

# Long Cycles and Paths in Distance Graphs

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**Abstract.** For  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ , the distance graph  $P_n^D$  has vertex set  $\{0, 1, \dots, n-1\}$  and edge set  $\{ij \mid 0 \leq i, j \leq n-1, |j-i| \in D\}$ . Note that the important and very well-studied circulant graphs coincide with the regular distance graphs.

A fundamental result concerning circulant graphs is that for these graphs, a simple greatest common divisor condition, their connectivity, and the existence of a Hamiltonian cycle are all equivalent. Our main result suitably extends this equivalence to distance graphs. We prove that for a set  $D$ , there is a constant  $c_D$  such that the greatest common divisor of the integers in  $D$  is 1 if and only if for every  $n$ ,  $P_n^D$  has a component of order at least  $n - c_D$  if and only if for every  $n$ ,  $P_n^D$  has a cycle of order at least  $n - c_D$ . Furthermore, we discuss some consequences and variants of this result.

**Keywords.** Circulant graph; distance graph; connectivity; Hamiltonian cycle; Hamiltonian path

## 1 Introduction

Circulant graphs form an important and very well-studied class of graph [3, 18, 20, 21, 24, 28, 29]. They are Cayley graphs of cyclic groups and have been proposed for numerous network applications such as local area computer networks, large area communication networks, parallel processing architectures, distributed computing, and VLSI design. Their connectivity and diameter [3, 6, 20, 21, 30, 34], cycle and path structure [1, 2, 4, 5, 7, 11, 27], and further graph-theoretical properties have been studied in great detail. Polynomial time algorithms for isomorphism testing and recognition of circulant graphs have been long-standing open problems which were completely solved only recently [17, 26].

For  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ , the *circulant graph*  $C_n^D$  has vertex set  $[0, n-1] = \{0, 1, \dots, n-1\}$  and the neighbourhood  $N_{C_n^D}(i)$  of a vertex  $i \in [0, n-1]$  in  $C_n^D$  is given by

$$N_{C_n^D}(i) = \{(i+d) \bmod n \mid d \in D\} \cup \{(i-d) \bmod n \mid d \in D\}.$$

Clearly, we may assume  $\max(D) \leq \frac{n}{2}$  for every circulant graph  $C_n^D$ .

Our goal here is to extend some of the fundamental results concerning circulant graphs to the similarly defined yet more general class of distance graphs: For  $n \in \mathbb{N}$  and  $D \subseteq \mathbb{N}$ , the *distance graph*  $P_n^D$  has vertex set  $[0, n-1]$  and

$$\begin{aligned} N_{P_n^D}(i) = & \{i+d \mid d \in D \text{ and } (i+d) \in [0, n-1]\} \\ & \cup \{i-d \mid d \in D \text{ and } (i-d) \in [0, n-1]\} \end{aligned}$$

for all  $i \in [0, n-1]$ . Clearly, we may assume  $\max(D) \leq n-1$  for every distance graph  $P_n^D$ .

Every distance graph  $P_n^D$  is an induced subgraph of the circulant graph  $C_{n+\max(D)}^D$ . More specifically, distance graphs are the subgraphs of sufficiently large circulant graphs induced by sets of consecutive vertices. Conversely, our following simple observation from [13] shows that every circulant graph is in fact a distance graph.

**Proposition 1** ([13]) *A graph is a circulant graph if and only if it is a regular distance graph.*

*Proof:* Clearly, every circulant graph  $C_n^D$  is regular and isomorphic to the distance graph  $P_n^{D'}$  for  $D' = D \cup \{n-d \mid d \in D\}$ .

Now let  $P_n^D$  be a regular distance graph. Let  $D = \{d_1, d_2, \dots, d_k\}$  with  $d_1 < d_2 < \dots < d_k \leq n-1$ . Since the vertex 0 has exactly  $k$  neighbours  $D$ ,  $P_n^D$  is  $k$ -regular.

