On stability of linear switched differential algebraic equations

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***Abstract*— This paper studies linear switched differential algebraic equations (DAEs), i.e., systems defined by a finite family of linear DAE subsystems and a switching signal that governs the switching between them. We show by examples that switching between stable subsystems may lead to instability, and that the presence of algebraic constraints leads to a larger variety of possible instability mechanisms compared to those observed in switched systems described by ordinary differential equations (ODEs). We prove two sufficient conditions for stability of switched DAEs based on the existence of suitable Lyapunov functions. The first result states that a common Lyapunov function guarantees stability under arbitrary switching when an additional condition involving consistency projectors holds (this extra condition is not needed when there are no jumps, as in the case of switched ODEs). The second result shows that stability is preserved under switching with sufficiently large dwell time.**

I. Introduction

We consider linear *switched differential algebraic equations (switched DAEs)* of the form

$$E_\sigma \dot{x} = A_\sigma x, \tag{1}$$

where $\sigma : \mathbb{R} \to \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$, is some switching signal, and $E_p, A_p \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, are constant matrices for each parameter $p \in \{1, 2, \ldots, N\}$. The goal of this paper is to develop sufficient conditions for stability of such switched DAEs systems, based on the existence of appropriate Lyapunov functions.

When each matrix E_p is invertible, (1) reduces to a more familiar switched ordinary differential equation (switched ODE), or switched system. The stability theory of switched ODEs has received considerable attention in the last couple of decades, and is now relatively mature. In particular, it is well known that switching among stable subsystems may lead to instability; a switched system is asymptotically stable under arbitrary switching if (and only if) the subsystems share a common Lyapunov function; and stability is preserved under sufficiently slow switching, as can be shown using multiple Lyapunov functions (one for each subsystem). We refer the reader to the book [1] for these and other results on switched systems and for an extensive literature overview.

On the other hand, an investigation of stability questions for switched DAEs by similar methods has not yet appeared in the literature. In this paper, we begin such an investigation by establishing Lyapunov-based sufficient conditions for stability of switched DAEs. In the special case of switched ODEs, our results reduce to the known results mentioned above. However, we will demonstrate by means of examples that the presence of algebraic constraints leads to new types of instability mechanisms. It also poses additional technical challenges related to interpreting system solutions and defining suitable Lyapunov functions.

Note that the solutions of each individual DAE $E_p \dot{x} = A_p x$, $p \in \{1, \ldots N\}$ evolve within a so-called consistency space, which is a subspace of $\mathbb{R}^n$. In general, at a switching time $t \in \mathbb{R}$ there does not exist a continuous extension of the solution into the future, because the value $x(t-)$ immediately before the switch need not be within the consistency space corresponding to the DAE after the switch. Therefore, it is necessary to allow for solutions with jumps. However, this leads to difficulties in evaluating the derivative of the solutions. To resolve this problem we adopt the distributional framework introduced in [2], i.e. as solutions of the switched DAE (1) distributions (generalized functions), in particular Dirac impulses, are considered. For this, we have to assume that the switching signal has only a locally finite set of switching times. Furthermore, to ensure existence and uniqueness of solutions we have to assume that each matrix pair (E_p, A_p) is regular, i.e. the polynomial $\det(Es - A)$ is not identically zero (see also [3]). Finally, we will make one more assumption which ensures that no impulses occur in the solutions of the switched DAE (1); for details see Section IV and Theorem 8. A consequence of these assumptions is that although a distributional solution framework is necessary as a theoretical basis for treating switched DAEs (1), the only solutions that arise in this paper are normal (piecewise-smooth) functions.

The structure of the paper is as follows. Section II summarizes relevant results for classical DAEs (i.e. DAEs with constant coefficients), most of which can be found in, or follow from, [4]. In particular, Lyapunov functions for classical DAEs are introduced. The notion of a consistency projector, needed for studying switched DAEs, is also defined there. Section III gives several examples which motivate the stability problem for switched DAEs by illustrating some stable as well as unstable behaviors that can occur. In Section IV the stability of switched DAEs is studied. The main result is Theorem 9, where sufficient conditions for asymptotic stability of the switched DAE (1) under arbitrary switching are given. Theorem 11 shows that a switching signal is not destabilizing if it has a large enough dwell time,

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which is a generalization of the same result for switched ODEs to switched DAEs.

