Funnel control with saturation: linear MIMO systems

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Abstract

Tracking – by the system output – of a reference signal r (assumed bounded with essentially bounded derivative) is considered in the context of linear m-input u, m-output y systems (A, B, C) in the presence of input saturation (i.e. $\|u(t)\| \le \widehat{u}$ for all t). The system is assumed to have strict relative degree one with positive-definite high-frequency gain (i.e. $CB > 0$) and stable zero dynamics. Prespecified is a parameterized performance funnel $\mathcal{F}(\psi) = \{(t, \xi) \mid \|\xi\| < \psi(t)\}$, where $\lambda > 0$ and $\psi\colon [0, \infty) \to [\lambda, \infty)$ is globally Lipschitz with Lipschitz constant Λ. The tracking error $e = y - r$ is required to evolve within the funnel (i.e. graph$(e) \subset \mathcal{F}(\psi)$): transient and asymptotic behaviour of the tracking error is influenced through choice of parameter values which define the funnel. The proposed control structure is a saturating error feedback of the form $u(t) = -\text{sat}_{\widehat{u}}(k(t)e(t))$ wherein the gain function $k\colon t \mapsto 1/(\psi(t) - \|e(t)\|)$ evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data (A, B, C) and $\widehat{u}$, the funnel data (ψ, Λ, λ), the reference signal r, and the initial state x^0) is presented under which the tracking objective is achieved, whilst maintaining boundedness of the state x and gain function k.

Keywords. Output feedback, input saturation, linear systems, transient behaviour, tracking.

1. Introduction

In the early 1980s, a novel feature in classical adaptive control was introduced: adaptive strategies which do not require identification of the particular system being controlled. Pioneering contributions to the area include [1], [5], [6], [8], [11] (see, also, the survey [3] and references therein). The prototypical example for a system class – rather than a single system – is that

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of linear m-input, m-output systems with relative degree one, positive high-frequency gain and stable zero dynamics, i.e. minimum phase. The simple output feedback $u(t) = -k(t)\,y(t)$ stabilizes each system belonging to the above class provided $k(\cdot)$ is appropriately generated: e.g. by the differential equation $\dot{k}(t) = \|y(t)\|^2$ or variants thereof. The two major drawbacks of the latter strategy (and its variants) are (i) albeit bounded, the gain $k(t)$ is monotonically increasing and (ii) whilst asymptotic performance is guaranteed, transient behaviour is not generally taken into account (an exception being the contribution [7], wherein the issue of prescribed transient behaviour is successfully addressed). A fundamentally different approach – so called 'funnel control' – was introduced in [2] in the context of output tracking: this control ensures prespecified transient behaviour of the tracking error, has a non-monotone gain, is simpler than the above adaptive controller (insofar as the gain is not dynamically generated), and does not invoke any internal model. It has been successfully applied in experiments controlling the speed of electric devices [4]; see [3] for further applications.

The present paper adopts the funnel control viewpoint – but differs from its precursor [2] in an essential manner: here, the presence of an explicit input constraint is a distinguishing feature of the underlying system class. A feasibility relationship involving the system data, funnel data, reference signal data and the saturation bound is derived under which the efficacy of funnel control in the presence of input saturation is established.

By way of motivation, consider the simple scalar linear system

$$\dot{y} = a\,y + b\,u, \qquad a \in \mathbb{R}, \quad b > 0, \quad y(0) = y^0.$$

The control objective is tracking, of a (suitably regular) reference signal r, with prescribed transient and asymptotic behavior in the sense that, for some given function $\psi\colon [0,\infty) \to [\lambda,\infty)$, $\lambda > 0$, the tracking error is bounded by ψ:

$$|y(t) - r(t)| < \psi(t) \quad \forall\; t \geq 0.$$

For example, if ψ given by $\psi(t) = \max\{1 - \Lambda t, \lambda\}$, with $\Lambda > 0$ and $\lambda \in (0,1)$, then attainment of the tracking objective implies that a prescribed tracking accuracy, quantified by $\lambda > 0$, is achieved in prescribed time $t^* = (1-\lambda)/\Lambda$: specifically, $|y(t) - r(t)| < \lambda$ for all $t \geq t^*$. In the general case, if ψ is globally Lipschitz and bounded away from zero, and the reference signal r is a bounded absolutely continuous function with essentially bounded derivative, then it is known (see [2]) that the tracking objective is achieved by the following simple strategy

$$u(t) = -k(t)[y(t) - r(t)], \quad k(t) := \frac{1}{\psi(t) - |y(t) - r(t)|} \tag{1.1}$$

if, and only if, the following feasibility condition holds: $|y^0 - r(0)| < \psi(0)$. Moreover, the gain k, and hence the control u, is bounded.

