Pairs of Disjoint Dominating Sets and the Minimum Degree of Graphs

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Abstract. For a connected graph G of order n and minimum degree δ we prove the existence of two disjoint dominating sets D_1 and D_2 such that, if $\delta \geq 2$, then $|D_1 \cup D_2| \leq \frac{6}{7}n$ unless $G = C_4$, and, if $\delta \geq 5$, then $|D_1 \cup D_2| \leq 2\frac{1+\ln(\delta+1)}{\delta+1}n$. While for the first estimate there are exactly six extremal graphs which are all of order 7, the second estimate is asymptotically best-possible.

Keywords. domination; domination number; domatic partition; domatic number; inverse domination; disjoint domination number

1 Introduction

We consider graphs $G = (V, E)$ with vertex set V and edge set E which are finite, simple and undirected.

Let $G = (V, E)$ be a graph and let $u \in V$ be a vertex. The neighbourhood $N_G(u)$ of u in G is the set $\{v \in V \mid uv \in E\}$ and the degree $d_G(u)$ of u in G is the number of edges incident with u. The minimum and maximum degree of a vertex in G are denoted by $\delta(G)$ and $\Delta(G)$. The closed neighbourhood $N_G[u]$ of $u \in G$ is the set $\{u\} \cup N_G(u)$. For some $i \in \mathbb{N}$ let $V_i = \{u \in V \mid d_G(u) = i\}$ and $V_{\geq i} = \{u \in V \mid d_G(u) \geq i\}$.

A set of vertices $D \subseteq V$ is said to *dominate* a vertex $u \in V$, if $N_G[u] \cap D \neq \emptyset$. D is a *dominating set* of G, if D dominates all vertices in V and the minimum cardinality of a dominating set of G is the domination number $\gamma(G)$ of G. Similarly, a pair (D_1, D_2) of disjoint sets of vertices $D_1, D_2 \subseteq V$ is said to dominate a vertex $u \in V$, if D_1 and D_2 dominate u. (D_1, D_2) is a *dominating pair*, if (D_1, D_2) dominates all vertices in V. The (total) cardinality of a pair (D_1, D_2) is $|D_1| + |D_2|$ and the minimum cardinality of a dominating pair is the *disjoint domination number* $\gamma\gamma(G)$ of G.

A path of length $l \geq 0$ in G is a sequence $P : u_0u_1u_2 \ldots u_l$ of $l + 1$ distinct vertices of G such that $u_{i-1}u_i \in E$ for $1 \leq i \leq l$. A cycle of length $l \geq 3$ in G is a sequence $C : u_1u_2 \ldots u_lu_1$ such that $u_1, u_2, \ldots, u_l \in V$ are l distinct vertices, $u_{i-1}u_i \in E$ for $2 \leq i \leq l$, and $u_1u_l \in E$. A path of length $i + 1$ whose endvertices are of degree at least 3 and whose i internal vertices are all of degree 2 is called an *i-path*. A cycle of length $i + 1$ which contains i vertices of degree 2 and one vertex of degree at least 3 is called an *i-cycle*.

Domination is a classical and well-studied graph-theoretical notion [14, 15]. Among the most fundamental results on the domination number are upper bound for graphs which satisfy a minimum degree condition [1, 2, 4, 20–23].

The first such result is due to Ore [21] who observed that the complement of every minimal dominating set of a graph $G = (V, E)$ of minimum degree at least 1 is also a dominating set. This implies that every such graph has two disjoint dominating sets and hence

$$\gamma(G) \leq \frac{1}{2}|V|.$$

For graphs $G = (V, E)$ of minimum degree at least 2, Blank [4] and — independently — McCuaig and Shepherd [20] proved that

$$\gamma(G) \leq \frac{2}{5}|V|$$

unless G is one of the seven graphs $H_1, H_2, \ldots, H_7$ in Figure 3.

Several authors studied so-called *domatic partitions* which are partitions of the vertex set of a graph into dominating sets. The maximum number of disjoint dominating sets into which a graph can be partitioned is known as the *domatic number* [6] (cf. Zelinka's contribution to [15]). Furthermore, graphs G having two disjoint minimum dominating sets [3] — i.e. graphs G with $\gamma\gamma(G) = 2\gamma(G)$ — and also the minimum intersection of pairs of minimum dominating sets [5, 9, 13] were considered.

Recently several authors initiated the study of the cardinalities of pairs of disjoint dominating sets in graphs. Kulli and Sigarkanti [19] introduce the *inverse domination number* which is the minimum cardinality of a dominating set whose complement contains a minimum dominating set (cf. [8, 11]).

Motivated by the inverse domination number, Hedetniemi et al. [17] defined and studied the disjoint domination number $\gamma\gamma(G)$ of a graph G. By Ore's observation,

$$\gamma\gamma(G) \leq |V|$$

for every graph $G = (V, E)$ without isolated vertices and Hedetniemi et al. characterized all extremal graphs for this bound. They also proved that it is NP-hard to determine $\gamma\gamma(G)$ even for chordal graphs G. In [17] they list 22 open problems in connection with the disjoint domination number, 9 of which were solved in [18].

It is a natural question why to devote special attention to the case of two disjoint dominating sets rather than k disjoint dominating sets for general k. The reason is that, by Ore's observation, the trivial necessary minimum degree condition is also sufficient for the existence of two disjoint dominating sets. For all fixed $k \geq 3$, it is NP-complete [12] to decide the existence of k disjoint dominating sets and no minimum degree condition is sufficient for the existence of three disjoint dominating sets. As a simple example attributed to Zelinka consider a bipartite graph with one partite set A containing $3\delta - 2$ vertices and a second partite set B containing $\binom{3\delta-2}{\delta}$ vertices each of which is adjacent to a different set of δ vertices from A. Clearly, this graph has minimum degree δ but does not contain three disjoint dominating sets.

Feige et al. [10] (cf. also [7]) proved that every graph G can be partitioned into

$$(1-o(1))\,\frac{\delta(G)+1}{\ln \Delta(G)}$$

dominating sets where the $o(1)$-term tends to 0 as $\Delta(G)$ tends to infinity. Considering the smallest k of these sets implies that every graph G has k disjoint dominating sets whose total cardinality is

$$(1+o(1))\,\frac{k\ln \Delta(G)}{\delta(G)+1}|V|. \tag{1}$$

Our results in the present paper are

- a best-possible upper bound on the disjoint domination number of graphs of minimum degree at least 2 together with the characterization of the unique exceptional graph and the six extremal graphs (Theorem 6) and
- an asymptotically best-possible upper bound on the disjoint domination number of graphs of minimum degree at least 5 (Theorem 8).

