Connectivity and Diameter in Distance Graphs

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Abstract. For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the distance graph P_n^D has vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{ij \mid 0 \leq i, j \leq n-1, |j-i| \in D\}$. The class of distance graphs generalizes the important and very well-studied class of circulant graphs which have been proposed for numerous network applications. In view of fault tolerance and delay issues in these applications, the connectivity and diameter of circulant graphs have been studied in great detail.

Our main contributions are hardness results concerning computational problems related to the connectivity and diameter of distance graphs and a number-theoretic characterization of the connected distance graphs P_n^D for $|D| = 2$.

Keywords. Circulant graph; distance graph; multiple loop networks; connectivity; diameter

1 Introduction

Circulant graphs form an important and very well-studied class of graph [1, 9, 11, 12, 16, 18, 19]. They are Cayley graphs of cyclic groups and have been proposed for numerous network applications such as local area computer networks, large area communication networks, parallel processing architectures, distributed computing, and VLSI design. In view of fault tolerance and delay issues in these applications, the connectivity and diameter of circulant graphs have been studied in great detail [1, 2, 11, 12, 20, 24].

For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the *circulant graph* C_n^D has vertex set $[0, n-1] = \{0, 1, \ldots, n-1\}$ and the neighbourhood $N_{C_n^D}(i)$ of a vertex $i \in [0, n-1]$ in C_n^D is given by

$$N_{C_n^D}(i) = \{(i + d) \bmod n \mid d \in D\} \cup \{(i - d) \bmod n \mid d \in D\}.$$

Clearly, we may assume $\max(D) \leq \frac{n}{2}$ for every circulant graph C_n^D.

Our goal is to investigate how some of the fundamental results concerning circulant graphs generalize to the similarly defined yet more general class of distance graphs: For $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$, the *distance graph* P_n^D has vertex set $[0, n-1]$ and

$$\begin{aligned} N_{P_n^D}(i) &= \{i + d \mid d \in D \text{ and } (i + d) \in [0, n-1]\} \\ &\quad \cup \{i - d \mid d \in D \text{ and } (i - d) \in [0, n-1]\} \end{aligned}$$

for all $i \in [0, n-1]$. Clearly, we may assume $\max(D) \leq n - 1$ for every distance graph P_n^D.

Every distance graph P_n^D is an induced subgraph of the circulant graph $C_{n+\max(D)}^D$. More specifically, distance graphs are the subgraphs of sufficiently large circulant graphs induced by sets of consecutive vertices, i.e. they represent the structure of small segments of circulant networks. Conversely, the following simple observation shows that every circulant graph is in fact a distance graph.

Proposition 1 *A graph is a circulant graph if and only if it is a regular distance graph.*

Proof: Clearly, every circulant graph C_n^D is regular and isomorphic to the distance graph $P_n^{D'}$ for $D' = D \cup \{n - d \mid d \in D\}$.

Now let P_n^D be a regular distance graph. Let $D = \{d_1, d_2, \ldots, d_k\}$ with $d_1 < d_2 < \ldots < d_k \leq n - 1$. Since the vertex 0 has exactly k neighbours D, P_n^D is k-regular.

Let $i \in [1, k]$. The vertex $d_i - 1$ has exactly $i - 1$ neighbours j with $j < d_i - 1$. Hence $d_i - 1$ has exactly $k + 1 - i$ neighbours j with $j > d_i - 1$ which implies $(d_i - 1) + d_{k+1-i} \leq n - 1$. The vertex d_i has exactly i neighbours j with $j < d_i$. Hence d_i has exactly $k - i$ neighbours j with $j > d_i$ which implies $d_i + d_{k+1-i} > n - 1$.

We obtain $d_i + d_{k+1-i} = n$ for every $i \in [1, k]$ which immediately implies that P_n^D is isomorphic to the circulant graph $C_n^{D'}$ for $D' = \{d \in D \mid d \leq \frac{n}{2}\}$. $\square$

Originally motivated by coloring problems for infinite distance graphs studied by Eggleton, Erdős, and Skilton [6, 7], most research on distance graphs focused on colorings [3–5, 13, 14, 22, 23].