The following notation is used throughout the paper. $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ are the natural numbers, integers, real and complex numbers, respectively. The ring of polynomials with real coefficients and indeterminate s is $\mathbb{R}[s]$. For a matrix $M \in \mathbb{R}^{n\times m}$, $n, m \in \mathbb{N}$, the kernel (null space) of M is $\ker M$, the image (range, column space) of M is $\operatorname{im} M$, and the transpose of M is $M^\top \in \mathbb{R}^{m\times n}$. For a matrix $M \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subset \mathbb{R}^n$, the image of $\mathcal{S}$ under M is $M\mathcal{S} := \{\, Mx \in \mathbb{R}^n \mid x \in \mathcal{S} \,\}$ and the pre-image of $\mathcal{S}$ under M is $M^{-1}\mathcal{S} := \{\, x \in \mathbb{R}^n \mid \exists y \in \mathcal{S} : Mx = y \,\}$. For a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ the left-sided evaluation $\lim_{\varepsilon \searrow 0} f(t-\varepsilon)$ at $t \in \mathbb{R}$ is denoted by $f(t-)$.

II. Lyapunov functions for differential algebraic equations

Consider the classical DAE

$$E\dot{x} = Ax, \tag{2}$$

where the matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is regular, i.e. $\det(Es - A)$ is not the zero polynomial. A (classical) solution of (2) is any differentiable function $x : \mathbb{R} \to \mathbb{R}^n$ such that (2) is fulfilled.

Definition 1 (Consistency space): Let the *consistency space* of (2) be given by

$$\mathfrak{C}_{(E,A)} := \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } x \text{ of (2)} \\ \text{with } x(0) = x^0 \end{array} \right\}.$$

It is well known that for regular matrix pairs each solution of (2) is uniquely determined by any consistent initial condition $x(0) = x^0 \in \mathfrak{C}_{(E,A)}$. Since (2) is time invariant, all solutions x evolve within the consistency space, i.e. $x(t) \in \mathfrak{C}_{(E,A)}$ for all $t \in \mathbb{R}$. Furthermore, if (2) is an ordinary differential equation, i.e. $E \in \mathbb{R}^{n\times n}$ is an invertible matrix, then $\mathfrak{C}_{(E,A)} = \mathbb{R}^n$.

The following lemma gives a nice characterization of the consistency space in terms of the matrices E, A.

Lemma 2 ([4]): Consider the DAE (2) with regular matrix pair (E, A). Let $\mathcal{V}^0 = \mathbb{R}^n$ and $\mathcal{V}^{k+1} := A^{-1}(E\mathcal{V}^k)$ for $k \in \mathbb{N}$. Then

$$\exists k^* \in \mathbb{N} : \quad \mathcal{V}_0 \supset \mathcal{V}_1 \supset \cdots \supset \mathcal{V}_{k^*} = \mathcal{V}_{k^*+1} = \cdots$$

and $\mathfrak{C}_{(E,A)} = \mathcal{V}_{k^*}$. Furthermore, $\ker E \cap \mathfrak{C}_{(E,A)} = \{0\}$.

Definition 3 (Lyapunov function): Consider the DAE (2) with regular matrix pair (E, A) and corresponding consistency space $\mathfrak{C}_{(E,A)} \subseteq \mathbb{R}^n$. Assume there exist a positive definite matrix $P = \overline{P}^\top \in \mathbb{C}^{n\times n}$ and a matrix $Q = \overline{Q}^\top \in \mathbb{C}^{n\times n}$ which is positive definite on $\mathfrak{C}_{(E,A)}$ such that the *generalized Lyapunov equation*

$$A^\top PE + E^\top PA = -Q$$

is fulfilled. Then

$$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$$

is called a *Lyapunov function* for the DAE (2).

Note that this definition ensures that V is not increasing along solutions, i.e., for any solution $x : \mathbb{R} \to \mathbb{R}^n$ and all $t \in \mathbb{R}$,

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V\big(x(t)\big) = -x(t)^\top Q x(t) \leq 0$$

and equality only holds for $x(t) = 0$. Furthermore, the property $\ker E \cap \mathfrak{C}_{(E,A)} = \{0\}$ ensures that V is positive definite on $\mathfrak{C}_{(E,A)}$.

With some abuse of terminology, we call (2) *asymptotically stable* if, and only if, $x(t) \to 0$ as $t \to \infty$ for all solutions x of (2). Note that attractivity of the zero solution already implies attractivity and stability in the sense of Lyapunov for all solutions of (2), [5]. The following theorem shows the equivalence between asymptotic stability of (2) and the existence of a Lyapunov function.

Theorem 4 ([4], [5]): The DAE (2) with regular matrix pair (E, A) is asymptotically stable if, and only if, there exists a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ for (2).