The first result is inspired by McCuaig and Shepherd's [20] work and their seven exceptional graphs $H_1, H_2, \ldots, H_7$ play an important role. The second result improves (1) for $k = 2$ and relies on a beautiful probabilistic argument used by Alon and Spencer [1] to prove the asymptotically best-possible bound

$$\gamma(G) \leq \frac{1+\ln(\delta(G)+1)}{\delta(G)+1}|V|.$$

2 Graph of Minimum Degree at least 2

We first prove the desired bound for graphs which arise by suitably subdividing the edges of some multigraph which may contain multiple edges but no loops.

Theorem 1 *Let $G^* = (V^*, E^*)$ be a multigraph which may contain multiple edges but no loops such that every vertex is incident with at least 3 edges. Let $E_1^* \cup E_2^* \cup E_3^*$ be a partition of the edge set E^* of G^*.*

If the graph $G = (V, E)$ arises from G^* by subdividing every edge in E_i^* exactly i times for $1 \leq i \leq 3$, then G has a dominating pair (D_1, D_2) such that $V_{\geq 3} = V^* \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V|$.

Proof: Let G^* and G be as in the statement of the result. We will prove the desired statement by explicitely describing the construction of a suitable dominating pair (D_1, D_2) for G. Initially, let $(D_1, D_2) = (\emptyset, \emptyset)$.

Note that the edges in E_i^* correspond exactly to the i-paths of G. Let $p_i = |E_i^*|$ for $1 \leq i \leq 3$. Furthermore, let $n_i = |V_i|$ and $n_{\geq i} = |V_{\geq i}|$ for $i \in \mathbb{N}$. Clearly, counting the vertices of G and the edges of G^* we obtain

$$|V| = n_{\geq 3} + p_1 + 2p_2 + 3p_3 \text{ and} \tag{2}$$

$$|E^*| = p_1 + p_2 + p_3 \geq \frac{3}{2}n_3 + 2n_{\geq 4}. \tag{3}$$

As a first step, we add all vertices in $V_{\geq 3} = V^*$ to either D_1 or D_2.

If $u, v \in V_{\geq 3}$ are the endvertices of an i-path P, then we call P *good*, if either $i \in \{1, 3\}$ and u and v do not both lie in one of the two sets D_1 and D_2, or $i = 2$ and u and v both lie in one of the two sets D_1 and D_2, i.e.

$$\begin{array}{lll} \text{either} & i \in \{1,3\} & \text{and } |\{u,v\} \cap D_1| = |\{u,v\} \cap D_2| = 1, \\ \text{or} & i = 2 & \text{and } \{|\{u,v\} \cap D_1|, |\{u,v\} \cap D_2|\} = \{0,2\}. \end{array}$$

We call i-paths *bad*, if they are not good and denote the number of bad i-paths by b_i for $1 \leq i \leq 3$.

We assume that the vertices in $V_{\geq 3} = V^*$ are added to either D_1 or D_2 in such a way that the total number of bad i-paths is as small as possible, i.e.

$$(b_1 + b_2 + b_3) \quad \rightarrow \quad \min. \tag{4}$$

Next, for every good i-path, we add $i - 1$ of the internal vertices to either D_1 or D_2 and for every bad i-path, we add all i internal vertices to either D_1 or D_2 in such a way that (D_1, D_2) dominates all vertices of degree 2 and as many vertices of degree at least 3 as possible, i.e. if $\dot{V}_i$ and $\dot{V}_{\geq i}$ denote the sets of vertices in V_i and $V_{\geq i}$ which are not — yet — dominated by (D_1, D_2), $\dot{n}_i = |\dot{V}_i|$, and $\dot{n}_{\geq i} = |\dot{V}_{\geq i}|$, then

$$\dot{n}_{\geq 3} \quad \rightarrow \quad \min. \tag{5}$$

Clearly, we may assume that the internal vertices of all i-paths are added to either D_1 or D_2 as indicated in Figure 1 where all vertices within squares belong to one of the two sets D_1 or D_2 and all vertices within cycles belong to the other set.

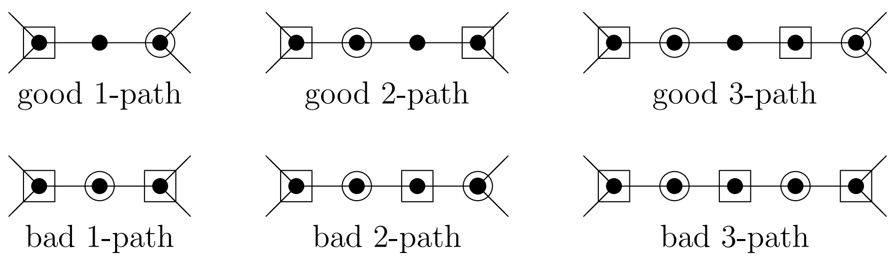


Figure 1

Let $\ddot{V}_j$ and $\ddot{V}_{\geq j}$ denote the set of vertices in V_j and $V_{\geq j}$ which do not belong to a bad i-path or a good 3-path. Let $\ddot{n}_j = |\ddot{V}_j|$ and $\ddot{n}_{\geq j} = |\ddot{V}_{\geq j}|$. Since all vertices in $V_{\geq 3}$ which lie on a bad i-path or a good 3-path are already dominated by (D_1, D_2), we have

$$\dot{n}_3 \leq n_3 \tag{6}$$

and

$$\dot{n}_{\geq 3} \leq \ddot{n}_{\geq 3}. \tag{7}$$

Claim 1

$$(b_1 + b_2 + b_3) \leq \frac{1}{2}(p_1 + p_2 + p_3) - \frac{1}{4}n_3 - \ddot{n}_{\geq 4} - \frac{1}{2}\ddot{n}_3 \tag{8}$$

Proof of Claim 1: It follows by the handshaking lemma that

$$2(p_1 + p_2 + p_3) = \sum_{i \geq 3} i n_i.$$

Furthermore, by (4), every vertex in $V_{\geq 3}$ belongs at least to as many good i-paths than bad i-paths. Therefore, another application of the handshaking lemma yields

$$\begin{aligned} 2\left(\sum_{i=1}^{3} p_i - \sum_{i=1}^{3} b_i\right) &\geq \sum_{i \geq 3} i\ddot{n}_i + \sum_{i \geq 3} \left\lceil \frac{i}{2} \right\rceil (n_i - \ddot{n}_i) \\ &= \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i + \sum_{i \geq 3} \left\lceil \frac{i}{2} \right\rceil n_i. \end{aligned}$$

