Cycle Length Parities and the Chromatic Number

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Abstract. In 1966 Erdős and Hajnal proved that the chromatic number of graphs whose odd cycles have lengths at most l is at most $l+1$. Similarly, in 1992 Gyárfás proved that the chromatic number of graphs which have at most k odd cycle lengths is at most $2k+2$ which was originally conjectured by Bollobás and Erdős.

Here we consider the influence of the parities of the cycle lengths modulo some odd prime on the chromatic number of graphs. As our main result we prove the following: Let p be an odd prime, $k \in \mathbb{N}$ and $I \subseteq \{0, 1, \ldots, p-1\}$ with $|I| \leq p-1$. If G is a graph such that the set of cycle lengths of G contains at most k elements which are not in I modulo p, then $\chi(G) \leq \left(1 + \frac{|I|}{p-|I|}\right) k + p(p-1)(r(2p, 2p)+1)+1$ where $r(p,q)$ denotes the ordinary Ramsey number.

Keywords. Graph; colouring; chromatic number; cycle

1 Introduction

We consider finite, simple and undirected graphs $G = (V, E)$ and denote by $\mathcal{L}(G)$ the set of cycle lengths of G. In this paper we continue the study of the influence of $\mathcal{L}(G)$ on the chromatic number $\chi(G)$ of G which was essentially initiated by Erdős and Hajnal in 1966.

Theorem 1 (Erdős and Hajnal [2]) *If G is a graph and l is the maximum odd element in $\mathcal{L}(G)$, then $\chi(G) \leq l + 1$.*

Bollobás and Erdős [1] conjectured the following strengthening which was eventually proved by Gyárfás in 1992.

Theorem 2 (Gyárfás [3]) *If G is a graph and $\mathcal{L}(G)$ contains k odd elements, then $\chi(G) \leq 2k+2$.*

In 2004 Mihok and Schiermeyer proved an analogous result for even cycle lengths.

Theorem 3 (Mihok and Schiermeyer [7]) *If G is a graph and $\mathcal{L}(G)$ contains k even elements, then $\chi(G) \leq 2k+3$.*

The extremal graphs for all three results are complete graphs of suitable order. The result of Erdős and Hajnal has recently been refined by excluding large cliques [4]. Similarly, the result of Gyárfás has been strengthened for graphs G for which $\mathcal{L}(G)$ contains two specified odd elements [5, 6, 9].

The starting point for our new results was the following observation which we made together with Frank Göring.

Observation 4 *Let p be an odd prime and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If G is a graph such that $\mathcal{L}(G)$ contains at most k elements which are not divisible by p, then $\chi(G) \leq 2k + 3$.*

Proof: We prove the result by induction on the order n of G. For $n \leq 2k + 3$ the result is trivial and we may assume that $n > 2k + 3$.

If G has a vertex u of degree at most $2k + 2$, then we can extend a $(2k + 3)$-coloring of $G - u$ to G. Hence we may assume that the minimum degree δ of G satisfies $\delta \geq 2k + 3$.

Let $P : v_1 v_2 \ldots v_l$ be a longest path in G.

Let $x_1, x_2, \ldots, x_{2k+3}$ be the first $2k + 3$ neighbours of x_0 on P ordered according to increasing distance from x_0 on P, i.e. if for $0 \leq i < j \leq 2k + 3$ the subpath $P(i, j)$ of P between x_i and x_j is of length $d(i, j)$, then

$$1 = d(0, 1) < d(0, 2) < \ldots < d(0, 2k + 3).$$

Since $P(1, i)$ together with the two edges $x_0 x_1$ and $x_0 x_i$ forms a cycle of length $d(1, i)+2$, there are at least $(2k + 3) - 1 - k = k + 2$ indices i with $2 \leq i \leq 2k + 3$ and

$$d(1, i) + 2 \equiv 0 \bmod p.$$

Let $i_0 < i_1 < \ldots < i_{k+1}$ be $k + 2$ such indices. Now

$$d(i_0, i_j) + 2 = (d(1, i_j) + 2) - (d(1, i_0) + 2) + 2 \equiv 2 \bmod p$$

and $P(i_0, i_j)$ together with the two edges $x_0 x_{i_0}$ and $x_0 x_{i_j}$ forms a cycle of length $d(i_0, i_j)+2$ for every $1 \leq j \leq k + 1$.