Consider again the above scalar system, with the same control objective, but now with saturation in the input channel:

$$\dot{y} = ay + b\,\mathsf{sat}_{\widehat{u}}(u), \qquad a \in \mathbb{R}, \quad b,\, \widehat{u} > 0, \quad y(0) = y^0, \tag{1.2}$$

where $\mathsf{sat}_{\widehat{u}}$ is the saturation function given by $\mathsf{sat}_{\widehat{u}} = \widehat{u}\,\mathsf{sgn}(u)$ if $|u| > \widehat{u}$ and $\mathsf{sat}_{\widehat{u}}(u) = u$ otherwise. Again, $|y^0 - r(0)| < \psi(0)$ is a necessary condition for feasibility. However, a moment's reflection confirms that the latter condition is not sufficient: the question of feasibility of the tracking objective in the presence of input saturation is delicate and inevitably involves addressing the interplay between the plant data (a, b, y^0), the reference signal r, the function ψ and the saturation bound $\widehat{u}$. For example, if $a > 0$, then it is readily seen that $b\widehat{u} \geq a|y^0|$ is a necessary condition for feasibility. Moreover, it is clear that, for feasibility, the saturation level $\widehat{u}$ should also be commensurate with the magnitude of the reference signal r and its derivative $\dot{r}$. To illustrate the interplay between $\widehat{u}$ and the function ψ, consider the case wherein $a = 0$, $r(\cdot) = 0$ and ψ is given, as above, by $\psi(t) = \max\{1 - \Lambda t, \lambda\}$ with $\Lambda > 0$ and $\lambda \in (0, 1)$ (and so ψ is globally Lipschitz, with Lipschitz constant Λ). Assume feasibility of the tracking objective. Then, writing $t^* := (1 - \lambda)/\Lambda$, we have

$$1 - \lambda = \psi(0) - \psi(t^*) < \psi(0) - y(t^*) = 1 - y^0 + y^0 - y(t^*) \leq 1 - y^0 + t^* b\widehat{u}$$

and since this must hold for all $|y^0| < 1$, we may conclude that $1 - \lambda \leq t^* b\widehat{u}$. Therefore, $b\widehat{u} \geq \lambda$ is a necessary condition for feasibility.

The purpose of the present paper is to extend the above investigations to a more general context of m-input u, m-output y, n-dimensional linear systems (A, B, C) subject to input saturation: $\|u(t)\| \leq \widehat{u}$ for all t. The system (A, B, C) is assumed (i) to have strict relative degree one with positive-definite high-frequency gain (i.e. $CB > 0$) and (ii) to satisfy a minimum-phase condition. Prespecified is a performance funnel $\mathcal{F}(\psi) = \{(t, \xi) |\ \|\xi\| < \psi(t)\}$, parameterized by $\lambda > 0$ and a globally Lipschitz function $\psi\colon [0, \infty) \to [\lambda, \infty)$ with Lipschitz constant Λ;

see Figure 3.1. The control objective is output tracking: determine a feedback structure which ensures that, for a given reference signal $r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ (the space of bounded locally absolutely continuous functions $r\colon \mathbb{R}_+ := [0,\infty) \to \mathbb{R}^m$ with essentially bounded derivative $\dot{r}$), the output tracking error $e = y - r$ evolve within the funnel (i.e. graph$(e) \subset \mathcal{F}(\psi)$): transient and asymptotic behaviour of the tracking error is influenced through choice of parameter values which define the funnel. The proposed control structure is a saturating error feedback of the form $u(t) = -\mathrm{sat}_{\widehat{u}}(k(t)e(t))$ wherein the gain function $k\colon t \mapsto 1/(\psi(t) - \|e(t)\|)$ evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data (A, B, C) and $\widehat{u}$, the funnel data (ψ, Λ, λ), the reference signal r, and the initial state x^0) is presented under which the tracking objective is achieved, whilst maintaining boundedness of the state x and gain function k.

In the highly specialized context of the motivating scalar system (1.2), the main result of the paper translates into the following: if

$$|y^0 - r(0)| < \psi(0) \qquad \text{and} \qquad b\,\widehat{u} \geq |a|\,[\|\psi\|_\infty + \|r\|_\infty] + \|\dot{r}\|_\infty + \Lambda, \tag{1.3}$$

wherein $\|\cdot\|_\infty$ denotes the L^∞-norm, then the simple control strategy

$$u(t) = -\mathrm{sat}_{\widehat{u}}(k(t)e(t)), \quad k(t) = \frac{1}{\psi(t) - |e(t)|}, \quad e(t) = y(t) - r(t),$$

ensures attainment of the tracking objective (and, moreover, the gain function k is bounded). Furthermore, if the first inequality in (1.3) is replaced by

$$|y^0 - r(0)| < \left(\frac{\widehat{u}}{1+\widehat{u}}\right)\psi(0),$$

then input saturation does not occur and so the control strategy coincides with (1.1).

We proceed to make precise the class of systems and performance funnels.

2. The System Class

Consider the m-input, m-output linear system

$$\left.\begin{aligned} &\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x^0 \in \mathbb{R}^n, \\ &y(t) = Cx(t) \end{aligned}\right\} \tag{2.4}$$

with $(A,B,C) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times n}$ and assume that the *minimum-phase* condition holds:

$$s\in\mathbb{C},\ \operatorname{Re} s\geq 0 \implies \det\begin{pmatrix} sI-A & B\\ C & 0\end{pmatrix}\neq 0. \tag{2.5}$$

Furthermore, we assume and the matrix $CB \in \mathbb{R}^{m\times m}$ is *positive definite* (not necessarily symmetric):

$$\exists\,\gamma>0\ \forall\, v\in\mathbb{R}^m\ :\ \langle v, CBv\rangle \geq \gamma\|v\|^2\ . \tag{2.6}$$