Combining these two observations, we obtain

$$\begin{aligned} 2(b_1 + b_2 + b_3) &\leq 2(p_1 + p_2 + p_3) - \sum_{i \geq 3} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i - \sum_{i \geq 3} \left\lceil \frac{i}{2} \right\rceil n_i \\ &= (p_1 + p_2 + p_3) + \sum_{i \geq 3} \frac{i}{2} n_i - \sum_{i \geq} \left\lfloor \frac{i}{2} \right\rfloor \ddot{n}_i - \sum_{i \geq 3} \left\lceil \frac{i}{2} \right\rceil n_i \\ &\leq (p_1 + p_2 + p_3) - \frac{1}{2} n_3 - 2\ddot{n}_{\geq 4} - \ddot{n}_3 \end{aligned}$$

which is equivalent to the statement of the claim. □

For the purpose of the present proof we will consider a suitable directed graph

$$\vec{G}^*$$

with vertex set $V^* = V_{\geq 3}$ which contains a directed edge (u, v) from u to v for every good 2-path $P : uxyv$ in G such that $y \in D_1 \cup D_2$, i.e. a directed edge "(u, v)" indicates that v is already properly dominated by the vertices on P. (Note that $\vec{G}^*$ can contain multiple directed edges.)

For a vertex $u \in \dot{V}_{\geq 3}$ let

$$T_u$$

denote the set of vertices $v \in V_{\geq 3}$ such that $\vec{G}^*$ contains a directed path from u to v.

Claim 2 *If $v \in T_u$ for some $u \in \dot{V}_{\geq 3}$, then v is not contained in a bad i-path or a good 3-path in G and v is not the endvertex of two directed edges in $\vec{G}^*$.*

Proof of Claim 2: For contradiction, we assume that vertices u and v as stated in the claim exist.

Let $P : u_0u_1 \dots u_l$ be a directed path in $\vec{G}^*$ from $u = u_0$ to $v = u_l$. By definition, every directed edge (u_{r-1}, u_r) for some $1 \leq r \leq l$ corresponds to a good 2-path $P_r : u_{r-1}x_ry_ru_r$ with $y_r \in D_s$ for some fixed $s \in \{1, 2\}$. If we replace the vertex y_r in D_s with x_r for $1 \leq r \leq l$, then, by the assumption, all vertices which were dominated by (D_1, D_2) — in particular v — are still dominated by the new pair and the total number of bad i-path remains unchanged. Since u is dominated by the new pair, $\dot{n}_{\geq 3}$ is reduced by 1, which is a contradiction to (5). □

By Claim 2, the sets T_u for $u \in \dot{V}_{\geq 3}$ induce disjoint rooted treed $\vec{T}_u$ within $\vec{G}^*$ with root u. Furthermore, again by Claim 2, every leaf of $\vec{T}_u$ which is different from u is the endvertex of at least two good 1-paths. Clearly, the sum of the number of good 1-paths which contain u and the number of leaves in $\vec{T}_u$ is at least $d_G(u) \geq 3$. Therefore, we can associate 3 good 1-paths to every vertex in $\dot{V}_{\geq 3}$ such that every good 1-path is associated at most twice to vertices in $\dot{V}_{\geq 3}$. By double counting, we obtain

$$\dot{n}_{\geq 3} \;\leq\; \frac{2}{3}(p_1 - b_1) \leq \frac{2}{3}p_1. \tag{9}$$

We now turn (D_1, D_2) into a dominating pair of G by adding at most $\dot{n}_{\geq 3}$ vertices to the two sets and possibly moving some vertices from D_s to D_{3-s}, if all their neighbours belong to D_s.

We are ready to estimate the cardinality of (D_1, D_2).

$$\begin{aligned}
|D_1 \cup D_2| &\leq n_{\geq 3} + b_1 + p_2 + b_2 + 2p_3 + b_3 + \dot{n}_{\geq 3} \\
&\overset{(8)}{\leq} n_{\geq 3} + \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 - \frac{1}{4}n_3 - \ddot{n}_{\geq 4} - \frac{1}{2}\ddot{n}_3 + \dot{n}_{\geq 3} \\
&= \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 + (\dot{n}_{\geq 4} - \ddot{n}_{\geq 4}) + \frac{1}{2}\left(\dot{n}_3 - \ddot{n}_3\right) \\
&\overset{(7)}{\leq} \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 \\
&\overset{(9)}{\leq} \frac{1}{2}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{3}{4}n_3 + n_{\geq 4} + \frac{1}{2}\dot{n}_3 + \left(\frac{1}{4}p_1 - \frac{3}{8}\dot{n}_3\right) \\
&\overset{(6)}{\leq} \frac{3}{4}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{7}{8}n_3 + n_{\geq 4} \\
&\overset{(3)}{\leq} \frac{3}{4}p_1 + \frac{3}{2}p_2 + \frac{5}{2}p_3 + \frac{7}{8}n_3 + n_{\geq 4} + \left(\frac{1}{14}\left(p_1 + p_2 + p_3\right) - \frac{3}{28}n_3 - \frac{1}{7}n_{\geq 4}\right) \\
&= \frac{23}{28}p_1 + 2 \cdot \frac{11}{14}p_2 + 3 \cdot \frac{6}{7}p_3 + \frac{43}{56}n_3 + \frac{6}{7}n_{\geq 4} \\
&\overset{(2)}{\leq} \frac{6}{7}|V|,
\end{aligned}$$

where equality is only possible if $p_1 = p_2 = n_3 = 0$, i.e. every vertex in G belongs to a 3-path and no vertex has degree exact 3.

In this case

$$|V| = 3p_3 + n_{\geq 4}, \quad (10)$$
$$p_3 \geq 2n_{\geq 4} \quad (11)$$

and we construct a dominating pair (D_1, D_2) for G in the following way: First we add all vertices in $V_{\geq 4}$ to either D_1 or D_2 in such a way that the number of bad 3-paths is minimum as in (4). Clearly, every vertex in $V_{\geq 4}$ belongs to a good 3-path. Therefore, we can turn (D_1, D_2) to a dominating pair of G by adding exactly two internal vertices of every 3-path to either D_1 or D_2 as indicated in Figure 2.