Remark 5: The above definition of a Lyapunov function might seem unsatisfactory because it is not clear how the definition can be generalized to switched DAEs (1) or nonlinear differential algebraic equations. One can say that Definition 3 is just a “sufficient” definition. Furthermore, in the proof of Theorem 9 we will construct a “common Lyapunov function”, but we will not precisely define what a Lyapunov function for (1) is. It should be possible to formulate a more general definition of a Lyapunov function for (switched) DAEs and similar results as formulated in the next section will hold, but it would get more technical without adding more insight.

For the switched DAE (1) so-called consistency projectors will play an important role; these projectors describe how an inconsistent initial value jumps to a consistent one in the event of a switch. It turns out that the consistency projectors can easily be calculated in terms of the original matrices (E_p, A_p), $p = 1, \ldots, N$ (no transformation into some normal form is necessary). In particular, the jumps cannot be defined arbitrarily, as they are already uniquely determined by the given matrix pairs.

Definition 6 ([3]): For a regular matrix pair (E, A), let $V \in \mathbb{R}^{n\times m}$, $m \in \mathbb{N}$, be a full rank matrix with $\operatorname{im} V = \mathfrak{C}_{(E,A)}$ and let $W \in \mathbb{R}^{n\times(n-m)}$ be a full rank matrix with $\operatorname{im} W = \mathcal{W}^*$, where $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}} \mathcal{W}_i$ and

$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i),\ i \in \mathbb{N}.$$

The *consistency projector* corresponding to (E, A) is defined as

$$\Pi_{(E,A)} := [V, W] \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} [V, W]^{-1},$$

where $I \in \mathbb{R}^{m\times m}$ is the identity matrix of size $m \times m$.

It is easy to see that, analogously to Lemma 2, there exists $k^* \in \mathbb{N}$ such that $\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{k^*} = \mathcal{W}_{k^*+1} = \ldots$, so the matrices V and W can be calculated easily for a given matrix pair (E, A). Note that $\operatorname{im}\Pi_{(E,A)} = \mathfrak{C}_{(E,A)}$ and $\Pi_{(E,A)}(x) = x$ for all $x \in \mathfrak{C}_{(E,A)}$.

Remark 7: It is well known that for regular matrix pairs (E, A) there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that (E, A) is transformed into the so-called *Weierstrass normal form*:

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

where J is some matrix, N is a nilpotent matrix, and I is the identity matrix of appropriate size (in fact, for a "strict" Weierstrass normal form, it is additionally assumed that J and N are in Jordan canonical form, but this is not needed here). It can be shown [6] that for V and W as in Definition 6, the matrices $S := [EV, AW]^{-1}$ and $T := [V, W]$ put (E, A) into the Weierstrass normal form. It is easy to see that a DAE in Weierstrass form consists of two independent parts: an "ODE part" given by $\dot{y} = Jy$ and a "pure DAE part" given by $N\dot{z} = z$, where the pure DAE part only has the solution $z = 0$. Hence the classical solutions of a regular DAE (E, A) are given by the "underlying ODE" $\dot{y} = Jy$ and the coordinate transformation $x = T \left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right)$. This motivates the definition of the consistency projector.

III. Switched DAEs: motivating examples

For switched ODEs there exist several well known examples of destabilizing switching. Of course, these examples are also examples for switched DAEs (because every ODE is a special DAE), but in the following we will give examples which are specific to switched DAEs. For all examples we consider a switching signal with a constant interval $\Delta t > 0$ between switching times as illustrated in Figure 1.

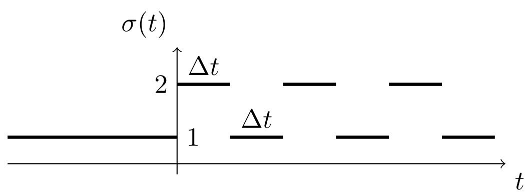

Fig. 1. Switching signal with constant interval $\Delta t > 0$ between switches.

Example 1

Let

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right),$$
$$(E_2, A_2) = \left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

The solutions of the corresponding switched DAE (1) are shown in Figure 2. For small enough Δt all solutions grow unbounded and for large enough Δt the solutions converge to zero. Furthermore, there exists a value of Δt for which all solutions are periodic.