As is well known, if (2.6) holds, then there exist $V\in\mathbb{R}^{n\times(n-m)}$ and $N\in\mathbb{R}^{(n-m)\times n}$ with

$$\operatorname{im} V = \ker C \quad \text{and} \quad N := (V^TV)^{-1}V^T[I_n - B(CB)^{-1}C]. \tag{2.7}$$

such that the similarity transformation

$$S := \begin{pmatrix} C\\ N\end{pmatrix},\quad \text{with inverse}\quad S^{-1} = \big(B(CB)^{-1},\ V\big)$$

takes system (2.4) into the form

$$\left.\begin{aligned} \dot y(t) &= A_1 y(t) + A_2 z(t) + CBu(t), & y(0) &= Cx^0\\ \dot z(t) &= A_3 y(t) + A_4 z(t), & z(0) &= Nx^0\,, \end{aligned}\right\} \tag{2.8}$$

where

$$A_1 := CAB(CB)^{-1},\quad A_2 := CAV,\quad A_3 := NAB(CB)^{-1},\quad A_4 = NAV. \tag{2.9}$$

Moreover, if (2.5) holds, then A_4 is a *Hurwitz* matrix, that is,

$$\operatorname{spec} A_4 \subset \{s\in\mathbb{C} \mid\ \operatorname{Re} s<0\}, \tag{2.10}$$

in which case, there exist positive constants $\alpha,\beta>0$ such that

$$\|\exp(A_4 t)\| \leq \beta\, e^{-\alpha t}\quad \forall\, t\geq 0. \tag{2.11}$$

Finally, we assume that the input function u is subject to a saturation constraint: in particular, for some $\widehat{u}>0$,

$$\|u(t)\| \leq \widehat{u}\quad \forall\, t\geq 0. \tag{2.12}$$

With the input constraint parameter $\widehat{u}$, we associate the saturation function

$$\operatorname{sat}_{\widehat{u}}\colon \mathbb{R}^m \to \{w\in\mathbb{R}^m \mid\ \|w\|\leq\widehat{u}\},\quad v\mapsto \operatorname{sat}_{\widehat{u}}(v) := \begin{cases} \widehat{u}\|v\|^{-1}v, & \|v\|>\widehat{u}\\ v, & \text{otherwise.}\end{cases}$$

3. THE PERFORMANCE FUNNEL

A central ingredient of our approach is the concept of a funnel given by

$$\mathcal{F}(\psi) := \left\{ (t,\xi) \in \mathbb{R}_+ \times \mathbb{R}^m \mid \|\xi\| < \psi(t) \right\} , \tag{3.13}$$

determined by a function $\psi(\cdot)$ belonging to

$$\mathcal{G}(\Lambda,\lambda) := \left\{ \psi \colon \mathbb{R}_+ \to [\lambda,\infty) \Big| \;\; \psi \text{ bounded and globally Lipschitz with Lipschitz constant } \Lambda \right\}$$

parameterized by $\Lambda \geq 0$ and $\lambda > 0$. The control objective is a feedback structure which – given

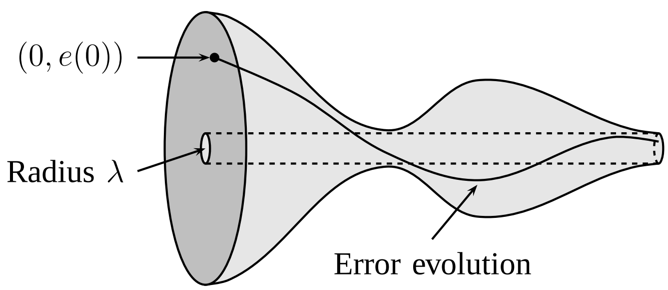


Fig. 3.1. Prescribed performance funnel $\mathcal{F}(\psi)$.

a reference signal $r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ and under appropriate feasibility conditions – ensures that the closed-loop system has unique global bounded solution $x \colon \mathbb{R}_+ \to \mathbb{R}^n$ and the tracking error $e = y - r$ evolves within the performance funnel.

4. THE MAIN RESULT

We summarize the main contributions of the paper in the following theorem, a proof of which can be found in the Appendix.

Theorem 4.1: Let $(A,B,C) \in (\mathbb{R}^{n\times n},\; \mathbb{R}^{n\times m},\; \mathbb{R}^{m\times n})$ such that (2.5) holds. Select $V \in \mathbb{R}^{n\times(n-m)}$ such that $\operatorname{im} V = \ker C$ and let N, A_1, A_2, A_3, A_4, $\alpha > 0$ and $\beta > 0$ be as in (2.7), (2.9) and (2.11). Assume further that (2.6) holds with associated constant $\gamma > 0$. Let $\Lambda \geq 0$, $\lambda > 0$ and $\psi \in \mathcal{G}(\lambda, \Lambda)$ define the performance funnel $\mathcal{F}(\psi)$.
If $x^0 \in \mathbb{R}^n$ and $r \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}^m)$ are such that

$$\|Cx^0 - r(0)\| < \psi(0) \tag{4.14}$$

and $\widehat{u} > 0$ such that

$$\gamma\widehat{u} > L := \left[\|A_1\| + \|A_2\|\|A_3\|\frac{\beta}{\alpha}\right]\left[\|\psi\|_\infty + \|r\|_\infty\right] + \beta\|A_2\|\|Nx^0\| + \|\dot{r}\|_\infty + \Lambda, \tag{4.15}$$

then application of the feedback strategy

$$u(t) = -\mathsf{sat}_{\widehat{u}}(k(t)e(t)), \quad k(t) = \frac{1}{\psi(t) - \|e(t)\|}, \quad e(t) = Cx(t) - r(t) \tag{4.16}$$

to (2.4) (subject to the input constraint (2.12)) yields a closed-loop initial-value problem with the following properties.