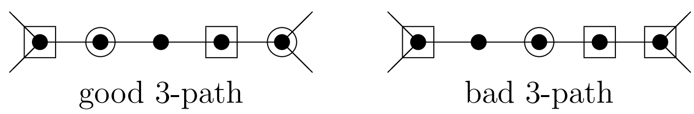
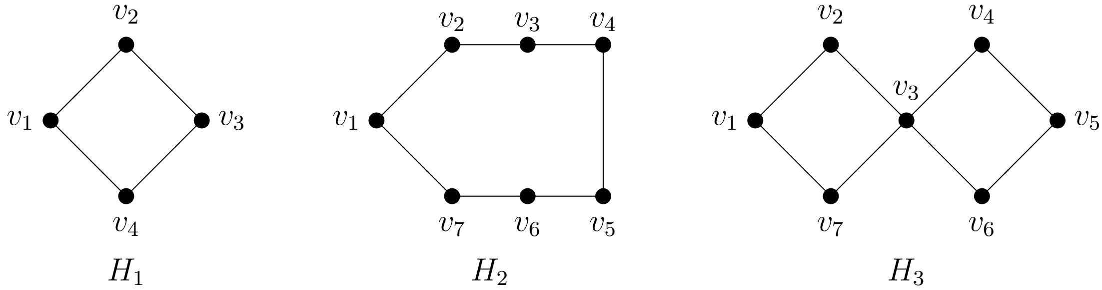


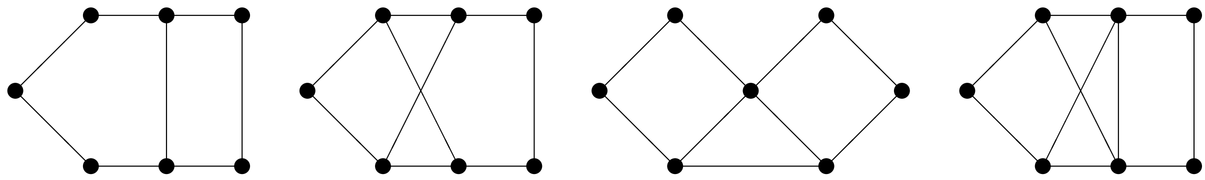
Figure 2

Now

$$\begin{aligned}|D_1 \cup D_2| &\leq n_{\geq 4} + 2p_3 \\ &\overset{(11)}{\leq} n_{\geq 4} + 2p_3 + \left(\frac{1}{7}p_3 - \frac{2}{7}n_{\geq 4}\right) \\ &= \frac{5}{7}n_{\geq 3} + \frac{15}{7}p_3 \\ &\overset{(10)}{\leq} \frac{5}{7}|V| \\ &< \frac{6}{7}|V|,\end{aligned}$$

and the proof is complete. □





H_4 H_5 H_6 H_7

Figure 3

Lemma 2 *(i)* $\gamma\gamma(H_1) = 4$, $\gamma\gamma(H_2) = \ldots = \gamma\gamma(H_7) = 6$.

(ii) *If* $G = (V, E) \in \{H_1, H_2, H_3\}$ *and* $v \in V$*, then* G *has a minimum dominating pair* (D_1, D_2) *such that* $v \in D_1$.

(iii) *If* $G = (V, E) \in \{H_1, H_2, H_3\}$ *and* $v \in V$*, then there is a pair* (D_1, D_2) *of disjoint sets of vertices of* G *such that* $|D_1 \cup D_2| = \gamma\gamma(G) - 1$*,* $v \in D_1$*,* D_1 *is a dominating set, and* $V \setminus \{v\} \subseteq N_G[D_2]$.

(iv) *If* G *arises from a path* $P : v_1v_2 \ldots v_rv_{r+1} \ldots v_{r+s}$ *by adding the edge* v_1v_r *such that* $r \in \{3, 4, 5\}$ *and* $s \in \{1, 3, 4, 5\}$*, then* G *has a minimum dominating pair* (D_1, D_2) *with* $v_{r+s} \in D_1$*,* $v_{r+s-1} \in D_2$*, and* $v_r \subseteq D_1 \cup D_2$*. Furthermore,* $\gamma\gamma(G) \leq \frac{6}{7}|V|$ *with equality if and only if* $(r, s) = (4, 3)$.

Proof: Since (i) is easily verified, we proceed to (ii).

Clearly, $(\{v_1, v_3\}, \{v_2, v_4\})$ is a dominating pair of H_1, $(\{v_1, v_3, v_6\}, \{v_2, v_4, v_7\})$ is a dominating pair of H_2, and $(\{v_1, v_5, v_6\}, \{v_3, v_4, v_7\})$ is a dominating pair of H_3. By symmetry - considering suitable automorphisms of the graphs, (ii) follows.

If $G = H_1$, then let $(D_1, D_2) = (\{v_1, v_2\}, \{v_3\})$, and, if $G = H_2$, then let $(D_1, D_2) = (\{v_1, v_4, v_5\}, \{v_3, v_6\})$. In both cases $v_1 \in D_1$, D_1 is dominating, and $V \setminus \{v_1\} \subseteq N_G[D_2]$ which, by symmetry, implies (iii) for $G \in \{H_1, H_2\}$.

If $G = H_3$ and $(D_1, D_2) = (\{v_1, v_4, v_6\}, \{v_3, v_5\})$, then $v_1 \in D_1$, D_1 is dominating and $V \setminus \{v_1\} \subseteq N_G[D_2]$. If $G = H_3$ and $(D_1, D_2) = (\{v_2, v_3, v_6\}, \{v_5, v_7\})$, then $v_2 \in D_1$, D_1 is dominating and $V \setminus \{v_2\} \subseteq N_G[D_2]$. Finally, if $G = H_3$ and $(D_1, D_2) = (\{v_3, v_6, v_7\}, \{v_1, v_5\})$, then $v_3 \in D_1$, D_1 is dominating and $V \setminus \{v_3\} \subseteq N_G[D_2]$. By symmetry, the above observations imply (iii) for $G = H_3$.