Our main contributions in the present paper are hardness results concerning computational problems related to the connectivity and diameter of distance graphs and a number-theoretic characterization of the connected distance graphs P_n^D for $|D| = 2$.

2 Results

Boesch and Tindell [2] observed that a circulant graph C_n^D is connected if and only if the greatest common divisor $\gcd(\{n\} \cup D)$ of the integers in $\{n\} \cup D$ equals 1. In fact, since C_n^D is vertex-transitive, it is connected if and only if it contains a path from the vertex 0 to the vertex 1 which is equivalent to the existence of integers l and l_d for $d \in D$ such that $1 = ln + \sum_{d \in D} l_d d$. It is a well-known consequence of the Euclidean algorithm that the existence of such integers is equivalent to the above gcd-condition. Therefore, deciding the connectivity of a circulant graph requires a simple polynomial time gcd-computation.

The most fundamental connectivity problem for distance graphs is the following.

Connectivity of P_n^D

Instance: $n \in \mathbb{N}$ and $D \subseteq \mathbb{N}$.

Question: Is P_n^D connected?

We have not been able to determine the complexity of Connectivity of P_n^D and pose the following conjecture.

Conjecture 2 Connectivity of P_n^D *is NP-hard.*

Clearly, P_n^D is connected if and only if for every $i \in [0, n-2]$, there is a path in P_n^D from i to $i+1$. Equivalently, for every $i \in [0, n-2]$, there are integers $x_1, x_2, \ldots, x_l$ such that

$$|x_i| \in D \text{ for all } i \in [1, l], \tag{1}$$

$$1 = \sum_{j=1}^{l} x_j, \text{ and} \tag{2}$$

$$i + \sum_{j=1}^{k} x_j \in [0, n-1] \text{ for all } k \in [0, l]. \tag{3}$$

As noted before, 1 is an integral linear combination of the elements of D if and only if $\gcd(D) = 1$. Hence the existence of integers x_i which satisfy (1) and (2) can be decided in polynomial time. Unfortunately, these integers are by far not unique. Furthermore, given integers x_i which satisfy (1) and (2), deciding the existence of an ordering of them which satisfies (3) is in general a hard problem as we show next.

Bounded Partial Sums

Instance:	$x_0, x_1, x_2, \ldots, x_l \in \mathbb{Z}$ and $n \in \mathbb{N}$.
Question:	Is there a permutation $\pi \in S_l$ such that $$x_0 + \sum_{j=1}^{k} x_{\pi(j)} \in [0, n-1]$$ for all $k \in [0, l]$?

Proposition 3 BOUNDED PARTIAL SUMS *is NP-complete.*

Proof: Clearly, BOUNDED PARTIAL SUMS is in NP. We will reduce the classical NP-complete problem PARTITION [10] to BOUNDED PARTIAL SUMS:

PARTITION

Instance:	$x_1, x_2, \ldots, x_k \in \mathbb{N}$.
Question:	Is there a set $I \subseteq [1, k]$ such that $\sum_{i \in I} x_i = \sum_{i \in [1,k] \setminus I} x_i$?

In order to relate to the preceding discussion we will reduce to instances of BOUNDED PARTIAL SUMS which satisfy (2).

Let $x_1, x_2, \ldots, x_{l-2} \in \mathbb{N}$ be an instance of PARTITION. Let $X = \sum_{i=1}^{l-2} x_i$.

Let $x_0 = 0$, $x_{l-1} = -X$, $x_l = -X + 1$, and $n = X + 1$. It is easy to see that the instance $x_1, x_2, \ldots, x_{l-1}$ of PARTITION is "yes"-instance if and only if the instance of BOUNDED PARTIAL SUMS defined by

$$x_0, 2x_1, 2x_2, \ldots, 2x_{l-2}, x_{l-1}, x_l$$

and n is a "yes"-instance. This completes the proof. $\Box$

Clearly, if $|D| = 1$, then P_n^D is connected if and only if $D = \{1\}$. Already for $|D| = 2$, the following number-theoretic characterization of the pairs (n, D) for which P_n^D is connected is not simple.