(i) There exists precisely one maximal solution $x\colon [0,\omega) \to \mathbb{R}^n$ and this solution *global* (i.e. $\omega = \infty$).

(ii) The global solution x is bounded and the tracking error $e = Cx - r$ evolves within the performance funnel $\mathcal{F}(\psi)$; more precisely,

$$\psi(t) - \|e(t)\| \geq \varepsilon := \min\left\{\frac{\lambda}{2}, \frac{\lambda}{2\widehat{u}}, \psi(0) - \|e(0)\|\right\} \quad \forall\, t \geq 0\,. \tag{4.17}$$

(iii) The gain function k is bounded, with $\|k\|_\infty \leq 1/\varepsilon$.

(iv) There exists $\tau \geq 0$ such that $\|u(\tau)\| < \widehat{u}$ (i.e. the input is unsaturated at some time τ).

(v) If $\tau \geq 0$ is such that $\|u(\tau)\| < \widehat{u}$, then $\|u(t)\| < \widehat{u}$ for all $t \geq \tau$ (i.e. if the input is unsaturated at time τ, then it remains unsaturated thereafter).

(vi) The input is globally unsaturated (i.e. $\|u(t)\| < \widehat{u}$ for all $t \geq 0$) if, and only if,

$$\|Cx^0 - r(0)\| < \psi(0)\,\widehat{u}/(1 + \widehat{u})\,. \tag{4.18}$$

(In which case, the first of equations (4.16) takes the simple form $u(t) = -k(t)e(t)$).

Remark 4.2: Some commentary on the content of the above theorem are warranted.

(a): In view of the potential singularity in (4.16), some care must be exercised in formulating the initial-value problem (2.4), (4.16). This we do as follows. Define

$$\kappa\colon \mathcal{F}(\psi) \to \mathbb{R}^m, \quad (t,\xi) \mapsto \kappa(t,\xi) := \frac{1}{\psi(t) - \|\xi\|},$$

$$\widetilde{\mathcal{D}} := \left\{(t,\eta) \in \mathbb{R}_+ \times \mathbb{R}^n) \mid \; (t, C\eta - r(t)) \in \mathcal{F}(\psi)\right\},$$

$$\tilde{F}\colon \widetilde{\mathcal{D}} \to \mathbb{R}^n, \quad (t,\eta) \mapsto \tilde{F}(t,\eta) := A\eta - B\mathsf{sat}_{\widehat{u}}\big(\kappa(t, C\eta - r(t))(C\eta - r(t))\big).$$

The closed-loop initial-value problem (2.4), (4.16) is now interpreted as

$$\dot{x}(t) = \tilde{F}(t, x(t)), \quad x(0) = x^0, \quad (0, x^0) \in \widetilde{\mathcal{D}}, \tag{4.19}$$

By a solution of (4.19) we mean a continuously differentiable function $x\colon [0,\omega) \to \mathbb{R}^n$ which satisfies (4.19) and has graph in $\widetilde{\mathcal{D}}$; x is maximal if it has no right extension that is also a solution; x is global if $\omega = \infty$. Assertion (i) of the theorem confirms the existence of precisely one maximal solution x of (4.19) and, moreover, this solution is global. Note that the requirement that graph(x) is in $\widetilde{\mathcal{D}}$ implies that the graph of the tracking error $e = Cx - r$ is in $\mathcal{F}(\psi)$: this – together with boundedness of x – is the content of Assertion (ii). Assertion (ii) establishes boundedness of both the control gain function $k(\cdot) = \kappa(\cdot, e(\cdot))$. Assertions (iv) and (v) imply that the control input cannot remain saturated for all $t \geq 0$ and, when it becomes unsaturated then it remains so thereafter. Assertion (vi) is an immediate consequence of Assertions (iv) and (v) and consists of the observation that, if the control is initially unsaturated (i.e. if $\|u(0)\| < \widehat{u}$), then the saturation bound is never attained.

(b): The first feasibility condition (4.14) is a necessary condition for attainment of the control objective and is equivalent to the requirement that $(0, x^0) \in \widetilde{\mathcal{D}}$.