Now let G be as in (iv). It is easy to verify that the Table 1 defines suitable minimum dominating pairs for G which completes the proof. $\Box$

r	s	D_1	D_2
3	1	$\{v_2, v_4\}$	$\{v_3\}$
3	3	$\{v_3, v_6\}$	$\{v_2, v_5\}$
3	4	$\{v_2, v_4, v_7\}$	$\{v_3, v_6\}$
3	5	$\{v_3, v_5, v_8\}$	$\{v_2, v_4, v_7\}$
4	1	$\{v_2, v_5\}$	$\{v_3, v_4\}$
4	3	$\{v_2, v_4, v_7\}$	$\{v_1, v_3, v_6\}$
4	4	$\{v_2, v_5, v_8\}$	$\{v_3, v_4, v_7\}$
4	5	$\{v_3, v_4, v_6, v_9\}$	$\{v_2, v_5, v_8\}$
5	1	$\{v_2, v_4, v_6\}$	$\{v_3, v_5\}$
5	3	$\{v_3, v_5, v_8\}$	$\{v_2, v_4, v_7\}$
5	4	$\{v_2, v_4, v_6, v_9\}$	$\{v_3, v_5, v_8\}$
5	5	$\{v_3, v_5, v_7, v_{10}\}$	$\{v_2, v_4, v_6, v_9\}$

Table 1

Lemma 3 *If $G = (V, E)$ is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G is connected,

(iii) $V_{\geq 3}$ is independent, and

(iv) $G \notin \{H_1, H_2, H_3\}$,

then G has a dominating pair (D_1, D_2) with $V_{\geq 3} \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V|$.

Proof: For contradiction, we assume that $G = (V, E)$ is a counterexample of minimum order. It is easy to check that $|V| \geq 5$.

Claim 1 *There is no path $P : v_1v_2v_3v_4v_5$ in G such that the vertices v_1, v_2, v_3, and v_4 are of degree 2 and $v_1v_5 \notin E$.*

Proof of Claim 1: For contradiction, we assume that a path P as described in the claim exists. The graph

$$\begin{aligned} G' &= G[V \setminus \{v_2, v_3, v_4\}] + v_1v_5 \\ &= (V \setminus \{v_2, v_3, v_4\}, (E \setminus \{v_1v_2, v_2v_3, v_3v_4, v_4v_5\}) \cup \{v_1v_5\}) \end{aligned}$$

satisfies (i)-(iii) of the hypothesis.

If $G' \in \{H_1, H_2, H_3\}$, then G is either H_2, or a cycle of length 10 or arises from H_3 by subdividing one edge three times. In all three cases the desired result follows easily. Hence, we may assume that $G' \notin \{H_1, H_2, H_3\}$.

By the choice of G, this implies the existence of a dominating pair (D_1', D_2') of G' with $V_{\geq 3} = V_{\geq 3}' \subseteq D_1' \cup D_2'$ and $|D_1' \cup D_2'| < \frac{6}{7}(|V| - 3)$. Since $d_{G'}(v_1) = 2$, either v_1 or v_5 belong to $D_1' \cup D_2'$.

If $v_1 \notin D_1' \cup D_2'$ and $v_5 \in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_3\}, D_2' \cup \{v_2\})$, if $v_1 \in D_1'$ and $v_5 \in D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_4\}, D_2' \cup \{v_2\})$, and if $v_1 \in D_1'$ and $v_5 \notin D_2'$, then let $(D_1, D_2) = (D_1' \cup \{v_4\}, D_2' \cup \{v_3\})$. In all three cases (D_1, D_2) is a dominating pair of G with

$$|D_1 \cup D_2| = |D_1' \cup D_2'| + 2 < \frac{6}{7}(|V| - 3) + 2 < \frac{6}{7}|V|$$

which is a contradiction. By symmetry, this completes the proof. □

Claim 2 *There is no cycle $C : v_1v_2v_3v_4v_1$ in G such that $d_G(v_1) + d_G(v_3) \geq 7$, $d_G(v_2) = d_G(v_4) = 2$ and $G[V \setminus \{v_2, v_4\}]$ has two components with vertex sets $\{v_1\} \cup U_1$ and $\{v_3\} \cup U_3$ such that $v_1 \notin U_1$ and $v_3 \notin U_3$. (Note that one of the two sets U_1 and U_3 may be empty.)*

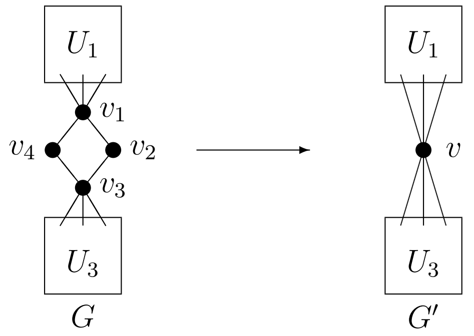


Figure 4

Proof of Claim 2: For contradiction, we assume that a cycle C as described in the claim exists. The graph G' which arises by contracting the cycle C to a single vertex v satisfies (i)-(iii) of the hypothesis. Since $d_{G'}(v) \geq 3$, the graph G' is different from H_1. Therefore, by Lemma 2 (i) and the choice of G, G' has a dominating pair (D_1', D_2') such that $v \in D_1'$ and $|D_1' \cup D_2'| \leq \frac{6}{7}(|V| - 3)$. By symmetry, we may assume that v has a neighbour v' in $D_2' \cap V_1$. Now (D_1, D_2) with

$$\begin{aligned} D_1 &= \{v_1, v_2\} \cup (D_1' \cap U_1) \cup (D_2' \cap U_3) \text{ and} \\ D_2 &= \{v_3\} \cup (D_2' \cap U_1) \cup (D_1' \cap U_3) \end{aligned}$$

is a dominating pair of G with

$$|D_1 \cup D_2| = |(D_1' \setminus \{v\}) \cup D_2'| + 3 \leq \left(\frac{6}{7}(|V| - 3) - 1\right) + 2 < \frac{6}{7}|V|,$$

which is a contradiction. □

Claim 3 *There are no six vertices $v_1, v_2, v_3, v_4, v_5, v_6 \in V$ such that*

$$v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_2v_6, v_4v_6 \in E,$$

v_1, v_3, v_5, and v_6 are of degree 2, v_2 and v_4 are of degree 3, $G[V \setminus \{v_2\}]$ is not connected.

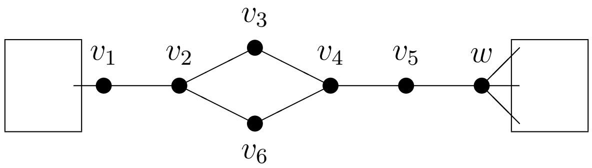


Figure 5

Proof of Claim 3: For contradiction, we assume that six vertices $v_1, v_2 \dots, v_6$ as described in the claim exist. Let w be the neighbour of v_5 different from v_4. The graph

$$G' = G[V \setminus \{v_2, v_3, v_4, v_5, v_6\}] + v_1 w$$

satisfies (i)-(iii) of the hypothesis.

Since the edge v_1w is a bridge of G', $G' \notin \{H_1, H_2, H_3\}$. By the choice of G, this implies the existence of a dominating pair (D'_1, D'_2) of G' with $V_{\geq 3} \setminus \{v_2, v_4\} \subseteq D'_1 \cup D'_2$ and $|D'_1 \cup D'_2| < \frac{6}{7}(|V| - 5)$. Since $d_{G'}(v_1) = 2$, either $v_1 \in D'_1 \cup D'_2$ or $w \in D'_1 \cup D'_2$.