Theorem 4 *Let $n, d_1, d_2 \in \mathbb{N}$ be such that $d_1 < d_2$. For $i \in [0, d_1-1]$, let $r_i = (id_2) \ mod \ d_1$ and $s_i = (n-1-r_i) \ mod \ d_1$. Furthermore, for $i \in [1, d_1-1]$, let*

$$\begin{aligned} d_i^+ &= \max \{r_i \mid i \in [0, i^*-1]\} \text{ and} \\ d_i^- &= \max \left\{ s_{-i \ mod \ d_1} \mid i \in [0, d_1 - i^* - 1] \right\}. \end{aligned}$$

Finally, let

$$d^* = \max_{i^* \in [1, d_1-1]} \min\{d_i^+, d_i^-\}.$$

(See Figure 1 for an example.)

$P_n^{\{d_1,d_2\}}$ *is connected if and only if* $\gcd(\{d_1, d_2\}) = 1$ *and* $d^* + d_2 \leq n-1$.

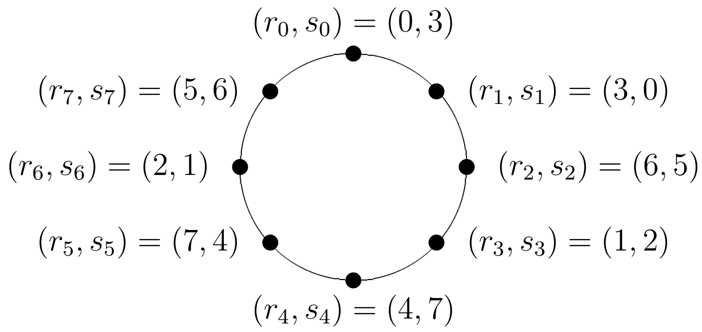

Figure 1. $(d_1, d_2, n) = (8, 11, 20)$, $(d_1^+, \ldots, d_7^+) = (0, 3, 6, 6, 6, 7, 7)$, $(d_1^-, \ldots, d_7^-) = (7, 7, 7, 6, 6, 6, 0)$, and $d^* = 6$.

Proof: "$\Rightarrow$": Let $D = \{d_1, d_2\}$ and let P_n^D be connected. As we have already noted before, this implies that $\gcd(D) = 1$.

Claim A *For every $v \in [1, n-1]$, there is a path $P : u_0 u_1 \ldots u_l$ in P_n^D such that $u_0 = 0$, $u_l = v$, and either*

$$\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, d_2\}$$

or

$$\{u_i - u_{i-1} \mid i \in [1,l]\} \subseteq \{d_1, -d_1, -d_2\}.$$

Proof of Claim A: Let $P : u_0u_1 \ldots u_l$ be a path in P_n^D such that $u_0 = 0$ and $u_l = v$. If $Q : u_xu_{x+1} \ldots u_{x+y}$ is a subpath of P with $\{u_{x+1} - u_x, u_{x+y} - u_{x+y-1}\} = \{-d_2, d_2\}$ and $u_{x+i} - u_{x+i-1} \in \{d_1, -d_1\}$ for $i \in [2, y-1]$, then $(u_{x+y} - u_x) \bmod d_1 = 0$, i.e. u_x and u_{x+y} differ by a multiple of d_1.

Therefore, Q can be replaced within P by a path $Q' : v_0v_1 \ldots v_z$ in P_n^D with $v_0 = u_x$, $v_z = u_{x+y}$, and either $v_i - v_{i-1} = d_1$ for $i \in [1, z]$ or $v_i - v_{i-1} = -d_1$ for $i \in [1, z]$.