Let  $i \in [1, k]$ . The vertex  $d_i - 1$  has exactly  $i - 1$  neighbours  $j$  with  $j < d_i - 1$ . Hence  $d_i - 1$  has exactly  $k + 1 - i$  neighbours  $j$  with  $j > d_i - 1$  which implies  $(d_i - 1) + d_{k+1-i} \leq n - 1$ . The vertex  $d_i$  has exactly  $i$  neighbours  $j$  with  $j < d_i$ . Hence  $d_i$  has exactly  $k - i$  neighbours  $j$  with  $j > d_i$  which implies  $d_i + d_{k+1-i} > n - 1$ .

We obtain  $d_i + d_{k+1-i} = n$  for every  $i \in [1, k]$  which immediately implies that  $P_n^D$  is isomorphic to the circulant graph  $C_n^{D'}$  for  $D' = \{d \in D \mid d \leq \frac{n}{2}\}$ .  $\square$

Distance graphs lack the symmetry and algebraic interpretation of circulant graphs and the algebraic methods used in [17, 26] will not apply to them. In view of Proposition 1, the recognition of distance graphs will be at least as difficult as the recognition of circulant graphs.

Originally motivated by coloring problems for infinite distance graphs studied by Eggleton, Erdős, and Skilton [15, 16], most research on distance graphs focused on colorings [9, 12, 14, 22, 23, 31, 32].

One of the most fundamental results for circulant graphs is the following beautiful equivalence.

**Theorem 2 (Boesch and Tindell [6], Burkard and Sandholzer [8], Garfinkel [19])**  
For  $n \in \mathbb{N}$  and a finite set  $D \subseteq \mathbb{N}$ , the following statements are equivalent.

- (i)  $C_n^D$  is connected.
- (ii) The greatest common divisor  $\gcd(\{n\} \cup D)$  of the integers in  $\{n\} \cup D$  equals 1.
- (iii)  $C_n^D$  has a Hamiltonian cycle.

In 1970 Lovász [10, 25, 33] asked whether every connected vertex-transitive graph has a Hamiltonian path. Since circulant graphs are clearly vertex-transitive, Theorem 2 is a positive example for this well-studied problem [10, 33].

In the present paper we suitably extend Theorem 2 to distance graphs. While connectivity and hamiltonicity of circulants are equivalent to a simple necessary gcd-condition, we prove that a similar condition for distance graphs is only equivalent to the existence of a large component and a long cycle. We also discuss consequences and variants of our result.

## 2 Cycles and Paths in Distance Graphs

We immediately proceed to our main result. The residue of an integer  $n \in \mathbb{Z}$  modulo  $d \in \mathbb{N}$  will be denoted by  $n \bmod d$ .

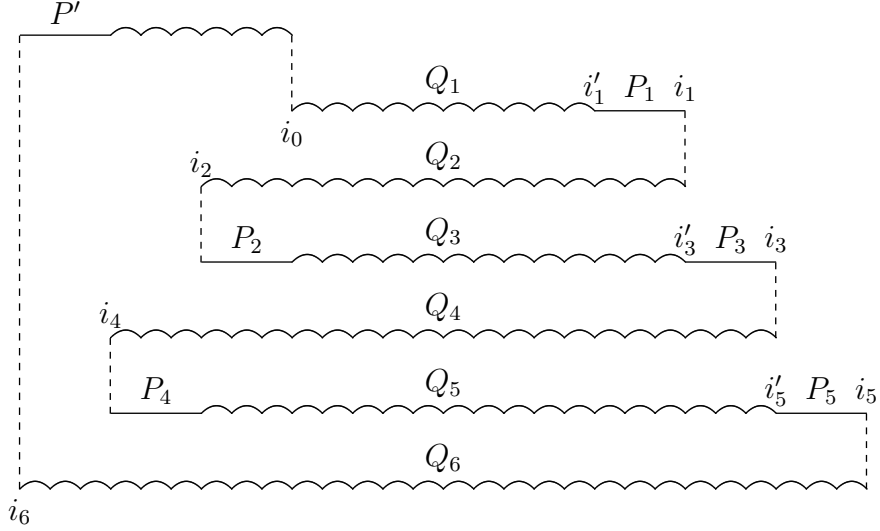
**Theorem 3** For a finite set  $D \subseteq \mathbb{N}$ , the following statements are equivalent.