If $v_1 \notin D'_1 \cup D'_2$ and $w \in D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_4, v_6\}, D'_2 \cup \{v_2, v_3\})$, if $v_1 \in D'_1$ and $w \notin D'_1 \cup D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_2, v_5\}, D'_2 \cup \{v_3, v_4\})$, if $v_1 \in D'_1$ and $w \in D'_1$, then let $(D_1, D_2) = (D'_1 \cup \{v_4\}, D'_2 \cup \{v_2, v_5\})$, and if $v_1 \in D'_1$ and $w \in D'_2$, then let $(D_1, D_2) = (D'_1 \cup \{v_4, v_5\}, D'_2 \cup \{v_2, v_3\})$. In all four cases (D_1, D_2) is a dominating pair of G with

$$|D_1 \cup D_2| \leq |D'_1 \cup D'_2| + 4 \leq \frac{6}{7}(|V| - 5) + 4 < \frac{6}{7}|V|$$

which is a contradiction. By symmetry, this completes the proof. □

By Claim 1, for every i-path in G we have $i \in \{1, 2, 3\}$ and for every i-cycle in G we have $i \in \{2, 3, 4\}$.

If G has no i-cycle, then the desired result follows from Theorem 1. Hence, we may assume that

$$C : v_1 v_2 \dots v_r v_1$$

with $r \in \{3, 4, 5\}$ is an $(r-1)$-cycle and $d_G(v_r) \geq 3$. If $d_G(v_r) = 3$, then there is an $(s-1)$-path

$$P : v_r v_{r+1} \dots v_{r+s}$$

in G with $s \in \{2, 3, 4\}$, $v_{r+1} \notin \{v_1, v_{r-1}\}$, and $d_G(v_{r+s}) \geq 3$. If $d_G(v_r) \geq 4$, then let $s = 0$, i.e. $s \in \{0, 2, 3, 4\}$.

Claim 4 *$d_G(v_r) \leq 4$ and, if $d_G(v_r) = 3$, then $d_G(v_{r+s}) = 3$.*

Proof of Claim 4: For contradiction, we assume that $d_G(v_r) \geq 5$ or that $d_G(v_r) = 3$ and $d_G(v_{r+s}) \geq 4$. The graph $G' = G[V \setminus \{v_1, v_2, \dots, v_{r+s-1}\}]$ satisfies (i)-(iii) of the hypothesis and is different from H_1 and H_2. Therefore, by Lemma 2 (i) and the choice of G, G' has a dominating pair (D'_1, D'_2) such that $v_{r+s} \in D'_1$ and $|D'_1 \cup D'_2| \leq \frac{6}{7}(|V| - (r + s - 1))$.

Table 2 summarizes how to construct a suitable dominating pair (D_1, D_2) for G which yields a contradiction and completes the proof of the claim. □

r	s	$D_1 \setminus D_1'$	$D_2 \setminus D_2'$
3	0	$\emptyset$	$\{v_1\}$
3	2	$\{v_2\}$	$\{v_3\}$
3	3	$\{v_3\}$	$\{v_2, v_4\}$
3	4	$\{v_2, v_4\}$	$\{v_1, v_5\}$
4	0	$\{v_3\}$	$\{v_2\}$
4	2	$\{v_1, v_3\}$	$\{v_2, v_4\}$
4	3	$\{v_3, v_4\}$	$\{v_2, v_5\}$
4	4	$\{v_2, v_5\}$	$\{v_1, v_3, v_6\}$
5	0	$\{v_3\}$	$\{v_1, v_4\}$
5	2	$\{v_2, v_4\}$	$\{v_3, v_5\}$
5	3	$\{v_3, v_5\}$	$\{v_2, v_4, v_6\}$
5	4	$\{v_2, v_4, v_6\}$	$\{v_1, v_3, v_7\}$

Table 2

By Claim 4, v_{r+s} has exactly two neighbours $x, y \not\in \{v_1, v_2, \ldots, v_{r+s-1}\}$. By (iii), $d_G(x) = d_G(y) = 2$.

If $xy \in E$, then $V = \{v_1, v_2, \ldots, v_{r+s}, x, y\}$ and the result follows easily using Lemma 2 (iv). Therefore, the unique neighbour z of y different from v_{r+s} is different from x.

If $xz \in E$, then Claim 2 and Claim 3 imply that $V = \{v_1, v_2, \ldots, v_{r+s}, x, y, z\}$ and the result follows easily. Therefore, $xz \not\in E$.

The graph

$$G' = (V', E') = G[V \setminus \{v_1, v_2, \ldots, v_{r+s}, y\}] + xz$$

satisfies (i)-(iii) of the hypothesis.

If $G' \in \{H_1, H_2, H_3\}$, then the desired result follows easily by combining Lemma 2 (iii) and (iv). Hence, we may assume that $G' \not\in \{H_1, H_2, H_3\}$. This implies, by the choice of G, that G' has a dominating pair (D_1', D_2') with $V'_{\geq 3} \subseteq D_1' \cup D_2'$ and $|D_1' \cup D_2'| < \frac{6}{7}|V'|$. In this case, Lemma 3 (iv) easily implies that G has a dominating pair (D_1, D_2) with $V_{\geq 3} \subseteq D_1 \cup D_2$ and $|D_1 \cup D_2| < \frac{6}{7}|V|$ which is a contradiction and completes the proof. $\Box$

Lemma 4 *If $G = (V, E)$ is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G connected,

(iii) G is edge-minimal with respect to (i)-(ii), and

(iv) $G \not\in \{H_1, H_2, H_3\}$,

then $\gamma\gamma(G) < \frac{6}{7}|V|$.

Proof: Let $c(G)$ denote the number of 3-cycles of G with exactly one vertex of degree 3. For contradiction, we assume that $G = (V, E)$ is a counterexample for which $|V| + c(G)$ is minimum. Clearly, we may assume again that $|V| \geq 5$.

In view of Lemma 3, we may assume that $V_{\geq 3}$ is not independent, i.e. $v'v'' \in E$ for some $v', v'' \in V_{\geq 3}$. By (iii) of the hypothesis, the edge $v'v''$ must be a bridge, i.e. G arises from the disjoint union of two graphs $G' = (V', E')$ and $G'' = (V'', E'')$ by adding the bridge $v'v''$ where $v' \in V'$ and $v'' \in V''$. Note that G' and G'' satisfy (i)-(iii) of the hypothesis.