Note that replacing Q with Q' within P results in a walk P' which might not be a path. Since there is a path from u_0 to u_l whose edge set is a subset of the edge set of P', this easily implies the claim. $\Box$

Note that, by the definition of r_i and s_i, r_i is the smallest integer at least 0 which has the same residue modulo d_1 as id_2 and $(n - 1 - s_i)$ is the largest integer at most $n - 1$ which has the same residue modulo d_1 as id_2.

Let $i^* \in [1, d - 1]$.

First, let $P : u_0u_1 \ldots u_l$ be a path in P_n^D such that $u_0 = 0$, $u_l \bmod d_1 = r_{i^*}$, and $\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, d_2\}$. By the definition of the r_i and the choice of the path P, we obtain that for every $i \in [1, i^*]$, the path P contains an edge $u_{j_i-1}u_{j_i}$ with $u_{j_i-1} \bmod d_1 = r_{i-1}$, $u_{j_i} \bmod d_1 = r_i$, and $u_{j_i} - u_{j_i-1} = d_2$. This implies

$$n - 1 \geq u_{j_i-1} + d_2 \geq r_{i-1} + d_2$$

for all $i \in [1, i^*]$ and hence $n - 1 \geq d_{i*}^+ + d_2$.

Now, let $P : u_0u_1 \ldots u_l$ be a path in P_n^D such that $u_0 = 0$, $u_l \bmod d_1 = r_{i^*}$, and $\{u_i - u_{i-1} \mid i \in [1, l]\} \subseteq \{d_1, -d_1, -d_2\}$. By the definition of the r_i and the choice of the path P, we obtain that for every $i \in [1, d_1 - i^*]$, the path P contains an edge $u_{j_i-1}u_{j_i}$ with $u_{j_i-1} \bmod d_1 = r_{(d_1-i+1) \bmod d_1}$, $u_{j_i} \bmod d_1 = r_{(d_1-i) \bmod d_1}$, and $u_{j_i} - u_{j_i-1} = -d_2$. This implies

$$d_2 \leq u_{j_i-1} \leq n - 1 - s_{(d_1-i+1) \bmod d_1}$$

for all $i \in [1, d_1 - i^*]$ and hence $n - 1 \geq d_{i*}^- + d_2$.

Altogether, this easily implies that $d^* + d_2 \leq n - 1$ which completes the proof of this implication.

"$\Leftarrow$": Let $\gcd(D) = 1$ and $d^* + d_2 \leq n - 1$.

Since $\gcd(D) = 1$, we obtain that

$$\{r_i \mid i \in [0, d_1 - 1]\} = \{s_i \mid i \in [0, d_1 - 1]\} = [0, d_1 - 1],$$

i.e. the r_i and s_i represent all residues modulo d_1.

Therefore, it suffices to prove that for every $i \in [0, d_1 - 1]$, the graph P_n^D contains a path from 0 to some vertex v with $v \bmod d_1 = r_i$.

Let $i^* \in [0, d_1 - 1]$.

By the definition of d^* and by symmetry, we may assume that $\max\{r_i \mid i \in [0, i^* - 1]\} \leq d^*$. This implies that

$$
\begin{aligned}
&0 d_2 (d_2 - d_1)(d_2 - 2d_1) \ldots \\
&r_1 (r_1 + d_2)(r_1 + d_2 - d_1)(r_1 + d_2 - 2d_1) \ldots \\
&\ldots \\
&r_{i^*-1}(r_{i^*-1} + d_2)
\end{aligned}
$$

is a path in P_n^D from 0 to some vertex v with $v \bmod d_1 = r_{i^*}$. This completes the proof. □

It is unclear whether Theorem 4 yields a polynomial time algorithm to check connectivity for P_n^D with $|D| = 2$. Furthermore, it would be interesting how the number-theoretic characterizations of the connectivity of circulant graphs given by Boesch and Tindell [2] and van Dorne [20] could be generalized to distance graphs.