- (i) There is a constant  $c_1(D)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a component of order at least  $n - c_1(D)$ .
- (ii)  $\gcd(D) = 1$ .
- (iii) There is a constant  $c_2(D)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a cycle of order at least  $n - c_2(D)$ .

*Proof:* (i)  $\Rightarrow$  (ii): Let  $n$  be such that  $n$  is even and  $n > 2c_1(D)$ . By (i), more than half the vertices are in the same component of  $P_n^D$ . By the pigeonhole principle, there is some  $i \in [0, n - 2]$  such that the two vertices  $i$  and  $i + 1$  are in the same component of  $P_n^D$ . This implies that there is a path in  $P_n^D$  from  $i$  to  $i + 1$ . Hence 1 is an integral linear combination of the elements in  $D$ . It is a well-known consequence of the Euclidean algorithm that this is equivalent to (ii).

(ii)  $\Rightarrow$  (iii): The essential idea in order to obtain a cycle which contains almost all vertices of  $P_n^D$  is to use increasing and decreasing paths which only use edges  $uv$  such that  $v - u$  is one fixed element  $d^*$  of  $D$ . Because the vertices on these paths always remain in the same residue class modulo  $d^*$ , such paths can be overlaid without intersecting. In order to connect these paths to a cycle, we use short paths which are close to 0 or  $n - 1$  and

whose end vertices are in different residue classes modulo  $d^*$ . In this way the cycle can collect all vertices of  $P_n^D$  in some middle section and only misses vertices close to 0 or  $n - 1$  in terms of  $D$  (cf. Figure 1).



**Figure 1**

In view of the constant  $c_2(D)$  in (iii), we may tacitly assume in the following that  $n$  is sufficiently large in terms of  $D$ .

Let  $d_{\max} = \max(D)$  and  $D^- = D \setminus \{d_{\max}\}$ . Since  $\gcd(D) = 1$ , 1 is an integral linear combination of the elements of  $D$ . Hence there are integers  $n_d$  for  $d \in D^-$  such that

$$1 = \left( \sum_{d \in D^-} n_d d \right) \bmod d_{\max} = \left( \sum_{d \in D^-} (n_d \bmod d_{\max}) d \right) \bmod d_{\max}. \quad (1)$$

This implies the existence of a path

$$P : v_0 v_1 \dots v_k$$

in  $P_n^D$  such that  $v_i - v_{i-1} \in D^-$  for all  $i \in [1, k]$  and

$$\{v_i \bmod d_{\max} \mid i \in [0, k]\} = [0, d_{\max} - 1],$$

i.e.  $P$  is a monotonously increasing path which only uses edges  $uv$  with  $v - u \in D^-$  and contains a vertex from every residue class modulo  $d_{\max}$ .

We assume that  $P$  is chosen so as to be shortest possible. This implies that the residues modulo  $d_{\max}$  of the vertices  $v_0$  and  $v_k$  appear exactly once on  $P$ . Let

$$r_1, r_2, \dots, r_{d_{\max}}$$

denote the residues modulo  $d_{\max}$  in the order in which they appear for the first time when traversing  $P$  from  $v_0$  to  $v_l$ . Clearly,

$$\begin{aligned} r_1 &= v_0 \bmod d_{\max}, \\ r_2 &= v_1 \bmod d_{\max}, \text{ and} \\ r_{d_{\max}} &= v_k \bmod d_{\max}. \end{aligned}$$

Furthermore,  $P$  is the concatenation of  $(d_{\max} - 1)$  edge-disjoint paths

$$P = P_1 P_2 \dots P_{d_{\max}-1}$$

such that for  $i \in [1, d_{\max} - 1]$ , the path  $P_i$  begins at the smallest vertex  $v_j$  on  $P$  with  $r_i = v_j \bmod d_{\max}$  and ends at the smallest vertex  $v_{j'}$  on  $P$  with  $r_{i+1} = v_{j'} \bmod d_{\max}$ ; let  $l_i = v_{j'} - v_j$  for these indices.