Hence G contains $k + 1$ cycles of different lengths not divisible by p which is a contradiction and the proof is complete. $\square$

In the next section we prove bounds for the chromatic number of graphs

- whose cycle lengths are either small or have a fixed parity modulo some odd prime p (Theorem 5),
- whose cycle lengths are either small or have a parity different from 2 modulo p (Proposition 6),
- whose cycle lengths are either small or have a parity modulo p belonging to some set of allowed parities (Corollary 8), and
- that have a bounded number of cycle lengths which do not have a parity modulo p belonging to some set of allowed parities (Theorem 7).

2 Results

We immediately proceed to our results.

Theorem 5 *Let p be an odd prime, $k \in \mathbb{N}$ with $k \geq 3$ and $q \in \{0, 1, \ldots, p-1\}$.*

If G is a graph in which all cycles of length at least kp have a length which is equivalent to q modulo p, then $\chi(G) \leq kp - \epsilon$ for $\epsilon = \begin{cases} 0 & , \text{ if } q = 0 \\ 1 & , \text{ if } q > 0. \end{cases}$

Proof: We prove the result by induction on the order $n = |V|$. As in the previous proof, we may assume that $n > kp - \epsilon$ and also that the minimum degree δ of G satisfies $\delta \geq kp - \epsilon$.

Let $P : v_1 v_2 \ldots v_l$ be a longest paths in G.

If $v_i \in N_G(v_1)$ and $i \geq 3$, then $v_1 v_2 \ldots v_i v_1$ is a cycle of length $\equiv i \bmod p$. Therefore, all neighbours v_i of v_1 with $i \geq kp$ satisfy $i \equiv q \bmod p$, i.e.

$$\{i \mid v_i \in N_G(v_1) \text{ and } kp \leq i \leq l\} \subseteq (k + \mathbb{N}_0)\, p + q.$$

Since $\delta \geq kp - \epsilon$, the vertex v_1 has at least one neighbour v_i with $i \geq kp$ and

$$t = \max\{i \in \mathbb{N}_0 \mid v_{(k+i)p+q} \in N_G(v_1)\}$$

is well-defined. Note that $q = 0$ implies $t \geq 1$.

If $v_i \in N_G(v_1)$ for some $2 \leq i \leq tp + q + 2$ with $i \not\equiv 2 \bmod p$, then $v_1 v_i v_{i+1} \ldots v_{(k+t)p+q} v_1$ is a cycle of length $(k+t)p + q - i + 2 \geq kp$ which is not equivalent to q modulo p. Hence

$$\{i \mid v_i \in N_G(v_1) \text{ and } 1 \leq i \leq tp + q + 2\} \subseteq \mathbb{N}_0 p + 2.$$

If $tp + q + 2 < kp$, then

$$\begin{array}{rcl} \delta & \leq & |N_G(v_1)| \\ & = & |N_G(v_1) \cap \{v_i \mid 1 \leq i \leq tp + q + 2\}| \\ & & + |N_G(v_1) \cap \{v_i \mid tp + q + 3 \leq i \leq kp - 1\}| \\ & & + |N_G(v_1) \cap \{v_i \mid kp \leq i \leq l\}| \\ & \leq & (t+1) + ((kp - 1) - (tp + q + 3) + 1) + (t + 1) \\ & = & kp - q - t(p-2) - 1 \\ & < & kp - \epsilon. \end{array}$$

Hence, we may assume that $tp + q + 2 \geq kp$.