(c): The second feasibility condition (4.15) is a sufficient condition for attainment of the control objective. It quantifies a saturation bound (sufficiently large to ensure performance) in terms of plant data, funnel data, initial data and reference signal data. The nature of the dependence of the saturation bound on these data is not surprising. For example, (i) the minimum-phase condition ensures exponential stability of the zero-dynamics of the linear triple (A, B, C) – this translates into the condition (2.11) on the matrix A_4 in (2.9) – the parameter α quantifies the exponential decay rate of the zero dynamics and is inversely related to the saturation bound; (ii) it is to be expected that tracking of “large and rapidly varying” reference signals r would require control inputs capable of taking sufficiently large values – this is reflected in the dependence of the saturation bound on both $\|r\|_\infty$ and $\|\dot{r}\|_\infty$; (iii) transient and asymptotic behaviour of the tracking error is influenced by the choice of funnel $\mathcal{F}(\psi)$ determined by the globally Lipschitz function $\psi \in \mathcal{G}(\Lambda, \lambda)$ – a stringent requirement that transient behaviour decays rapidly would be reflected in a large Lipschitz constant Λ which, not unexpectedly, appears as an additive term in the saturation bound.

5. EXAMPLE

For purposes of illustration, we choose a single-input, single-output system in normal form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 0 & -3 \end{pmatrix} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t), \quad \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = \begin{pmatrix} y^0 \\ 0 \\ 0 \end{pmatrix} \tag{5.20}$$

subject to the saturation constraint

$$|u(t)| \le \widehat{u} := 10 \quad \forall\, t \ge 0.$$

It is readily verified that (2.5) and (2.6) hold (with $\gamma = 1$ in the latter) and that

$$\|A_1\| = 1, \quad \|A_2\| = \|A_3\| = \sqrt{2}, \quad \text{and} \quad \left\|e^{A_4 t}\right\| \le e^{-\alpha t} \quad \forall\, t \ge 0,$$

where $\alpha = (5-\sqrt{2})/2$. As reference signal we choose $r(\cdot) = \xi_1(\cdot)$ the first component of the solution of the Lorentz system

$$\dot\xi_1 = \xi_2 - \xi_1, \qquad \dot\xi_2 = (28\xi_1/10) - (\xi_2/10) - \xi_1\xi_3, \qquad \dot\xi_3 = \xi_1\xi_2 - (8\xi_3/30),$$

with the initial values $(\xi_1(0),\, \xi_2(0),\, \xi_3(0)) = (1,\, 0,\, 3)$. It is shown in [9, App. C] that this solution is chaotic and yields a bounded $r(\cdot)$ with bounded derivative. Note that $r(0) = 1$ and numerical computation yields $\|r\|_\infty \le 9/5$, and $\|\dot r\|_\infty \le 6/5$.

Setting $\lambda = 0.1$ and $\Lambda = 0.2$, the funnel $\mathcal{F}(\psi)$ is determined by the function $\psi \in \mathcal{G}(\lambda, \Lambda)$ given by

$$\psi(t) := \max\{2e^{-0.1\,t},\ 0.1\} \qquad \forall\, t \ge 0.$$

Note that this prescribes exponential (exponent 0.1) decay of the tracking error in the transient phase $[0, T]$, where $T = 10 \ln 20 \approx 30$, and a tracking accuracy quantified by $\lambda = 0.1$ thereafter.

The constant L is given by

$$L = \frac{926 + 76\sqrt{2}}{105} < 10 = \widehat{u}$$

and so the second feasibility condition is satisfied. In order to satisfy the other feasibility condition, the initial datum y^0 must be such that $|e(0)| = |y^0 - r(0)| < 2$ and so $y^0 \in (-1, 3)$. To illustrate the occurrence of saturation of the control input in our simulations, we choose $y^0 \in (-1, 3)$ to be such that the inequality in Assertion (vi) fails to hold (in which case, there exists $\tau > 0$ such that the control is saturated on $[0, \tau)$). For this reason, we choose $y^0 = -0.9$, in which case ε is given by $\varepsilon = \lambda/(2\widehat{u}) = 0.005$.

Figure 5.2 depicts the behaviour of the closed-loop system (5.20), (4.16). The simulations confirm the result of Theorem 4.1: the tracking error remains uniformly bounded away from the funnel boundary; moreover, the second picture suggests that that the calculated bound $\varepsilon = 0.005$ is conservative. Non-monotonicity of gain function $k(\cdot)$ is also evident: it increases when the error approaches the funnel boundary and decreases when the error recedes from the boundary. The final picture confirms that the input is initially saturated: it remains so on an interval of short duration and thereafter remains unsaturated.

6. Appendix: Proof of Theorem 4.1

Reiterating comments in Remark 4.2(a), some care must be exercised in formulating the initial-value problem (2.4), (4.16) (equivalently, (2.8), (4.16)). Define

$$\mathcal{D} := \big\{(t,\mu,\zeta) \in \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{n-m})| \quad (t, \mu - r(t)) \in \mathcal{F}(\psi)\big\},$$

and

$$f\colon \mathcal{D} \to \mathbb{R}^m, \quad (t,\mu,\zeta) \mapsto f(t,\mu,\zeta) := A_1\mu + A_2\zeta - CB\mathrm{sat}_{\widehat{u}}(\kappa(t,\mu - r(t))(\mu - r(t)))\,.$$

The initial-value problem (2.8), (4.16) may now be expressed in the form

$$\left.\begin{array}{ll} \dot{y}(t) = f(t,y(t),z(t)), & y(0) = Cx^0 \\ \dot{z}(t) = A_3 y(t) + A_4 z(t), & z(0) = Nx^0. \end{array}\right\} \tag{6.21}$$