The consistency spaces $\mathfrak{C}_p := \mathfrak{C}_{(E_p, A_p)}$, $p = 1, 2$ are given by

$$\mathfrak{C}_1 = \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathfrak{C}_2 = \operatorname{im} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Furthermore, the matrices $T_1 := [V_1, W_1]$ and $T_2 := [V_2, W_2]$ (where V_p and W_p for each subsystem come from Definition 6) are

$$T_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

and the corresponding consistency projectors are

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Note that (see also Remark 7 with $S_1 = S_2 = I$)

$$(E_1T_1, A_1T_2) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) = (E_2T_2, A_2T_2),$$

hence both DAEs are governed by the same underlying scalar ODE $\dot{y} = -y$; in particular, both DAEs are asymptotically stable. Furthermore, it is easy to see that

$$V : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}, \quad x \mapsto x^\top x$$

restricted to the corresponding consistency space is a Lyapunov function for both subsystems. In spite of this, the switched system is not stable under arbitrary switching.

Example 2

Let

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 8\pi & 0 \\ \frac{1}{2}\pi & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right),$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right).$$

The consistency spaces are

$$\mathfrak{C}_1 = \mathbb{R}^3, \quad \mathfrak{C}_2 = \operatorname{im} \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and the matrices T_1 and T_2 (defined as in Example 1) are

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

In view of Remark 7 it follows that $S_2 = \frac{1}{4}I$ and the underlying two dimensional ODE of subsystem 2 is given by

$$\dot{y} = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix} y.$$

We select $\Delta t = 1/4$, which ensures that the switching only occurs at that moment when the solution is located in the intersection of the consistency spaces (i.e. in $\mathfrak{C}_2$). Hence the solution of the switched DAE exhibits no jumps. The solutions of the unswitched DAEs are shown in the left part of Figure 3 and the unstable solutions of the switched DAE are illustrated in the right part of Figure 3.

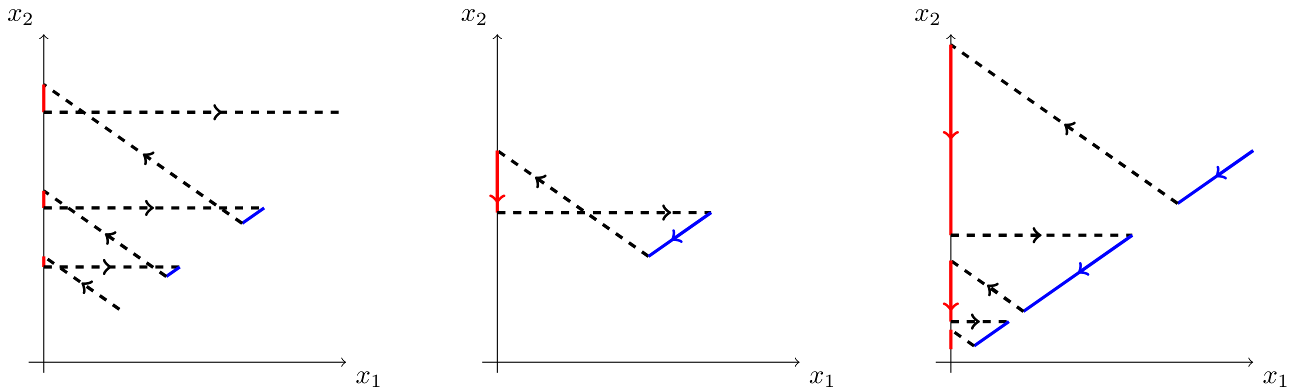

Fig. 2. Solutions for Example 1 for different switching signals (dashed lines mean jumps induced by the switching), left: $\Delta t < \frac{1}{2}\ln 2$, all nontrivial solutions grow unbounded; middle: $\Delta t = \frac{1}{2}\ln 2$, all solutions are periodic on $[0,\infty)$; right: $\Delta t > \frac{1}{2}\ln 2$, all solution tend to zero.