By the choice of  $P$ , for  $i \in [1, d_{\max} - 1]$ , all internal vertices of  $P_i$  have residues modulo  $d_{\max}$  in  $\{r_j \mid 2 \leq j \leq i - 1\}$ .

Let

$$l = l_1 + l_2 + \dots + l_{d_{\max}-1}.$$

Note that  $l = v_k - v_0$ .

We now describe a long cycle  $C$  in  $P_n^D$ . The general structure of  $C$  is illustrated in Figure 1. For simplicity we will first assume that  $d_{\max}$  is even.

Let  $i_0$  be the smallest integer at least

$$(d_{\max} + l_2) + (d_{\max} + l_4) + \dots + (d_{\max} + l_{d_{\max}-2}) + (d_{\max} + l)$$

of residue  $r_1$  modulo  $d_{\max}$ . Furthermore, let  $i'_1$  be the largest integer at most

$$(n - 1) - l_1 - (d_{\max} + l_3) - (d_{\max} + l_5) - \dots - (d_{\max} + l_{d_{\max}-1})$$

of residue  $r_1$  modulo  $d_{\max}$ .

We start  $C$  with an increasing path  $Q_1$  only using edges  $uv$  with  $v - u = d_{\max}$  which begins at  $i_0$  and ends at  $i'_1$ . We continue  $C$  with  $P_1$  shifted by a multiple of  $d_{\max}$  such that it begins at  $i'_1$  and ends at a vertex  $i_1$ .

For  $j$  from 1 up to  $\frac{d_{\max}}{2} - 1$ , we proceed as follows: We assume that we have already constructed  $C$  until the end of a shifted path  $P_{2j-1}$  which ends at a vertex  $i_{2j-1}$ . We continue  $C$  with a decreasing path  $Q_{2j}$  which only uses edges  $uv$  with  $v - u = d_{\max}$  and ends at the largest integer  $i_{2j}$  at most  $i_{2j-2} - l_{2j}$  with residue  $r_{2j}$  modulo  $d_{\max}$ . We continue  $C$  with  $P_{2j}$  shifted by a multiple of  $d_{\max}$  such that it begins at  $i_{2j}$ . We continue  $C$  with an increasing path  $Q_{2j+1}$  which only uses edges  $uv$  with  $v - u = d_{\max}$  and ends at the smallest integer  $i'_{2j+1}$  at least  $i_{2j-1}$  with residue  $r_{2j+1}$  modulo  $d_{\max}$ . We continue  $C$  with  $P_{2j+1}$  shifted by a multiple of  $d_{\max}$  such that it begins at  $i'_{2j+1}$  and ends at a vertex  $i_{2j+1}$ . At this point, we increase  $j$  until it reaches  $\frac{d_{\max}}{2} - 1$ .

To complete  $C$ , we may assume now that we have already constructed  $C$  until the end of the shifted path  $P_{d_{\max}-1}$  which ends at a vertex  $i_{d_{\max}-1}$ . We continue  $C$  with a decreasing

path  $Q_{d_{\max}}$  which only uses edges  $uv$  with  $v - u = d_{\max}$  and ends at the largest integer  $i_{d_{\max}}$  at most  $i_{d_{\max}-2} - l$  with residue  $r_{d_{\max}}$  modulo  $d_{\max}$ .