While deciding connectivity is easy for circulant graphs, the exact calculation and minimization of the diameter of C_n^D are very difficult and well-studied problems even for the case $|D| = 2$ [1,11,12,25]. Many of the general upper and lower bounds on the diameter of circulant graphs easily generalize to distance graphs. The arguments used by Wong and Coppersmith [24] to obtain their classical estimates (cf. Theorems 4.6 and 4.7 in [11]) imply

$$\begin{aligned} \mathrm{diam}\left(P_n^D\right) &\geq \frac{1}{2}\left(|D|!n\right)^{\frac{1}{|D|}} - |D| \text{ and} \\ \mathrm{diam}\left(P_{d^k}^{\{1,d,\ldots,d^{k-1}\}}\right) &\leq k(d-1). \end{aligned}$$

For our final hardness result, we consider the following decision problem which closely relates to the diameter of distance graphs.

Short Path in P_n^D

Instance: $n \in \mathbb{N}$, $D \subseteq \mathbb{N}$ and $l \in \mathbb{N}$.

Question: Is there some $u \in [0, n-2]$ such that P_n^D contains a path of length at most l between u and $u+1$?

A natural certificate for a "yes"-instance of Short Path in P_n^D would be a path P of length at most l between two vertices u and $u+1$ of P_n^D. The hardness of Bounded Partial Sums implies that an encoding of P which can be checked in polynomial time would most likely have to use at least $\Omega(l)$ bits which would not be polynomially bounded in the encoding length of the triple (n, D, l).

The construction in the following proof is inspired by van Emde Boas's proof [21] that

WEAK PARTITION is NP-complete.

Theorem 5 SHORT PATH IN P_n^D *is NP-hard.*

Proof: For an instance I of PARTITION, we will construct an instance I' of SHORT PATH IN P_n^D such that the encoding length of I' is polynomially bounded in the encoding length of I, and I is a "yes"-instance if and only if I' is a "yes"-instance.

Let $x_1, x_2, \ldots, x_k \in \mathbb{Z}$ be an instance of PARTITION.

Let

$$\begin{aligned} l &:= 2(k+1), \\ d &:= 2(k+1)\max(\{|x_1|, |x_2|, \ldots, |x_k|\}) + 1, \text{ and} \\ n &:= 2d^{3k+4} + 1. \end{aligned}$$

For $i \in [1, k]$ let

$$\begin{array}{rclcccccccc} x_{i,1} &:=& x_i d &+& d^{3i-1} &+& d^{3i} &+& 0 &+& 0, \\ x_{i,2} &:=& 0 &+& 0 &+& d^{3i} &+& 0 &+& d^{3i+2}, \\ x_{i,3} &:=& 0 &+& d^{3i-1} &+& 0 &+& d^{3i+1} &+& 0, \\ x_{i,4} &:=& x_i d &+& 0 &+& 0 &+& d^{3i+1} &+& d^{3i+2}, \end{array}$$

and let

$$\begin{array}{rclcccccc} x_{k+1,1} &:=& 1 &+& d^{3(k+1)-1} &+& d^{3(k+1)} & & , \\ x_{k+1,2} &:=& 0 &+& 0 &+& d^{3(k+1)} &+& d^2, \end{array}$$

Writing the $x_{i,j}$ as d-ary numbers, we obtain the following pattern.

	d^0	d^1	d^2	d^3	d^4	d^5	d^6	d^7	d^8	$\dots$	d^{3k+2}	d^{3k+3}
$x_{1,1}$		x_1	1	1								
$x_{1,2}$				1		1						
$x_{1,3}$			1		1							
$x_{1,4}$		x_1			1	1						
$x_{2,1}$		x_2				1	1					
$x_{2,2}$							1		1			
$x_{2,3}$						1		1				
$x_{2,4}$		x_2						1	1			
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
$x_{k+1,1}$	1										1	1
$x_{k+1,2}$			1									1

Let

$$D := \{x_{i,j} \mid i \in [1,k], j \in [1,4]\} \cup \{x_{k+1,1}, x_{k+1,2}\}.$$

The encoding length of the Partition instance is

$$\Omega\left(k + \sum_{i=1}^{k} \log(x_i)\right)$$

and the encoding length of (D, n, l) is

$$O\left(k^2 \log(k) \log\left(\max\left(\{|x_1|, |x_2|, \dots, |x_k|\}\right)\right)\right),$$

i.e. the latter is polynomially bounded in terms of the former.

