \{v_1, v_2, \dots, v_{r+s-1}\}]$ satisfies (i)-(iii) of the hypothesis and is different from H_1 and H_2 . Therefore, by Lemma 2 (i) and the choice of G , G' has a dominating pair (D'_1, D'_2) such that $v_{r+s} \in D'_1$ and $|D'_1 \cup D'_2| \leq \frac{6}{7}(|V| - (r + s - 1))$.

Table 2 summarizes how to construct a suitable dominating pair (D_1, D_2) for G which yields a contradiction and completes the proof of the claim. \square

r	s	$D_1 \setminus D'_1$	$D_2 \setminus D'_2$
3	0	\emptyset	$\{v_1\}$
3	2	$\{v_2\}$	$\{v_3\}$
3	3	$\{v_3\}$	$\{v_2, v_4\}$
3	4	$\{v_2, v_4\}$	$\{v_1, v_5\}$
4	0	$\{v_3\}$	$\{v_2\}$
4	2	$\{v_1, v_3\}$	$\{v_2, v_4\}$
4	3	$\{v_3, v_4\}$	$\{v_2, v_5\}$
4	4	$\{v_2, v_5\}$	$\{v_1, v_3, v_6\}$
5	0	$\{v_3\}$	$\{v_1, v_4\}$
5	2	$\{v_2, v_4\}$	$\{v_3, v_5\}$
5	3	$\{v_3, v_5\}$	$\{v_2, v_4, v_6\}$
5	4	$\{v_2, v_4, v_6\}$	$\{v_1, v_3, v_7\}$

Table 2

By Claim 4, v_{r+s} has exactly two neighbours $x, y \notin \{v_1, v_2, \dots, v_{r+s-1}\}$. By (iii), $d_G(x) = d_G(y) = 2$.

If $xy \in E$, then $V = \{v_1, v_2, \dots, v_{r+s}, x, y\}$ and the result follows easily using Lemma 2 (iv). Therefore, the unique neighbour z of y different from v_{r+s} is different from x .

If $xz \in E$, then Claim 2 and Claim 3 imply that $V = \{v_1, v_2, \dots, v_{r+s}, x, y, z\}$ and the result follows easily. Therefore, $xz \notin E$.

The graph

$$G' = (V', E') = G[V \setminus \{v_1, v_2, \dots, v_{r+s}, y\}] + xz$$

satisfies (i)-(iii) of the hypothesis.

If $G' \in \{H_1, H_2, H_3\}$, then the desired result follows easily by combining Lemma 2 (iii) and (iv). Hence, we may assume that $G' \notin \{H_1, H_2, H_3\}$. This implies, by the choice of G , that G' has a dominating pair (D'_1, D'_2) with $V'_{\geq 3} \subseteq D'_1 \cup D'_2$ and $|D'_1 \cup D'_2| < \frac{6}{7}|V'|$. In this case, Lemma 3 (iv) easily implies that G has a dominating pair (D_1, D_2) with $V_{\geq 3} \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V|$ which is a contradiction and completes the proof. \square

Lemma 4 *If $G = (V, E)$ is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G connected,

(iii) G is edge-minimal with respect to (i)-(ii), and

(iv) $G \notin \{H_1, H_2, H_3\}$,

then $\gamma\gamma(G) < \frac{6}{7}|V|$.

Proof: Let $c(G)$ denote the number of 3-cycles of G with exactly one vertex of degree 3. For contradiction, we assume that $G = (V, E)$ is a counterexample for which $|V| + c(G)$ is minimum. Clearly, we may assume again that $|V| \geq 5$.

In view of Lemma 3, we may assume that $V_{\geq 3}$ is not independent, i.e. $v'v'' \in E$ for some $v', v'' \in V_{\geq 3}$. By (iii) of the hypothesis, the edge $v'v''$ must be a bridge, i.e. G arises from the disjoint union of two graphs $G' = (V', E')$ and $G'' = (V'', E'')$ by adding the bridge $v'v''$ where $v' \in V'$ and $v'' \in V''$. Note that G' and G'' satisfy (i)-(iii) of the hypothesis.