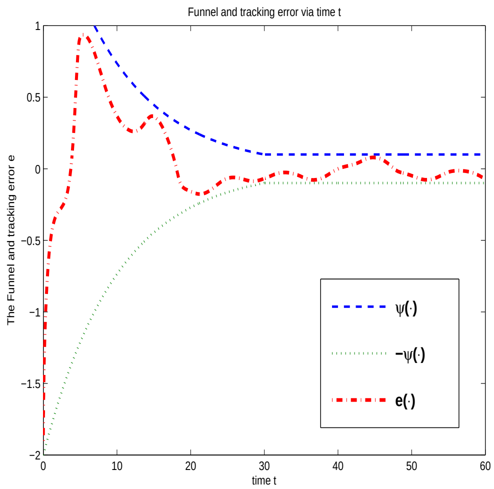
Clearly, $(y,z)\colon [0,\omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ is a (maximal/global) solution of (6.21) if, and only if, $x = B(CB)^{-1}y + Vz\colon [0,\omega) \to \mathbb{R}^n$ is a (maximal/global) solution of (4.19).

Now, it is readily verified that

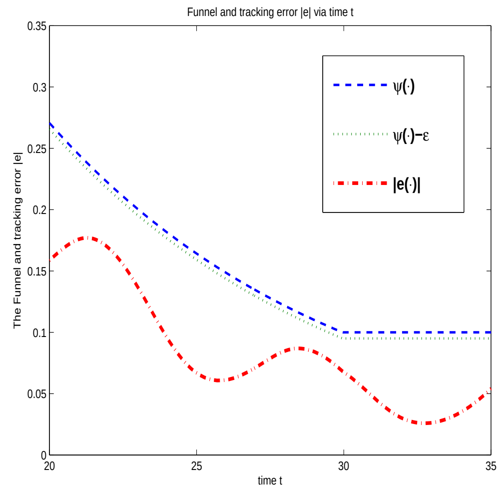
$$F\colon (t,\mu,\zeta) \mapsto \big(f(t,\mu,\zeta)\,,\, A_3\mu + A_4\zeta\big)$$

satisfies a local Lipschitz condition on the (relatively open) domain $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}^m \times \mathbb{R}^{n-m}$, in the sense that, for each $(\tau,\mu,\zeta) \in \mathcal{D}$, there exists an open neighbourhood U of (τ,μ,ζ) and a constant K such that

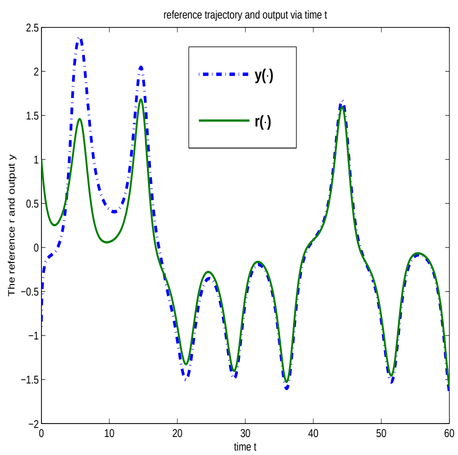
$$\|F(t,y,z) - F(t,\mu,\zeta)\| \le K\big(\|y-\mu\| + \|z-\zeta\|\big) \quad \forall\,(t,y,z) \in U.$$



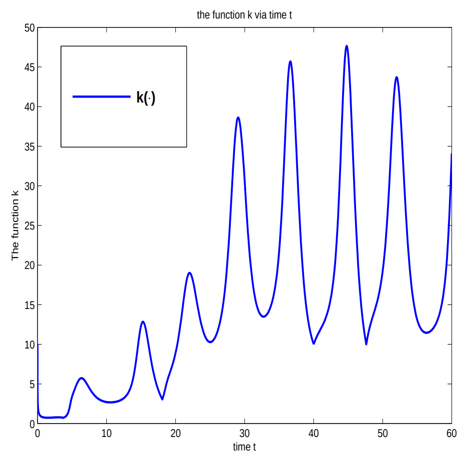
Funnel and tracking error e



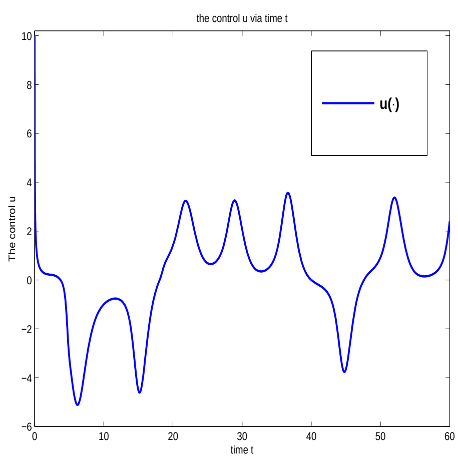
Funnel and tracking error $|e|$ - zoomed in



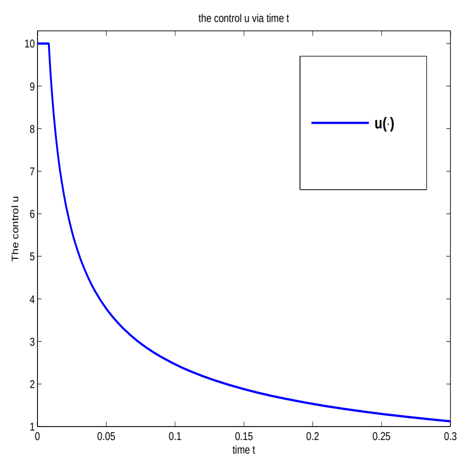
Reference signal r and output y



Gain function k



Control u



Control u - zoomed in.