Let

$$P' : u_0 u_1 \dots u_{k'}$$

be a path in  $P_n^D$  such that  $u_0 = i_{d_{\max}}$ ,  $u_i - u_{i-1} \in D^-$  for all  $i \in [1, k']$ ,  $r_1 = u_{k'} \bmod d_{\max}$ , and  $l' = u_{k'} - u_0$  is minimum possible.

Clearly,  $l' \leq l$ . Furthermore, no internal vertex of  $P'$  has residue  $r_{\max}$  modulo  $d_{\max}$ . We continue  $C$  with  $P'$ . Finally, we complete  $C$  with an increasing path which only uses edges  $uv$  with  $v - u = d_{\max}$ , begins at  $u_{k'}$  and ends at  $i_0$ .

At this point we have completely described  $C$  as the concatenation of paths. Clearly, the choices of  $i_0$  and  $i'_1$  imply that  $C$  never leaves  $[0, n - 1]$ , i.e.  $C$  is in fact a closed walk within  $P_n^D$ . In order to show that  $C$  is a cycle, it remains to prove that it visits no vertex twice. This follows easily from the facts that

- the vertices on  $Q_i$  all have residue  $r_i$  modulo  $d_{\max}$  for all  $i \in [1, d_{\max}]$ ,
- the end vertices of the shifted paths  $P_i$  are the first vertices on  $C$  - traversed as constructed above - which have residue  $r_{i+1}$  modulo  $d_{\max}$  for all  $i \in [1, d_{\max} - 1]$ ,
- all internal vertices of  $P_i$  have residues modulo  $d_{\max}$  in  $\{r_j \mid 2 \leq j \leq i - 1\}$  for all  $i \in [1, d_{\max} - 1]$ , and
- no internal vertex of  $P'$  has residue  $r_{\max}$  modulo  $d_{\max}$ .

Since  $C$  contains all vertices between  $i_0$  and  $i'_1$ , it misses at most

$$\begin{aligned} & 2d_{\max} + l_1 + (d_{\max} + l_2) + (d_{\max} + l_3) + \dots + (d_{\max} + l_{d_{\max}-1}) + (d_{\max} + l) \\ &= d_{\max}(d_{\max} + 1) + 2l \end{aligned}$$

many vertices of  $P_n^D$ . Since this expression is bounded in terms of  $D$ , the proof of (iii) in the case that  $d_{\max}$  is even is complete.

Next, we consider the case that  $d_{\max}$  is odd.

Let  $i_0$  be the smallest integer at least

$$(d_{\max} + l_3) + (d_{\max} + l_5) + \dots + (d_{\max} + l_{d_{\max}-2}) + (d_{\max} + l)$$

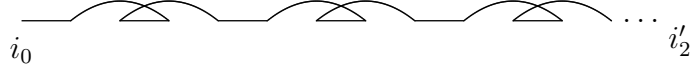
of residue  $r_1$  modulo  $d_{\max}$ . Furthermore, let  $i'_2$  be the largest integer at most

$$(n - 1) - l_2 - (d_{\max} + l_4) - (d_{\max} + l_6) - \dots - (d_{\max} + l_{d_{\max}-1})$$

of residue  $r_2$  modulo  $d_{\max}$ .