If $q \neq 2$, then

$$\begin{array}{rcl} \delta & \leq & |N_G(v_1)| \\ & = & |N_G(v_1) \cap \{v_i \mid 1 \leq i \leq kp - 1\}| \\ & & + |N_G(v_1) \cap \{v_i \mid kp \leq i \leq tp + q + 2\}| \\ & & + |N_G(v_1) \cap \{v_i \mid tp + q + 3 \leq i \leq l\}| \\ & \leq & k + 0 + k \\ & \leq & 2k \\ & < & kp - \epsilon. \end{array}$$

Hence, we may assume that $q = 2$ which implies

$$\{i \mid v_i \in N_G(v_1) \text{ and } 1 \leq i \leq l\} \subseteq \mathbb{N}_0 p + 2.$$

We assume that P is chosen such that

$$j = \min\{i \in \mathbb{N} \mid v_{ip+2} \in N_G(v_1)\}$$

is minimum possible. Since $v_{jp+1}v_{jp}\dots v_1 v_{jp+2}v_{jp+3}\dots v_l$ is a longest path, this implies that

$$\{i \mid v_i \in N_G(v_{jp+1}) \text{ and } 1 \leq i \leq l\} \subseteq \{jp\} \cup (j + \mathbb{N}_0)\, p + 2$$

Let

$$s = \min\{i \mid i \geq jp + 3, v_i \in N_G(v_1) \cup N_G(v_{jp+1})\}.$$

Let $x \in \{1, jp+1\}$ be such that $v_s \in N_G(v_x)$ and let $\{y\} = \{1, jp+1\} \setminus \{x\}$. Let

$$t = \max\{i \mid i \geq jp + 3, v_i \in N_G(v_y)\}.$$

Since $|\{i \mid i \geq jp + 3, v_i \in N_G(v_y)\}| \geq \delta - 2 \geq kp - 3$, we have $t - s \geq (kp - 3)p \geq kp$ and $v_x \dots v_y v_t \dots v_s v_x$ is a cycle of length more than kp which is equivalent 4 modulo p. This contradiction completes the proof. $\square$

Proposition 6 *Let p be an odd prime and $k \in \mathbb{N}$ with $k \geq 3$.*

If G is a graph in which all cycles of length at least kp have a length which is not equivalent to 2 modulo p, then $\chi(G) \leq kp + 1$.

Proof: We proceed as in the proof of Theorem 5. Therefore, we may assume that the minimum degree δ of G satisfies $\delta \geq kp + 1$ and consider a longest paths $P : v_1 v_2 \dots v_l$ in G.

By the pigeon hole principle, there is some $q \in \{0, 1, \dots, p-1\}$ such that the set

$$I = \{i \mid 1 \leq i \leq l, v_i \in N_G(v_1), i \equiv q \bmod p\}$$

has at least $k + 1$ elements. If $s = \min I$ and $t = \max I$, then $v_1 v_s v_{s+1} \dots v_t v_1$ is a cycle of length at least kp which is equivalent to 2 modulo p. This contradiction completes the proof. $\square$

Theorem 5 and Proposition 6 are best-possible in view of complete graphs of suitable order.

Theorem 7 *Let p be an odd prime, $k \in \mathbb{N}$ and $I \subseteq \{0, 1, \dots, p-1\}$ with $|I| \leq p - 1$. If G is a graph such that $\mathcal{L}(G)$ contains at most k elements which are not in I modulo p, then*

$$\chi(G) \leq \left(1 + \frac{|I|}{p - |I|}\right) k + p(p-1)(r(2p, 2p) + 1) + 1$$

where $r(p, q)$ denotes the ordinary Ramsey number.

Proof: Let

$$\begin{aligned} f_0(p) &= p(p-1)(r(2p,2p)+1)+1, \\ f_1(p) &= p(p-1)(r(2p,2p)-1)+1, \\ f_2(p) &= (p-1)(r(2p,2p)-1)+1, \\ f_3(p) &= r(2p,2p), \text{ and} \\ f_4(p) &= 2p. \end{aligned}$$

We prove the result by induction on the order n of G.

As in the previous proofs, we may assume that $n > \left(1+\frac{|I|}{p-|I|}\right)k + f_0(p)$ and also that the minimum degree δ of G satisfies $\delta \geq \left(1+\frac{|I|}{p-|I|}\right)k + f_0(p)$.

Let $P : v_1 v_2 \ldots v_l$ be a longest path in G.