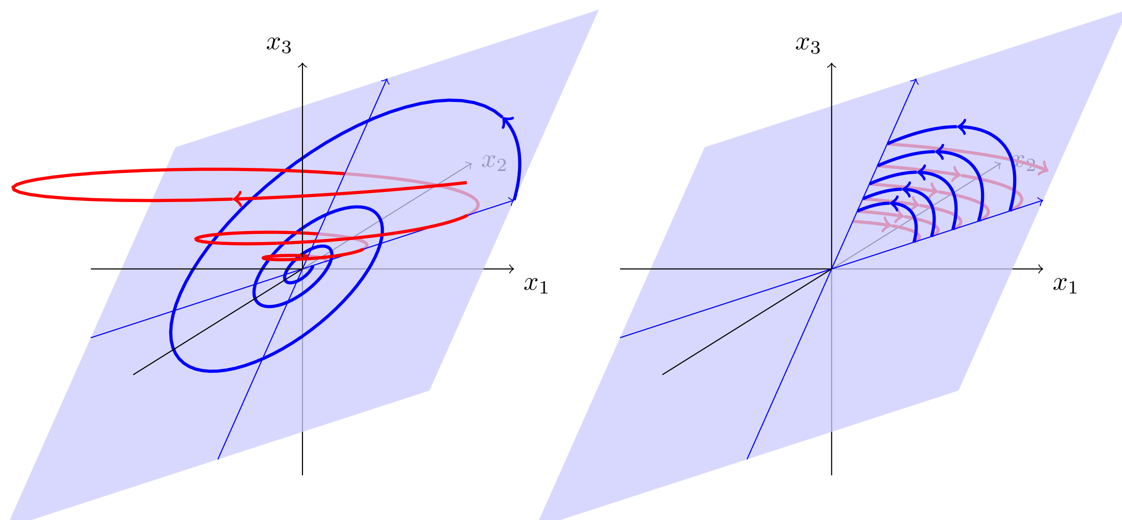

Fig. 3. Solutions for the Example 2. Left: Without switching, red: solution of subsystem 1, a three dimensional spiral converging to zero, blue: solution of subsystem 2, a two dimensional spiral converging to zero, Right: with switching, the solutions grow unbounded and exhibit no jumps.

Example 3

Let

$$(E_1, A_1) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2\pi & 0 \\ -2\pi & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right),$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 4\pi & -1 & 4\pi \\ -1 & \pi & -1 \\ 1 & 0 & 0 \end{bmatrix} \right).$$

The consistency spaces are

$$\mathfrak{C}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathfrak{C}_2 = \operatorname{im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and the matrices T_1 and T_2 (as in example 1) are

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

The corresponding consistency projectors are then given by

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

The underlying ODE for the first DAE can be read off directly from the matrix pair (E_1, A_1), the underlying ODE for the second DAE is given by ($S_2 = I$ in view of Remark 7)

$$\dot{y} = \begin{bmatrix} -1 & 4\pi \\ -\pi & -1 \end{bmatrix} y.$$

The solutions of the unswitched subsystems are illustrated in the left part of Figure 4. For $\Delta t = 1/2$ and an initial value at $t = 0$ which is located on the x_2 axis, the switching does not induce jumps and all solutions converge to zero. However, the choice $\Delta t = 1/4$ induces jumps and destabilizes the system, see the right part of Figure 4. Note that $V(x) = x^\top x$ is a common Lyapunov function on the intersection of the consistency spaces.

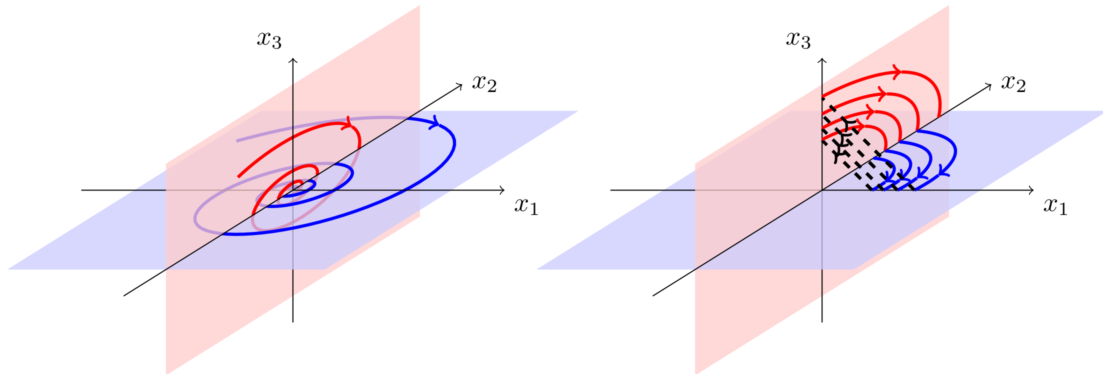

Fig. 4. Solutions for the Example 3. Left: Solutions of the individual subsystems, Right: Solutions of the switched system (dashed lines are jumps induced by the switches), the solutions grow unbounded.

IV. Stability of switched DAEs

As already mentioned and motivated in the introduction, we will make the following assumptions for the switched DAE (1).

- A1 The switching signal $\sigma : \mathbb{R} \to \{1, \dots, N\}$ is piecewise constant with a locally finite set of jump points and right-continuous.
- A2 Each matrix pair (E_p, A_p), $p \in \{1, \dots, N\}$ is regular, i.e. $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.
- A3 For the consistency projectors $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \dots, N\}$ corresponding to the regular matrix pairs (E_p, A_p) from (1), it holds that

$$\forall p, q \in \{1, \dots, N\} : E_p(I - \Pi_p)\Pi_q = 0.$$

Under these assumptions, the following result is known to hold.