First, we assume that $G', G'' \in \{H_1, H_2, H_3\}$. In this case let (D'_1, D'_2) and (D''_1, D''_2) be as in Lemma 2 (iii) with $v' \in D'_1$ and $v'' \in D''_1$. Clearly, $(D'_1 \cup D''_2, D'_1 \cup D''_2)$ is a dominating pair of G and $|D'_1 \cup D''_2 \cup D'_1 \cup D''_1| < \frac{6}{7}|V|$ which is a contradiction.

Next, we assume that $G' \notin \{H_1, H_2, H_3\}$ and $G'' \neq H_1$. Since $c(G'), c(G'') \leq c(G) + 1$ and $|V'|, |V''| \geq 3$, we obtain, by the choice of G , $\gamma\gamma(G') < \frac{6}{7}|V'|$ and $\gamma\gamma(G'') \leq \frac{6}{7}|V''|$. If (D'_1, D'_2) and (D''_1, D''_2) are minimum dominating pairs of G' and G'' , then $(D_1, D_2) = (D'_1 \cup D''_1, D'_2 \cup D''_2)$ is a dominating pair of G with $|D_1 \cup D_2| < \frac{6}{7}|V|$ which is a contradiction.

Therefore, we may assume that $G' \notin \{H_1, H_2, H_3\}$ and $G'' = H_1$, i.e. G'' is a 3-cycle of G with exactly one vertex of degree 3. Let

$$G'' = (\{v'' = v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\})$$

and let

$$G''' = G - v_1v_4 + v'v_4 = (V, (E \setminus \{v_1v_4\}) \cup \{v'v_4\}).$$

Clearly, G''' satisfies (i)-(iii) of the hypothesis, $G''' \notin \{H_1, H_2, H_3\}$ and $c(G''') < c(G)$. Therefore, by the choice of G , we obtain that $\gamma\gamma(G''') < \frac{6}{7}|V|$.

Let (D'''_1, D'''_2) be a minimum dominating pair of G''' . Note that

$$|(D'''_1 \cup D'''_2) \cap \{v', v_1, v_2, v_3, v_4\}| \geq 4$$

and that we may assume $v' \in D'''_1$. Now, (D_1, D_2) with

$$\begin{aligned} D_1 &= (D'''_1 \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_3\} \text{ and} \\ D_2 &= (D'''_2 \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_1, v_2\} \end{aligned}$$

is a dominating pair of G with $|D_1 \cup D_2| < \frac{6}{7}|V|$ which is a contradiction.

This completes the proof. \square

Lemma 5 (McCuaig and Sherpherd, cf. Lemma 2 in [20]) *If $G = (V, E)$ is a connected graph with $|V| \leq 7$, $\delta(G) \geq 2$, and $\gamma(G) > \frac{2}{5}|V|$, then*

$$G \in \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}.$$

Theorem 6 *If $G = (V, E)$ is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G connected, and

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) < \frac{6}{7}|V|$.

Proof: Let $G' = (V', E')$ be a graph with $V' = V$ and $E' \subseteq E$ such that

(i) $\delta(G') \geq 2$,

(ii) G' connected, and

(iii) G' is edge-minimal with respect to (i)-(ii).

Clearly, $\gamma\gamma(G') \geq \gamma\gamma(G)$, and thus, by Lemma 4, the statement of the theorem is true, if $G' \notin \{H_1, H_2, H_3\}$.

If $G' = H_1$, then it is straightforward to check that $\gamma\gamma(G) \leq \frac{3}{4}|V|$, because $G \neq H_1$. Therefore, we may assume that $G' \in \{H_2, H_3\}$.

If G has a hamiltonian cycle and $\gamma(G) \leq 2$, then $\gamma\gamma(G) \leq 5$, because for any 2 vertices $v_i, v_j \in V$ there exists a dominating set of G of cardinality 3 that does not contain v_i or v_j . Thus, if $G' = H_2$, then, by Lemma 5, $\gamma\gamma(G) \leq \frac{5}{7}|V|$, because $G \notin \{H_2, H_4, H_5, H_6, H_7\}$.