To simplify our terminology, we extend the order and parity of the indices to the vertices on P. (*For example, for $1 \leq i < j \leq l$ we say that v_i is smaller than v_j and write $v_i < v_j$. Similarly, we say that v_i is equivalent to some q modulo p if i is equivalent to q modulo p*).

Let N^1 denote the set of the $f_1(p)$ smallest neighbours of v_1 on P.

We assume that P is chosen such that $\max N^1$ is smallest possible.

By the pigeon hole principle, there is a set $N^2 \subseteq N^1$ with $|N^2| = f_2(p)$ such that all elements of N^2 are equivalent modulo p.

Let $v_i \in N^2$.

The path $v_{i-1}v_{i-2}\ldots v_1 v_i v_{i+1} \ldots v_l$ is a longest path in G. Hence the vertex v_{i-1} has all its neighbours on P. Furthermore, the choice of P implies that v_{i-1} has at most $f_1(p)$ neighbours which are smaller or equal to $\max N^1$.

Therefore, if $M^1(v_i)$ denotes the set of neighbours of v_{i-1} (*note the little index shift*) which are larger than $\max N^1$, then

$$|M^1(v_i)| \geq \left(1+\frac{|I|}{p-|I|}\right)k + f_0(p) - f_1(p) = \left(1+\frac{|I|}{p-|I|}\right)k + 2p(p-1).$$

By the hypothesis, there are at most k neighbours $v_j \in M^1(v_i)$ for which the cycle $v_{i-1}v_i \ldots v_j v_{i-1}$ has a length $j-i+2$ which is not in I modulo p. This implies that, if $M^2(v_i)$ denotes the set of all $v_j \in M^1(v_i)$ for which $j-i+2$ is in I modulo p, then

$$|M^2(v_i)| \geq \left(1+\frac{|I|}{p-|I|}\right)k + 2p(p-1) - k = \frac{|I|}{p-|I|}k + 2p(p-1).$$

By the pigeon hole principle, there is a set $M^3(v_i) \subseteq M^2(v_i)$ with

$$|M^3(v_i)| \geq \frac{k}{p-|I|} + 2p$$

such that all vertices in $M^3(v_i)$ are equivalent modulo p.

Again by the pigeon hole principle, there is a subset $N^3 \subseteq N^2$ with $|N^3| = f_3(p)$ such that all vertices in

$$\bigcup_{v_i \in N^3} M^3(v_i)$$

are equivalent modulo p.

By the Ramsey theorem, there is a subset $N^4 \subseteq N^3$ with $f_4(p)$ elements

$$N^4 = \left\{ v_{j^1} < v_{j^2} < \ldots < v_{j^{f_4(p)}} \right\}$$

such that, if

$$M^3(v_{j^i}) = \left\{ v_{j_1^i} < v_{j_2^i} < \ldots \right\}$$

for $1 \leq i \leq f_4(p)$, then either

$$j_1^1 < j_1^2 < \ldots < j_1^{f_4(p)}$$

or

$$j_1^1 \geq j_1^2 \geq \ldots \geq j_1^{f_4(p)}.$$

First, we assume that

$$j_1^1 < j_1^2 < \ldots < j_1^{f_4(p)}$$

holds (cf. Figure 1).

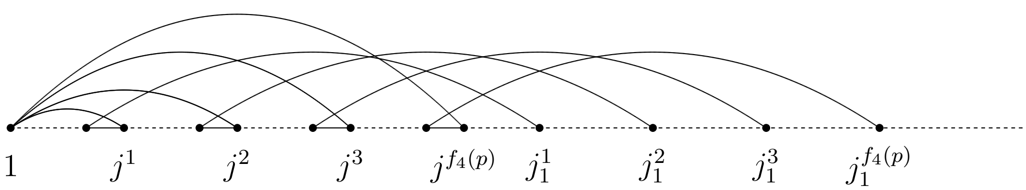

Figure 1

For $0 \leq r \leq p-1$ and $1 \leq s \leq \frac{k}{p-|I|} + 2p$ let $C(r,s)$ denote the cycle (cf. Figures 2 and 3)