Theorem 8 ([3]): Consider the switched DAE (1) satisfying Assumptions A1, A2 and A3. Then every distributional solution of (1) is impulse free and is represented by a piecewise-smooth function $x : \mathbb{R} \to \mathbb{R}^n$. Furthermore, for all solutions $x : \mathbb{R} \to \mathbb{R}^n$,

$$\forall t \in \mathbb{R} : \quad x(t) = \Pi_{\sigma(t)} x(t-).$$

In the following we call the switched DAE (1) *asymptotically stable* if, and only if, all distributional solutions are impulse free and each solution $x : \mathbb{R} \to \mathbb{R}^n$ fulfills $x(t) \to 0$ as $t \to \infty$.

Theorem 9: Consider the switched DAE (1) satisfying A1, A2 and A3. Let $\mathfrak{C}_p := \mathfrak{C}_{(E_p, A_p)} \subseteq \mathbb{R}^n$, $p = 1, \dots, N$, and $\Pi_p := \Pi_{(E_p, A_p)} \in \mathbb{R}^{n \times n}$ be the consistency spaces and projectors corresponding to the matrix pairs (E_p, A_p). Assume the classical DAE $E_p \dot{x} = A_p x$ is for every $p = 1, \dots, N$ asymptotically stable with Lyapunov function $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. If

$$\forall p, q = 1, \dots, N \ \forall x \in \mathfrak{C}_q : \quad V_p(\Pi_p x) \leq V_q(x) \tag{3}$$

then the switched DAE (1) is asymptotically stable for every switching signal.

Proof: Theorem 8 already shows that all (distributional) solutions of (1) are impulse free, hence it remains to show the convergence to zero.

Step 1: Definition of a common Lyapunov function candidate.
If $x \in \mathfrak{C}_p \cap \mathfrak{C}_q$ for some $p, q \in \{1, \dots, N\}$ then $x = \Pi_p x = \Pi_q x$ hence (3) implies $V_p(x) = V_q(x)$, therefore

$$V : \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \begin{cases} V_p(x), & x \in \mathfrak{C}_p, \\ 0, & \text{otherwise} \end{cases}$$

is well defined.

Step 2: $V\big(x(t)\big) \to 0$ as $t \to \infty$.
For $p = \{1, \dots, N\}$ let $P_p, Q_p \in \mathbb{C}^{n \times n}$ be the matrices as in Definition 3 corresponding to the DAE $E_p \dot{x} = A_p x$. Let furthermore

$$\lambda_p := \min_{x \in \mathfrak{C}_p \setminus \{0\}} \frac{x^T Q_p x}{V_p(x)} = \min_{\substack{x \in \mathfrak{C}_p \\ V_p(x) = 1}} x^\top Q_p x \ > 0,$$

where positivity follows from positive definiteness of V_p and Q_p on $\mathfrak{C}_p$. Consider a solution $x : \mathbb{R} \to \mathbb{R}^n$ of (1), then from [3, Lemma 4] it follows that on each open interval (s, t) which does not contain a switching time of σ the function x is smooth and a local solution of $E_p \dot{x} = A_p x$, where $p = \sigma(\tau)$, $\tau \in (s, t)$. From $x(\tau) \in \mathfrak{C}_p$ for all $\tau \in (s, t)$ it follows that $V(x(\tau)) = V_p(x(\tau))$ for all $\tau \in (s, t)$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_p\big(x(\tau)\big) = x(\tau)^\top Q_p x(\tau) \leq -\lambda_p V_p\big(x(\tau)\big).$$

Let $t \in \mathbb{R}$ be a jump of σ, then $x(t) = \Pi_{\sigma(t)} x(t-)$ and $x(t-) \in \mathfrak{C}_{\sigma(t-)}$, hence, by (3),

$$\begin{aligned} V(x(t)) = V_{\sigma(t)}\big(x(t)\big) &= V_{\sigma(t)}\big(\Pi_{\sigma(t)} x(t-)\big) \\ &\leq V_{\sigma(t-)}\big(x(t-)\big) = V\big(x(t-)\big) \end{aligned}$$

For $\lambda := \min_p \lambda_p$ and any $t_0 \in \mathbb{R}$ it therefore follows

$$\forall t \in \mathbb{R} : \quad V\big(x(t)\big) \leq e^{-\lambda(t - t_0)} V\big(x(t_0)\big),$$

which implies that $V\big(x(t)\big) \to 0$ for all solutions x of (1).