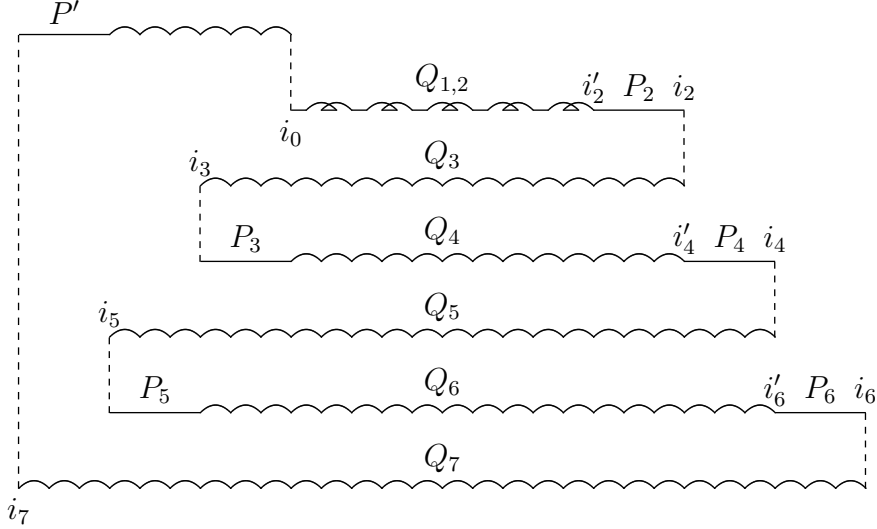
Let  $d' = v_1 - v_0$ . In the above construction, we replace the path  $Q_1$  by the path  $Q_{1,2}$  illustrated in Figure 2 with

$$\begin{aligned} Q_{1,2} : & i_0(i_0 + d')(i_0 + d_{\max} + d')(i_0 + d_{\max})(i_0 + 2d_{\max}) \\ & (i_0 + 2d_{\max} + d')(i_0 + 3d_{\max} + d')(i_0 + 3d_{\max})(i_0 + 4d_{\max}) \\ & \dots i'_2 \end{aligned}$$



**Figure 2**

Note that  $Q_{1,2}$  visits all but at most one of the vertices between  $i_0$  and  $i'_1$  of residues  $r_1$  or  $r_2$  modulo  $d_{\max}$ . The rest of the above construction is adapted accordingly as illustrated in Figure 3 replacing  $P_i$  and  $Q_i$  with  $P_{i+1}$  and  $Q_{i+1}$  for all  $i \in [1, d_{\max} - 1]$ .



**Figure 3**

Again the same residue properties as before imply that the closed walk  $C$  is a cycle. Also as before, the number of vertices of  $P_n^D$  missed by  $C$  is at most  $d_{\max}(d_{\max} + 1) + 2l$  which is bounded in terms of  $D$ . This completes the proof of the implication “(ii)  $\Rightarrow$  (iii)”.

(iii)  $\Rightarrow$  (i): Since this implication is trivial, the proof is complete.  $\square$

We add some comments concerning Theorem 3.

It is easy to see that a distance graph  $P_n^D$  with  $\gcd(D) = 1$  and  $n \geq 2 \max(D) + 1$  is actually connected. Hence (i) in Theorem 3 could be replaced by:

- (i)' *There is a constant  $c_3(D)$  such that for every  $n \in \mathbb{N}$  with  $n \geq c_3(D)$ , the distance graph  $P_n^D$  is connected.*

For (iii) in Theorem 3, a similar change is not possible, i.e. no lower bound on the order  $n$  would imply that  $P_n^D$  has a Hamiltonian cycle. If  $n$  as well as all elements of  $D$  are odd for instance, then  $P_n^D$  is bipartite and every cycle misses at least one vertex. In this sense, Theorem 3 is best-possible.

It follows easily from (1) that our proof yields the estimate

$$c_2(D) = O(\max(D)^2 + l) = O(\max(D)^3 |D|).$$

For the case that  $\gcd(D)$  is different from 1, Theorem 3 implies the following corollary.

**Corollary 4** For a finite set  $D \subseteq \mathbb{N}$  and  $d \in \mathbb{N}$ , the following statements are equivalent.

- (i) There is a constant  $c_4(D)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a component of order at least  $\frac{n}{d} - c_4(D)$ .
- (ii)  $\gcd(D) \leq d$ .
- (iii) There is a constant  $c_5(D)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a cycle of order at least  $\frac{n}{d} - c_5(D)$ .

Theorem 3 trivially implies yet another condition which is equivalent to (i), (ii) and (iii) in Theorem 3:

- (iv) There is a constant  $c_6(D)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a path of order at least  $n - c_6(D)$ .