First, we assume that $G', G'' \in \{H_1, H_2, H_3\}$. In this case let (D'_1, D'_2) and (D''_1, D''_2) be as in Lemma 2 (iii) with $v' \in D'_1$ and $v'' \in D''_1$. Clearly, $(D'_1 \cup D''_2, D''_1 \cup D'_2)$ is a dominating pair of G and $|D'_1 \cup D''_2 \cup D''_1 \cup D'_1| < \frac{6}{7}|V|$ which is a contradiction.

Next, we assume that $G' \not\in \{H_1, H_2, H_3\}$ and $G'' \neq H_1$. Since $c(G'), c(G'') \leq c(G) + 1$ and $|V'|, |V''| \geq 3$, we obtain, by the choice of G, $\gamma\gamma(G') < \frac{6}{7}|V'|$ and $\gamma\gamma(G'') \leq \frac{6}{7}|V''|$. If (D'_1, D'_2) and (D''_1, D''_2) are minimum dominating pairs of G' and G'', then $(D_1, D_2) = (D'_1 \cup D''_1, D'_2 \cup D''_2)$ is a dominating pair of G with $|D_1 \cup D_2| < \frac{6}{7}|V|$ which is a contradiction.

Therefore, we may assume that $G' \not\in \{H_1, H_2, H_3\}$ and $G'' = H_1$, i.e. G'' is a 3-cycle of G with exactly one vertex of degree 3. Let

$$G'' = (\{v'' = v_1, v_2, v_3, v_4\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_1\})$$

and let

$$G''' = G - v_1v_4 + v'v_4 = (V, (E \setminus \{v_1v_4\}) \cup \{v'v_4\}).$$

Clearly, G''' satisfies (i)-(iii) of the hypothesis, $G''' \not\in \{H_1, H_2, H_3\}$ and $c(G''') < c(G)$. Therefore, by the choice of G, we obtain that $\gamma\gamma(G''') < \frac{6}{7}|V|$.

Let (D'''_1, D'''_2) be a minimum dominating pair of G'''. Note that

$$|(D'''_1 \cup D'''_2) \cap \{v', v_1, v_2, v_3, v_4\}| \geq 4$$

and that we may assume $v' \in D'''_1$. Now, (D_1, D_2) with

$$\begin{aligned} D_1 &= (D'''_1 \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_3\} \text{ and} \\ D_2 &= (D'''_2 \setminus \{v_1, v_2, v_3, v_4\}) \cup \{v_1, v_2\} \end{aligned}$$

is a dominating pair of G with $|D_1 \cup D_2| < \frac{6}{7}|V|$ which is a contradiction.

This completes the proof. $\square$

Lemma 5 (McCuaig and Sherpherd, cf. Lemma 2 in [20]) *If $G = (V, E)$ is a connected graph with $|V| \leq 7$, $\delta(G) \geq 2$, and $\gamma(G) > \frac{2}{5}|V|$, then*

$$G \in \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}.$$

Theorem 6 *If $G = (V, E)$ is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G connected, and

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) < \frac{6}{7}|V|$.

Proof: Let $G' = (V', E')$ be a graph with $V' = V$ and $E' \subseteq E$ such that

(i) $\delta(G') \geq 2$,

(ii) G' connected, and

(iii) G' is edge-minimal with respect to (i)-(ii).

Clearly, $\gamma\gamma(G') \geq \gamma\gamma(G)$, and thus, by Lemma 4, the statement of the theorem is true, if $G' \notin \{H_1, H_2, H_3\}$.

If $G' = H_1$, then it is straightforward to check that $\gamma\gamma(G) \leq \frac{3}{4}|V|$, because $G \neq H_1$. Therefore, we may assume that $G' \in \{H_2, H_3\}$.

If G has a hamiltonian cycle and $\gamma(G) \leq 2$, then $\gamma\gamma(G) \leq 5$, because for any 2 vertices $v_i, v_j \in V$ there exists a dominating set of G of cardinality 3 that does not contain v_i or v_j. Thus, if $G' = H_2$, then, by Lemma 5, $\gamma\gamma(G) \leq \frac{5}{7}|V|$, because $G \notin \{H_2, H_4, H_5, H_6, H_7\}$.

Hence we may assume that G has no hamiltonian cycle and $G' = H_3$. If $G'' = (V'', E'')$ is a graph that arises from H_3 by adding an edge $e \in E \setminus E'$, then $\gamma\gamma(G'') \geq \gamma\gamma(G)$. By symmetry, $e \in \{v_1v_3, v_1v_4, v_1v_5, v_2v_4, v_2v_7\}$ (cf. Figure 3). Thus $\gamma\gamma(G'') \leq \frac{5}{7}$ or $G'' = H_6$ in which case G has a hamiltonian cycle — a contradiction. This completes the proof. $\square$

While Theorem 6 is best-possible in view of the graphs $H_2, H_3, \ldots, H_7$, we believe that the following considerable strengthening is possible.

Conjecture 7 *If* $G = (V, E)$ *is a graph such that*

(i) $\delta(G) \geq 2$,

(ii) G *connected, and*

(iii) $G \notin \{H_1, H_2, H_3, H_4, H_5, H_6, H_7\}$,

then $\gamma\gamma(G) \leq \frac{4}{5}|V|$.

By the results of McCuaig and Shepherd [20], there would be infinitely many extremal graphs for this estimate. In fact, we believe that the edge-minimal extremal graphs for the bound in Conjecture 7 are the same as those described in [20] for the bound $\gamma(G) \leq \frac{2}{5}|V|$.

3 Graph with Minimum Degree at least 5

In this section we prove an upper bound on $\gamma\gamma(G)$ for graphs G using the probabilistic method.

The proof builds on an elegant probabilistic argument given by Alon and Spencer [1]. Several times during the proof we will use Ore's observation [21] that the complement of a minimal dominating set in a graph of minimum degree at least 1 is also a dominating set.

Theorem 8 *If $G = (V, E)$ is a graph of order n and minimum degree $\delta \geq 5$, then*

$$\gamma\gamma(G) \leq 2\frac{1 + \ln(\delta + 1)}{\delta + 1}n.$$

Proof: Let $p = \frac{\ln(\delta+1)}{\delta+1}$. Note that $p \leq \frac{1}{2}$.

We construct a partition of V into three sets

$$V = D_1^0 \cup D_2^0 \cup Y$$

by assigning every vertex independently at random to the set D_1^0 with probability p, to the set D_2^0 with probability p, and to the set Y with probability $(1 - 2p)$.