Hence we may assume that G has no hamiltonian cycle and $G' = H_3$. If $G'' = (V'', E'')$ is a graph that arises from H_3 by adding an edge $e \in E \setminus E'$, then $\gamma\gamma(G'') \geq \gamma\gamma(G)$. By symmetry, $e \in \{v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_2v_7\}$ (cf. Figure 3). Thus $\gamma\gamma(G'') \leq \frac{5}{7}$ or $G'' = H_6$ in which case G has a hamiltonian cycle — a contradiction. This completes the proof. \square

While Theorem 6 is best-possible in view of the graphs H_2, H_3, \dots, H_7 , we believe that the following considerable strengthening is possible.

Conjecture 7 *If $G = (V, E)$ is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G connected, and

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) \leq \frac{4}{5}|V|$.

By the results of McCuaig and Shepherd [20], there would be infinitely many extremal graphs for this estimate. In fact, we believe that the edge-minimal extremal graphs for the bound in Conjecture 7 are the same as those described in [20] for the bound $\gamma(G) \leq \frac{2}{5}|V|$.

3 Graph with Minimum Degree at least 5

In this section we prove an upper bound on $\gamma\gamma(G)$ for graphs G using the probabilistic method.

The proof builds on an elegant probabilistic argument given by Alon and Spencer [1]. Several times during the proof we will use Ore's observation [21] that the complement of a minimal dominating set in a graph of minimum degree at least 1 is also a dominating set.

Theorem 8 *If $G = (V, E)$ is a graph of order n and minimum degree $\delta \geq 5$, then*

$$\gamma\gamma(G) \leq 2 \frac{1 + \ln(\delta + 1)}{\delta + 1} n.$$

Proof: Let $p = \frac{\ln(\delta+1)}{\delta+1}$. Note that $p \leq \frac{1}{2}$.

We construct a partition of V into three sets

$$V = D_1^0 \cup D_2^0 \cup Y$$

by assigning every vertex independently at random to the set D_1^0 with probability p , to the set D_2^0 with probability p , and to the set Y with probability $(1 - 2p)$.

Clearly, $\mathbf{E}[|D_1^0|] = \mathbf{E}[|D_2^0|] = np$.

Let

$$Z^1 = \{v \in V \mid N_G[v] \cap (D_1^0 \cup D_2^0) = \emptyset\}.$$

For a fixed vertex $v \in V$, we have

$$\mathbf{P}[v \in Z^1] = \mathbf{P}[N_G[v] \subseteq Y] = (1 - 2p)^{d_G(v)+1}.$$

Let

$$D_1^1$$

be a minimal dominating set of $G[Z^1]$ and let

$$D_2^1$$

be the union of $Z^1 \setminus D_1^1$ and a minimal set of vertices of G such that each isolated vertex in $G[Z^1]$ has a neighbour in D_2^1 . Clearly, $D_2^1 \subseteq Y \setminus D_1^1$ and (D_1^1, D_2^1) dominates every vertex in Z^1 .

Note that $|D_1^1| + |D_2^1| \leq 2|Z^1|$ and thus

$$\mathbf{E}[|D_1^1| + |D_2^1|] \leq 2 \sum_{v \in V} (1 - 2p)^{d_G(v)+1}.$$

Let

$$Z_1^2 = \{v \in V \mid N_G[v] \cap (D_1^0 \cup D_1^1) = \emptyset\}.$$

Note that $|N_G[v] \cap D_2^0| \geq 1$ for each $v \in Z_1^2$, since otherwise $v \in Z^1$ and thus $|N_G[v] \cap D_1^1| \geq 1$ - a contradiction to $v \in Z_1^2$.