Clearly, such a path can be obtained from the cycle in (iii) by deleting one edge. It traverses  $[0, n - 1]$  several times back and forth just like the cycle does. We believe that there is also always a path containing almost all vertices of  $P_n^D$  which is *essentially monotonic*, i.e. it traverses  $[0, n - 1]$  once. The following conjecture makes this precise.

**Conjecture 5** For a finite set  $D \subseteq \mathbb{N}$ , the following statements are equivalent.

- (i)  $\gcd(D) = 1$ .
- (ii) There are two constants  $c_7(D)$  and  $c_8(D)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a path  $u_0 u_1 \dots u_l$  of order at least  $n - c_7(D)$  such that  $u_j > u_i$  for all  $0 \leq i, j \leq l$  with  $j - i \geq c_8(D)$ .

A simple modification of the construction used in the proof of Theorem 3 implies the following weak version of Conjecture 5.

**Theorem 6** If  $D \subseteq \mathbb{N}$  is a finite set with  $\gcd(D) = 1$  and  $\epsilon > 0$ , then there are constants  $c_9(D, \epsilon)$  and  $c_{10}(D, \epsilon)$  such that for every  $n \in \mathbb{N}$ , the distance graph  $P_n^D$  has a path  $u_0 u_1 \dots u_l$  of order at least

$$(1 - \epsilon)n - c_9(D, \epsilon)$$

such that  $u_j > u_i$  for all  $0 \leq i, j \leq l$  with  $j - i \geq c_{10}(D, \epsilon)$ .

*Proof:* Since  $\gcd(D) = 1$ ,  $D$  contains at least one odd element  $d_{\text{odd}}$ . Replacing the increasing and decreasing paths  $Q_i$  from the proof of Theorem 3 with increasing and decreasing paths  $R_i$  which only use edges  $uv$  with  $v - u = d_{\text{odd}}$ , and using the paths  $P_1, P_2, \dots, P_{d_{\text{odd}}-1}$  and  $P'$  for parity changes as indicated in Figure 4, we obtain a path  $R$  from a vertex  $i_0$  to a vertex  $i_{d_{\text{odd}}} > i_0$  which both have residue  $r_1$  modulo  $d_{\text{odd}}$ .



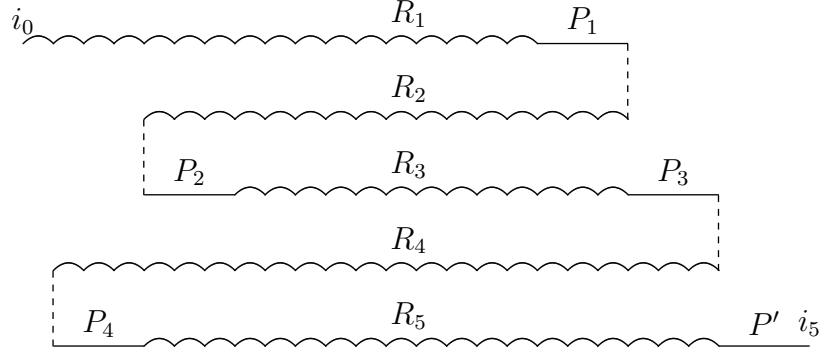


Figure 4

Note that  $R$  only visits vertices within  $[i_0, i_{d_{\text{odd}}}]$ . Furthermore, the number of vertices in  $[i_0, i_{d_{\text{odd}}}]$  which  $R$  misses is bounded in terms of  $D$ . Therefore, increasing all paths  $R_1, R_2, \dots, R_{d_{\text{odd}}}$  by the same sufficiently large multiple of  $d_{\text{odd}}$ , we can ensure that  $R$  has order  $(1 - \epsilon)(i_{d_{\text{odd}}} - i_0 + 1)$ . Concatenating shifted copies of  $R$  yields the desired path and completes the proof.  $\square$

Note that Conjecture 5 is trivial, if  $D$  contains only one element. If  $D$  contains exactly two elements, then Conjecture 5 easily follows from the following result.