Fig. 5.2. Behaviour of the closed-loop system (5.20), (4.16)

By the standard theory of ordinary differential equations (see, e.g. [10, Theorem III.10.VI]), the initial-value problem (6.21) has a unique maximal solution $(y,z)\colon [0,\omega) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ $0<\omega\leq\infty$; moreover, graph$((y,z)) = \{(t,y(t),z(t)) |\ t\in[0,\omega)\} \subset \mathcal{D}$ does not have compact closure in $\mathcal{D}$.

Next we show that the absolutely continuous tracking error e, defined by $e(t) := y(t)-r(t)$ for all $t\in[0,\omega)$, satisfies

$$\langle e(t),\dot e(t)\rangle \leq \|e(t)\|\big[L-\Lambda\big] - \langle e(t), CB\text{sat}_{\widehat u}(k(t)e(t))\rangle \quad \text{for a.a. } t\in[0,\omega), \tag{6.22}$$

wherein, for notational convenience, we have introduced the function

$$k\colon [0,\omega)\to\mathbb{R}_+,\quad t\mapsto k(t) := \kappa(t,e(t)) = \frac{1}{\psi(t)-\|e(t)\|}\,.$$

Since graph$((y,z))$ is in $\mathcal{D}$, it follows that graph(e) is in $\mathcal{F}(\psi)$ and so

$$\|e(t)\| < \psi(t) \leq \|\psi\|_\infty \quad \forall\, t\in[0,\omega). \tag{6.23}$$

By the second subsystem in (6.21), we have

$$z(t) = e^{A_4 t}Nx^0 + \int_0^t e^{A_4(t-s)}A_3\big(e(s)+r(s)\big)\,\mathrm{d}s \quad \forall\, t\in[0,\omega).$$

In view of (2.11) and (6.23), it follows that

$$\|z(t)\| \leq M^0 := \beta\|Nx^0\| + \frac{\beta}{\alpha}\|A_3\|\big[\|\psi\|_\infty + \|r\|_\infty\big] \quad \forall\, t\in[0,\omega). \tag{6.24}$$

By absolute continuity of e and the first subsystem in (6.21), we have

$$\dot e(t) = f(t, e(t)+r(t), z(t)) - \dot r(t) \quad \text{for a.a. } t\in[0,\omega),$$

whence

$$\langle e(t),\dot e(t)\rangle \leq \|e(t)\|\big[\|A_1\|\|e(t)\| + \|A_2\|\|z(t)\| + \|A_1\|\|r\|_\infty + \|\dot r\|_\infty\big] \\ - \langle e(t), CB\text{sat}_{\widehat u}(k(t)e(t))\rangle \quad \text{for a.a. } t\in[0,\omega),$$

The conjunction of (4.15), (6.23) and (6.24) yields

$$\|A_1\|\|e(t)\| + \|A_2\|\|z(t)\| + \|A_1\|\|r\|_\infty + \|\dot r\|_\infty \leq L-\Lambda \quad \forall\, t\in[0,\omega)$$

and so, we have (6.22).

Next, we show that, for ε as in (4.17),

$$\psi(t) - \|e(t)\| \geq \varepsilon \quad \forall\, t \in [0, \omega). \tag{6.25}$$

Seeking a contradiction, suppose there exists $t_1 \in [0, \omega)$ such that $\psi(t_1) - \|e(t_1)\| < \varepsilon$. Since $\psi(0) - \|e(0)\| \geq \varepsilon$, the following is well defined

$$t_0 := \max\{t \in [0, t_1) \,|\, \psi(t) - \|e(t)\| = \varepsilon\} \in (0, t_1).$$

Moreover,

$$\|e(t)\| \geq \psi(t) - \varepsilon \geq \lambda - \varepsilon \geq \lambda/2 \quad \forall\, t \in [t_0, t_1]$$

and so

$$k(t)\|e(t)\| = \frac{\|e(t)\|}{\psi(t) - \|e(t)\|} \geq \frac{\lambda}{2\varepsilon} \geq \widehat{u} \quad \forall\, t \in [t_0, t_1].$$

Therefore,

$$\text{sat}_{\widehat{u}}(k(t)e(t)) = \widehat{u}\|e(t)\|^{-1}e(t) \quad \forall\, t \in [t_0, t_1]$$

which, together with (2.6), implies that

$$\langle e(t), CB\, \text{sat}_{\widehat{u}}(k(t)e(t))\rangle \geq \gamma\widehat{u}\|e(t)\| \quad \forall\, t \in [t_0, t_1]\,,$$

and so, in view of (4.15) and (6.22), we may infer that

$$\langle e(t), \dot{e}(t)\rangle \leq -\Lambda\|e(t)\| \quad \text{for a.a. } t \in [t_0, t_1].$$