For a fixed vertex $v \in V$,

$$\begin{aligned}
\mathbf{P}[v \in Z_1^2] &= \mathbf{P}[N_G[v] \cap (D_1^0 \cup D_1^1) = \emptyset] \\
&\leq \mathbf{P}[(N_G[v] \cap D_1^0 = \emptyset) \wedge (N_G[v] \cap D_2^0 \neq \emptyset)] \\
&= \mathbf{P}[N_G[v] \cap D_1^0 = \emptyset] - \mathbf{P}[N_G[v] \cap (D_1^0 \cup D_2^0) = \emptyset] \\
&= (1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}.
\end{aligned}$$

Let

$$D_1^2$$

be a minimal set of vertices in $V \setminus (D_2^0 \cup D_2^1)$ such that each vertex $v \in Z_1^2$ which satisfies

$$|N_G[v] \cap (D_2^0 \cup D_2^1)| < d_G(v) + 1$$

is dominated by D_1^2 . Note that $|D_1^2| \leq |Z_1^2|$ and thus

$$\mathbf{E}[|D_1^2|] \leq \sum_{v \in V} ((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}).$$

Let

$$Z_2^2 = \{v \in V \mid N_G[v] \cap (D_2^0 \cup D_2^1) = \emptyset\}.$$

Note that $|N_G[v] \cap D_1^0| \geq 1$ for each $v \in Z_2^2$, since otherwise $v \in Z^1$ and thus $|N_G[v] \cap D_2^1| \geq 1$ - a contradiction to $v \in Z_2^2$.

For a fixed vertex $v \in V$,

$$\begin{aligned}
\mathbf{P}[v \in Z_2^2] &= \mathbf{P}[N_G[v] \cap (D_2^0 \cup D_2^1) = \emptyset] \\
&\leq \mathbf{P}[(N_G[v] \cap D_2^0 = \emptyset) \wedge (N_G[v] \cap D_1^0 \neq \emptyset)] \\
&= \mathbf{P}[N_G[v] \cap D_2^0 = \emptyset] - \mathbf{P}[N_G[v] \cap (D_2^0 \cap D_1^0) = \emptyset] \\
&= (1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}.
\end{aligned}$$

Let

$$D_2^2$$

be a minimal set of vertices in $V \setminus (D_1^0 \cup D_1^1 \cup D_1^2)$ such that each vertex $v \in Z_2^2$ which satisfies

$$|N_G[v] \cap (D_1^0 \cup D_1^1 \cup D_1^2)| < d_G(v) + 1$$

is dominated by D_2^2 . Note that $|D_2^2| \leq |Z_2^2|$ and thus

$$\mathbf{E}[|D_2^2|] \leq \sum_{v \in V} ((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}).$$

For $i \in \{1, 2\}$ let

$$D'_i = D_i^0 \cup D_i^1 \cup D_i^2.$$

Clearly, $D'_1 \cap D'_2 = \emptyset$.

For $i \in \{1, 2\}$ let

$$X_i = \{v \in V \mid N_G[v] \subseteq D'_i\}.$$

Let D_i^3 be a minimal dominating set of $G[X_{3-i}]$ for $i \in \{1, 2\}$.

Let

$$\begin{aligned} D_1 &= (D'_1 \setminus D_2^3) \cup D_1^3 \text{ and} \\ D_2 &= (D'_2 \setminus D_1^3) \cup D_2^3. \end{aligned}$$

Clearly, (D_1, D_2) is a dominating pair of G and, by the first moment method [1], we obtain

$$\begin{aligned} \gamma\gamma(G) &\leq \mathbf{E}[|D_1| + |D_2|] \\ &= \mathbf{E}[|(D'_1 \setminus D_2^3) \cup D_1^3|] + \mathbf{E}[|(D'_2 \setminus D_1^3) \cup D_2^3|] \\ &= \mathbf{E}[|D'_1|] + \mathbf{E}[|D'_2|] \\ &= \mathbf{E}[|D_1^0 \cup D_1^1 \cup D_1^2|] + \mathbf{E}[|D_1^0 \cup D_1^1 \cup D_1^2|] \\ &\leq 2np + 2 \sum_{v \in V} (1 - 2p)^{d_G(v)+1} + 2 \sum_{v \in V} ((1 - p)^{d_G(v)+1} - (1 - 2p)^{d_G(v)+1}) \\ &= 2np + 2 \sum_{v \in V} (1 - p)^{d_G(v)+1} \\ &\leq 2np + 2n(1 - p)^{\delta+1} \\ &\leq 2np + 2ne^{-p(\delta+1)} \\ &= 2n \frac{1 + \ln(\delta + 1)}{\delta + 1} \end{aligned}$$

which completes the proof. \square

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