**Proposition 7** *If  $d_1, d_2 \in \mathbb{N}$  are such that  $d_1 > d_2$  and  $\gcd(\{d_1, d_2\}) = 1$ , then  $P_{d_1+d_2+1}^{\{d_1, d_2\}}$  has a Hamiltonian path which begins at 0 and ends at  $d_1 + d_2$ .*

*Proof:* Consider the sequence  $i_0, i_1, \dots, i_{d_1+d_2}$  produced by Algorithm 1 below.

```

 $i_0 := 0;$ 
 $n := 0;$ 
 $n_1 := 0;$ 
 $n_2 := 0;$ 
while  $n < d_1 + d_2$  do
  if  $i_n \geq d_2$  and  $n_2 < d_1 - 1$  then
     $i_{n+1} := i_n - d_2;$ 
     $n_2 := n_2 + 1;$ 
  else
     $i_{n+1} := i_n + d_1;$ 
     $n_1 := n_1 + 1;$ 
  end
   $n := n + 1;$ 
end

```

Algorithm 1

Clearly,  $i_j \geq 0$  for  $j \in [0, d_1 + d_2]$ .

If  $n_1 > d_2 + 1$  after the termination of the algorithm, then  $n_2 = n - n_1 < d_1 - 1$  and hence

$$i_{d_1+d_2} = n_1 d_1 - n_2 d_2 \geq (d_2 + 2)d_1 - (d_1 - 2)d_2 = 2d_1 + 2d_2.$$

Let  $j \in [1, d_1 + d_2 - 1]$  be maximum such that  $i_{j+1} = i_j + d_1$ . Clearly,  $i_{j+1} \geq d_2$  and the algorithm would have set  $i_{j+1} = i_j - d_2$  instead, which is a contradiction. Hence  $n_1 \leq d_2 + 1$ . Since  $n_2 \leq d_1 - 1$ , we obtain  $n_1 = d_2 + 1$  and  $n_2 = d_1 - 1$ . This implies

$$i_{d_1+d_2} = (d_2 + 1)d_1 - (d_1 - 1)d_2 = d_1 + d_2.$$

If  $i_j > d_1 + d_2$  for some  $j \in [1, d_1 + d_2 - 1]$ , then let  $j$  be largest with this property. Clearly,  $i_{j-1} \geq d_2$  and at this moment of the execution of the algorithm  $n_2 < d_1 - 1$ . Therefore, the algorithm would have set  $i_j = i_{j-1} - d_2$  instead, which is a contradiction. Hence  $i_j \leq d_1 + d_2$  for all  $j \in [0, d_1 + d_2]$ ,

If  $i_r = i_s$  for some  $r, s \in [0, d_1 + d_2]$  with  $s > r$ , then  $i_s - i_r = a_1 d_1 - a_2 d_2 = 0$  for some  $a_2 \in [1, d_1 - 1]$ . This implies  $a_1 d_1 = a_2 d_2$ . Since  $\gcd(\{d_1, d_2\}) = 1$ , we obtain that  $a_2$  must be a multiple of  $d_1$ , which is a contradiction. Hence all  $d_1 + d_2 + 1$  integers  $i_0, i_1, \dots, i_{d_1+d_2}$  are distinct and define the desired Hamiltonian path of  $P_{d_1+d_2+1}^{\{d_1, d_2\}}$ . This completes the proof.  $\square$

It seems possible that Proposition 7 generalizes to sets  $D$  with more elements. For  $D = \{6, 10, 15\}$  for instance, the pattern 0, 6, 12, 2, 8, 14, 4, 10, 16, 1, 7, 13, 3, 9, 15, 5, 11, 17 yields a Hamiltonian path in  $P_{18}^D$  which begins at 0 and ends at 17.

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