$$\begin{aligned} C(r,s) \quad : \quad & v_1 v_{j^1} v_{j^1-1} v_{j_1^1} v_{j_1^1+1} \ldots \\ & v_{j_1^2} v_{j^2-1} v_{j^2} \ldots v_{j^3-1} v_{j_1^3} v_{j_1^3+1} \ldots \\ & v_{j_1^4} v_{j^4-1} v_{j^4} \ldots v_{j^5-1} v_{j_1^5} v_{j_1^5+1} \ldots \\ & \ldots \\ & v_{j_1^{2r}} v_{j^{2r}-1} v_{j^{2r}} \ldots v_{j^{2r+1}-1} v_{j_1^{2r+1}} v_{j_1^{2r+1}+1} \ldots \\ & v_{j_1^{f_4(p)}} v_{j_1^{f_4(p)}+1} \ldots v_{j_s^{f_4(p)}} v_{j^{f_4(p)}-1} v_{j^{f_4(p)}} v_1. \end{aligned}$$

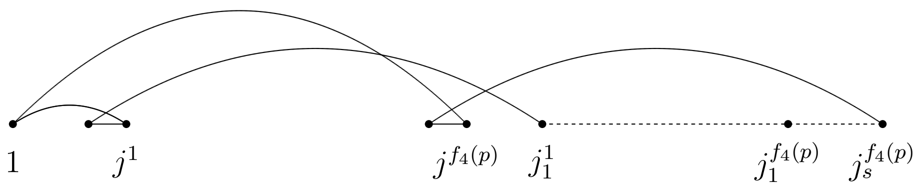

Figure 2 $C(0,s)$

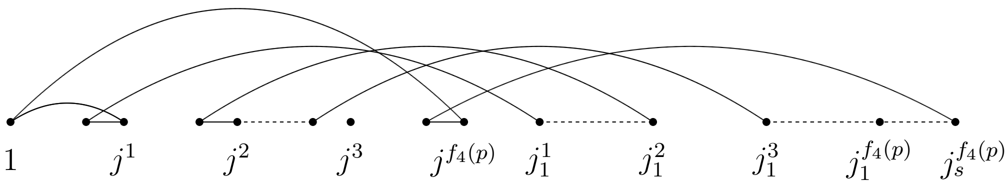

Figure 3 $C(1,s)$

For $0 \leq r \leq p-2$ and $1 \leq s_1, s_2 \leq \frac{k}{p-|I|} + 2p$ the lengths of the two cycles $C(r,s_1)$ and $C(r+1,s_2)$ differ exactly by 2 modulo p. Furthermore, for $0 \leq r \leq p-1$ and $1 \leq s_1 < s_2 \leq \frac{k}{p-|I|} + 2p$ the cycle $C(r,s_2)$ is longer than the cycle $C(r,s_1)$.

This implies the existence of

$$(p-|I|)\left(\frac{k}{p-|I|}+2p\right) > k$$

cycles of different lengths not in I modulo p, which is a contradiction.

In the second case

$$j_1^1 \geq j_1^2 \geq \ldots \geq j_1^{f_4(p)}$$

a very similar construction also leads to a contradicton. In order to ensure the appropriate vertices to be different one can discard the smallest $(2p-i)$ elements from every set $M^3(v_{j^i})$ for all $1 \leq i \leq 2p$. This completes the proof. $\Box$

Corollary 8 *Let p be an odd prime, $k \in \mathbb{N}$ and $I \subseteq \{0,1,\ldots,p-1\}$ with $|I| \leq p-1$.*

If G is a graph in which all cycles of length at least kp have a length which is in I modulo p, then

$$\chi(G) \leq kp + p(p-1)(r(2p,2p)+1)+1.$$

Proof: The result follows immediately from Theorem 7, because the hypothesis implies that $\mathcal{L}(G)$ contains at most $k(p-|I|)$ many cycle lengths which are not in I modulo p. $\Box$

While the complete graphs show that the term "$\left(1+\frac{|I|}{p-|I|}\right)$" in Theorem 7 and the term "kp" in Corollary 8 are best possible, the additive term depending only on p can clearly be improved. In fact, it is conceivable that it could be replaced by a constant.

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