Step 3: Solutions tend to zero.
Seeking a contradiction, assume $x(t) \not\to 0$. Then there exists $\varepsilon > 0$ and a sequence $(s_i)_{i \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ with $s_i \to \infty$ as $i \to \infty$ such that $\|x(s_i)\| > \varepsilon$ for all $i \in \mathbb{N}$.

There is at least one $p \in \{1,\dots,N\}$ such that the set $\{\ i \in \mathbb{N} \mid \sigma(s_i) = p\ \}$ has infinitely many elements, therefore assume that $\sigma(s_i) = p$ for some p and all $i \in \mathbb{N}$. Then $x(s_i) \in \mathfrak{C}_p \setminus \{\ \xi \in \mathfrak{C}_p \mid \|\xi\| < \varepsilon\ \}$ for all $i \in \mathbb{N}$ and since V_p is positive definite on $\mathfrak{C}_p$ there exists $\delta > 0$ such that $V\big(x(s_i)\big) > \delta$ for all $i \in \mathbb{N}$. This is a contradiction to $V\big(x(t)\big) \to 0$ as $t \to \infty$. Therefore $x(t) \to 0$ as $t \to \infty$. ■

Condition (3) implies that any two Lyapunov functions V_p and V_q coincide on the intersection $\mathfrak{C}_p \cap \mathfrak{C}_q$, hence Theorem 9 is a generalization of the switched ODE case where the existence of a common Lyapunov function is sufficient to ensure stability under arbitrary switching. However, the existence of a common Lyapunov function is not enough in the DAE case, as becomes clear from Example 1 in Section III. Under arbitrary switching, solutions will in general exhibit jumps; these jumps are described by the consistency projectors, and these projectors must "fit together" with the Lyapunov functions in the sense of (3) to ensure stability of the switched DAE under arbitrary switching. If one assumes that the switching signal is chosen in such a way that no jumps occur, then the conditions on the consistency projectors are not needed and we get the following corollary.

Corollary 10: Consider the switched DAE (1) satisfying A1, A2, A3 and assume each DAE $E_p\dot{x} = A_p x$, $p = 1,\dots,N$ is asymptotically stable with Lyapunov function V_p. Let

$$\Sigma_{x^0} := \left\{ \sigma : \mathbb{R} \to \{1,\dots,N\} \;\middle|\; \begin{array}{l} \sigma \text{ fulfills A1 and} \\ \exists \text{ solution } x \text{ of (1)} \\ \text{with } x(0) = x^0 \text{ and} \\ x \text{ has no jumps} \end{array} \right\}.$$

If

$$\forall p,q = 1,\dots,N\ \forall x \in \mathfrak{C}_p \cap \mathfrak{C}_q : \quad V_p(x) = V_q(x) \qquad (4)$$

then all solutions x of (1) with $x(0) = x^0 \in \mathbb{R}^n$ and $\sigma \in \Sigma_{x^0}$ converge to zero as $t \to \infty$.

Note that actually we do not need to impose Assumptions A1, A2 and A3 explicitly in the above corollary, because A1 is already induced by the definition of Σ_{x^0}, A2 follows from the assumption that each DAE $E_p\dot{x} = A_p x$ is asymptotically stable [5], and A3 is not needed any more, because the assumption that no jumps occur also implies that no impulses can occur [3].

Example 3 from Section III fulfills the assumptions of Corollary 10, hence if no jumps occur all solutions tend to zero. In contrast to this, in Example 1 only the constant switching signals yield non-jumping non-trivial solutions, hence Corollary 10 is not very useful in this case. For Example 2 it is not possible to find Lyapunov functions for both subsystems such that condition (4) is fulfilled.

For switched ODEs it is well known that switching between stable subsystems always yields a stable system provided the so-called dwell time is large enough. Consider therefore the following set of switching signals parameterized by a dwell time $\tau_d > 0$:

$$\Sigma^{\tau_d} := \left\{ \sigma : \mathbb{R} \to \{1,\dots,N\} \;\middle|\; \begin{array}{l} \forall \text{ switching times} \\ t_i \in \mathbb{R}, i \in \mathbb{Z}, \\ t_{i+1} - t_i \geq \tau_d \end{array} \right\}.$$

Theorem 11: Consider the switched DAE (1) satisfying A1, A2, A3 and assume that each DAE $E_p\dot{x} = A_p x$, $p = 1,\dots,N$, is asymptotically stable with Lyapunov functions V_p and corresponding matrices $Q_p \in \mathbb{C}^{n\times n}$. Let

$$\lambda := \min_p \min_{x \in \mathfrak{C}_p \setminus \{0\}} \frac{x^T Q_p x}{V_p(x)}.$$