Clearly, $\mathbf{E}[|D_1^0|] = \mathbf{E}[|D_2^0|] = np$.

Let

$$Z^1 = \left\{v \in V \mid N_G[v] \cap (D_1^0 \cup D_2^0) = \emptyset\right\}.$$

For a fixed vertex $v \in V$, we have

$$\mathbf{P}\left[v \in Z^1\right] = \mathbf{P}\left[N_G[v] \subseteq Y\right] = (1 - 2p)^{d_G(v)+1}.$$

Let

$$D_1^1$$

be a minimal dominating set of $G[Z^1]$ and let

$$D_2^1$$

be the union of $Z^1 \setminus D_1^1$ and a minimal set of vertices of G such that each isolated vertex in $G[Z^1]$ has a neighbour in D_2^1. Clearly, $D_2^1 \subseteq Y \setminus D_1^1$ and (D_1^1, D_2^1) dominates every vertex in Z^1.

Note that $|D_1^1| + |D_2^1| \leq 2|Z^1|$ and thus

$$\mathbf{E}\left[|D_1^1| + |D_2^1|\right] \leq 2\sum_{v \in V}(1 - 2p)^{d_G(v)+1}.$$

Let

$$Z_1^2 = \left\{v \in V \mid N_G[v] \cap (D_1^0 \cup D_1^1) = \emptyset\right\}.$$

Note that $|N_G[v] \cap D_2^0| \geq 1$ for each $v \in Z_1^2$, since otherwise $v \in Z^1$ and thus $|N_G[v] \cap D_1^1| \geq 1$ - a contradiction to $v \in Z_1^2$.

For a fixed vertex $v \in V$,

$$\begin{aligned}
\mathbf{P}\left[v \in Z_1^2\right] &= \mathbf{P}\left[N_G[v] \cap (D_1^0 \cup D_1^1) = \emptyset\right] \\
&\leq \mathbf{P}\left[(N_G[v] \cap D_1^0 = \emptyset) \wedge (N_G[v] \cap D_2^0 \neq \emptyset)\right] \\
&= \mathbf{P}\left[N_G[v] \cap D_1^0 = \emptyset\right] - \mathbf{P}\left[N_G[v] \cap (D_1^0 \cup D_2^0) = \emptyset\right] \\
&= (1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}.
\end{aligned}$$

Let

$$D_1^2$$

be a minimal set of vertices in $V \setminus (D_2^0 \cup D_2^1)$ such that each vertex $v \in Z_1^2$ which satisfies

$$|N_G[v] \cap (D_2^0 \cup D_2^1)| < d_G(v) + 1$$

is dominated by D_1^2. Note that $|D_1^2| \leq |Z_1^2|$ and thus

$$\mathbf{E}\left[|D_1^2|\right] \leq \sum_{v \in V} \left((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}\right).$$

Let

$$Z_2^2 = \left\{v \in V \mid N_G[v] \cap (D_2^0 \cup D_2^1) = \emptyset\right\}.$$

Note that $|N_G[v] \cap D_1^0| \geq 1$ for each $v \in Z_2^2$, since otherwise $v \in Z^1$ and thus $|N_G[v] \cap D_2^1| \geq 1$ - a contradiction to $v \in Z_2^2$.

For a fixed vertex $v \in V$,

$$\begin{aligned}
\mathbf{P}\left[v \in Z_2^2\right] &= \mathbf{P}\left[N_G[v] \cap (D_2^0 \cup D_2^1) = \emptyset\right] \\
&\leq \mathbf{P}\left[(N_G[v] \cap D_2^0 = \emptyset) \wedge (N_G[v] \cap D_1^0 \neq \emptyset)\right] \\
&= \mathbf{P}\left[N_G[v] \cap D_2^0 = \emptyset\right] - \mathbf{P}\left[N_G[v] \cap (D_2^0 \cap D_1^0) = \emptyset\right] \\
&= (1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}.
\end{aligned}$$

Let

$$D_2^2$$

be a minimal set of vertices in $V \setminus (D_1^0 \cup D_1^1 \cup D_1^2)$ such that each vertex $v \in Z_2^2$ which satisfies

$$|N_G[v] \cap (D_1^0 \cup D_1^1 \cup D_1^2)| < d_G(v) + 1$$

is dominated by D_2^2. Note that $|D_2^2| \leq |Z_2^2|$ and thus

$$\mathbf{E}\left[|D_2^2|\right] \leq \sum_{v \in V} \left((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}\right).$$

For $i \in \{1, 2\}$ let

$$D_i' = D_i^0 \cup D_i^1 \cup D_i^2.$$

Clearly, $D_1' \cap D_2' = \emptyset$.

For $i \in \{1,2\}$ let

$$X_i = \{v \in V \mid N_G[v] \subseteq D'_i\}\,.$$

Let D_i^3 be a minimal dominating set of $G[X_{3-i}]$ for $i \in \{1,2\}$.
Let

$$\begin{aligned} D_1 &= (D'_1 \setminus D_2^3) \cup D_1^3 \text{ and} \\ D_2 &= (D'_2 \setminus D_1^3) \cup D_2^3. \end{aligned}$$

Clearly, (D_1, D_2) is a dominating pair of G and, by the first moment method [1], we obtain

$$\begin{aligned} \gamma\gamma(G) &\leq \mathbf{E}\left[|D_1| + |D_2|\right] \\ &= \mathbf{E}\left[|(D'_1 \setminus D_2^3) \cup D_1^3|\right] + \mathbf{E}\left[|(D'_2 \setminus D_1^3) \cup D_2^3|\right] \\ &= \mathbf{E}\left[|D'_1|\right] + \mathbf{E}\left[|D'_2|\right] \\ &= \mathbf{E}\left[|D_1^0 \cup D_1^1 \cup D_1^2|\right] + \mathbf{E}\left[|D_1^0 \cup D_1^1 \cup D_1^2|\right] \\ &\leq 2np + 2\sum_{v \in V}(1-2p)^{d_G(v)+1} + 2\sum_{v \in V}\left((1-p)^{d_G(v)+1} - (1-2p)^{d_G(v)+1}\right) \\ &= 2np + 2\sum_{v \in V}(1-p)^{d_G(v)+1} \\ &\leq 2np + 2n(1-p)^{\delta+1} \\ &\leq 2np + 2ne^{-p(\delta+1)} \\ &= 2n\frac{1+\ln(\delta+1)}{\delta+1} \end{aligned}$$

which completes the proof. □

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