Integration, together with the Lipschitz property of ψ, now yields

$$\|e(t_1)\| - \|e(t_0)\| \leq -\Lambda[t_1 - t_0] \leq -|\psi(t_1) - \psi(t_0)| \leq \psi(t_1) - \psi(t_0),$$

whence the contradiction:

$$\varepsilon = \psi(t_0) - \|e(t_0)\| \leq \psi(t_1) - \|e(t_1)\| < \varepsilon\,.$$

Therefore, (6.25) holds. It immediately follows that the function k is bounded, with $k(t) \leq 1/\varepsilon$ for all $t \in [0, \omega)$. Moreover, in view of (6.23) and (6.24) and boundedness of r, we may infer boundedness of the solution

$$x\colon [0, \omega) \to \mathbb{R}^n, \quad t \mapsto x(t) = B(CB)^{-1}y(t) + Vz(t)\,.$$

To establish Assertions (i)-(iii), it remains only to show that $\omega = \infty$. Suppose that $\omega < \infty$ and define

$$\mathcal{C} := \big\{(t,\xi,\zeta) \in [0,\omega] \times \mathbb{R}^m \times \mathbb{R}^{n-m} \big| \;\; \psi(t) - \|\xi\| \geq \varepsilon,\; \|\xi\| \leq \|\psi\|_\infty,\; \|\zeta\| \leq M^0\big\}.$$

Then, in view of (6.23), (6.24) and (6.25), it follows that $\mathcal{C}$ is a compact set which contains graph$((e,z)) = \{(t,e(t),z(t)) |\; t \in [0,\omega)\}$, thereby contradicting the fact that the closure of the latter is not a compact subset of $\mathcal{D}$. Therefore, $\omega = \infty$.

Next, we show the Assertion (iv) holds, i.e. we establish the existence of $\tau \geq 0$ such that $\|u(\tau)\| < \widehat{u}$. Seeking a contradiction, suppose

$$k(t)\|e(t)\| \geq \widehat{u} \quad \forall\, t \geq 0\,.$$

Then $\text{sat}_{\widehat{u}}(k(t)e(t)) = \widehat{u}\|e(t)\|^{-1}e(t)$ for all $t \geq 0$ and so, by (2.6),

$$\langle e(t), CB\text{sat}_{\widehat{u}}(k(t)e(t))\rangle \geq \gamma\widehat{u}\|e(t)\| \quad \forall\, t \geq 0$$

which, in conjunction with (6.22), yields

$$\langle e(t), \dot{e}(t)\rangle \leq -[\gamma\widehat{u} + \Lambda - L]\,\|e(t)\| \quad \forall\, t \geq 0,$$

with $[\gamma\widehat{u} + \Lambda - L] > 0$. Integration gives the contradiction:

$$0 \leq \|e(t)\| \leq \|e(0)\| - [\gamma\widehat{u} + \Lambda - L]\,t \quad \forall\, t \geq 0.$$

We proceed to establish Assertion (v). Assume $\|u(\tau)\| < \widehat{u}$ for some $\tau \geq 0$. In view of (4.15), there exists $\delta \in (0,1)$ such that

$$\|u(\tau)\| \leq (1-\delta)\widehat{u} \;\; \text{and} \;\; (1-\delta)\gamma\widehat{u} \geq L.$$

Seeking a contradiction, suppose Assertion (v) is false. Then there exist $t_1 > t_0 \geq \tau$ such that

$$\|u(t_1)\| = \widehat{u} \;\; \text{and} \;\; \widehat{u} > \|u(t)\| \geq (1-\delta)\widehat{u} \geq L/\gamma \quad \forall\, t \in [t_0,t_1). \tag{6.26}$$

Then,

$$\begin{aligned}\langle e(t), \dot{e}(t)\rangle &\overset{(6.22)}{\leq} \|e(t)\|\,[L-\Lambda] - k(t)\langle e(t), CBe(t)\rangle \overset{(2.6)}{\leq} \|e(t)\|\,[L-\Lambda] - k(t)\gamma\|e(t)\|^2 \\ &= \big(L - \Lambda - \gamma\|u(t)\|\big]\,\|e(t)\| \overset{(6.26)}{\leq} -\Lambda\|e(t)\| \quad \text{for a.a. } t \in [t_0,t_1]\end{aligned}$$

which, on integration and invoking the Lipschitz property of ψ, yields

$$\|e(t_1)\| - \|e(t_0\| \leq -\Lambda\big[t_1 - t_0\big] \leq -|\psi(t_1) - \psi(t_0)| \leq \psi(t_1) - \psi(t_0),$$

whence the contradiction

$$\begin{aligned}\widehat{u} = \|u(t_1)\| = k(t_1)\|e(t_1)\| = \frac{\|e(t_1)\|}{\psi(t_1) - \|e(t_1)\|} \\ < \frac{\|e(t_0)\|}{\psi(t_0) - \|e(t_0)\|} = k(t_0)\|e(t_0)\| = \|u(t_0)\| < \widehat{u}.\end{aligned}$$

Finally, we turn to Assertion (vi). Note that $\|u(0)\| = \|e(0)\|/(\psi(0) - \|e(0)\|) < \widehat{u}$ is equivalent to $\|e(0)\| < \psi(0)\,\widehat{u}/(1+\widehat{u})$ and so the claim follows from Assertion (v) and setting $t = 0$. This completes the proof. □

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