Let $\mu \geq 1$ be such that

$$\forall p,q = 1,\dots,N\ \forall x \in \mathfrak{C}_q : \quad V_p(\Pi_p x) \leq \mu V_q(x). \qquad (5)$$

Then the switched DAE (1) with $\sigma \in \Sigma^{\tau_d}$ is asymptotically stable whenever

$$\tau_d > \frac{\ln \mu}{\lambda}.$$

Proof: First note that all solutions of (1) by Theorem 8 are impulse free. Fix a solution $x \in \mathbb{R} \to \mathbb{R}^n$ of (1) with a fixed switching signal $\sigma \in \Sigma^{\tau_d}$. If σ has only finitely many switching times then asymptotic stability of (1) is obvious, therefore assume that the set of switching times $\{\ t_i \in \mathbb{R} \mid i \in \mathbb{Z}\ \}$ of σ is infinite. Let $v : \mathbb{R} \to \mathbb{R}_{\geq 0}$, $t \mapsto V_{\sigma(t)}\big(x(t)\big)$ and $0 < \varepsilon := \tau_d - \frac{\ln \mu}{\lambda}$. Then as in the proof of Theorem 9 it follows that

$$v(t_{i+1}-) \leq e^{-\lambda(t_{i+1}-t_i)} v(t_i) \leq \frac{e^{-\lambda\varepsilon}}{\mu} v(t_i).$$

Furthermore, condition (5) yields

$$\begin{aligned} v(t_i) &= V_{\sigma(t_i)}\big(\Pi_{\sigma(t_i)} x(t_i-)\big) \\ &\leq \mu V_{\sigma(t_i-)}\big(x(t_i-)\big) = \mu v(t_i-). \end{aligned}$$

All together this yields for all $i \in \mathbb{Z}$,

$$v(t_{i+1}-) \leq e^{-\lambda\varepsilon} v(t_i-),$$

hence $v(t_i-) \to 0$ as $i \to \infty$. Since $v(t) \leq e^{-\lambda(t-t_i)} v(t_i) \leq \mu v(t_i-)$ for all $t \in [t_i, t_{i+1})$, $i \in \mathbb{Z}$, it also follows that $v(t) \to 0$ as $t \to \infty$. As in the proof of Theorem 9 it now follows that $x(t) \to 0$ as $t \to 0$. ■

Remark 12: Since each Lyapunov function V_q, $q \in \{1,\dots,N\}$, is a quadratic function which is positive definite on $\mathfrak{C}_q$ if follows that for all $p,q \in \{1,\dots,N\}$

$$\mu_{q,p} := \min_{x \in \mathfrak{C}_q \setminus \{0\}} \frac{V_p\big(\Pi_p x\big)}{V_q(x)} = \min_{x \in \mathfrak{C}_q : V_q(x) = 1} V_p\big(\Pi_p x\big) > 0,$$

hence (5) is always fulfilled for $\mu \geq \max_{q,p} \mu_{q,p}$. Therefore, Theorem 11 states that switching between asymptotically stable subsystems yields asymptotic stability provided the dwell time of the switching signal is large enough.

For Example 1 from Section III condition (5) is fulfilled for the (common) Lyapunov function $x \mapsto V(x) = x^\top x$ with $\lambda = 2$ and $\mu = 2$. Hence for dwell times larger than $\frac{1}{2}\ln 2$ the switched system (1) is asymptotically stable, see also Figure 2.

V. Conclusions

This paper studied linear switched differential algebraic equations (DAEs). We constructed several examples showing that switching between stable subsystems may lead to instability, and illustrating a large variety of possible instability mechanisms caused by the presence of algebraic constraints. We presented two sufficient conditions for stability of switched DAEs based on the existence of suitable Lyapunov functions. The first result (Theorem 9) says that a common Lyapunov function guarantees stability under arbitrary switching when an additional condition involving consistency projectors holds; this extra condition is not needed when the switching signal is chosen in such a way that no jumps occur (Corollary 10). The second result (Theorem 11) shows that stability is preserved under switching with sufficiently large dwell time.

The work reported here is just an initial step in the investigation of stability of switched DAEs by Lyapunov-based methods. Some avenues for future research include: stability under average dwell-time and other useful classes of switching signals; stability and stabilization under state-dependent switching; stability of nonlinear switched DAEs; converse Lyapunov theorems; and switched DAEs with inputs and/or outputs.

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