Claim A *(n, D, l) is a "yes"-instance of* Short Path in P_n^D *if and only if there are integers $l_{i,j}$ such that*

$$1 = l_{k+1,1}x_{k+1,1} + l_{k+1,2}x_{k+1,2} + \sum_{i=1}^{k}\sum_{j=1}^{4} l_{i,j}x_{i,j} \text{ and} \tag{4}$$

$$l \geq |l_{k+1,1}| + |l_{k+1,2}| + \sum_{i=1}^{k}\sum_{j=1}^{4} |l_{i,j}|. \tag{5}$$

Proof of Claim A: "⇒" Let P be a path in P_n^D of length at most l between vertices u and $u+1$. If, for $x_{i,j} \in D$, going from u to $u+1$ the path P uses $l^+_{i,j}$ edges vw with $w - v = x_{i,j}$ and $l^-_{i,j}$ edges vw with $v - w = x_{i,j}$, then the integers $l_{i,j} := l^+_{i,j} - l^-_{i,j}$ satisfy (4) and (5).

"⇐" Let the integers $l_{i,j}$ be such that (4) and (5) hold. Consider the Algorithm 1 below.

$r = 0$;
$u_r := 0$;
while $u_r \neq 1$ **do**
  if $u_r \leq \frac{n-1}{2}$ *and* $l_{i,j} > 0$ *for some pair* (i, j) **then**
    Choose a pair (i, j) such that $l_{i,j} > 0$;
    $u_{r+1} := u_r + x_{i,j}$;
    $l_{i,j} := l_{i,j} - 1$;
  else
    Choose a pair (i, j) such that $l_{i,j} < 0$;
    $u_{r+1} := u_r - x_{i,j}$;
    $l_{i,j} := l_{i,j} + 1$;
  end
end

Algorithm 1

By the definition of n and D, we have $\max(D) < \frac{n-1}{2}$. Using this fact, it is straightforward to check that the sequence $u_0, u_1, u_2, \ldots$ produced by Algorithm 1 is a path of length at most l between the vertices 0 and 1, i.e. (n, D, l) is a "yes"-instance of Short Path in P_n^D. □

Claim B *(n, D, l) is a "yes"-instance of* Short Path in P_n^D *if and only if $x_1, x_2, \ldots, x_k$*

is a "yes"-instance of PARTITION.

Proof: "⇒" If (n, D, l) is a"yes"-instance of SHORT PATH IN P_n^D, then Claim A implies the existence of integers $l_{i,j}$ satisfying (4) and (5). For $i \in [1, k]$, let

$$\begin{aligned} l_i &= |l_{i,1}| + |l_{i,2}| + |l_{i,3}| + |l_{i,4}| \text{ and} \\ l_{k+1} &= |l_{k+1,1}| + |l_{k+1,2}|. \end{aligned}$$

By the definition of d, forming the sum in (4) and representing the involved numbers as d-ary numbers, there is never any carry contribution from one digit to the next. (In view of [21] we could say that there are no inheritance problems in this case.)

Considering the different digits of the sum in (4), this implies that

$$\begin{aligned} l_{k+1,1} &= 1, \\ l_{k+1,2} &= -1, \text{ and} \\ l_{1,1} + l_{1,3} &= 1. \end{aligned}$$

Furthermore, for $i \in [1, k]$,

$$\begin{aligned} l_{i,2} &= -l_{i,1}, \\ l_{i,4} &= -l_{i,3}, \text{ and} \\ l_{i+1,1} + l_{i+1,3} &= -(l_{i,2} + l_{i,4}) \\ &= l_{i,1} + l_{i,3}. \end{aligned}$$

Therefore, for $i \in [1, k]$, if $l_i > 0$, then $l_i \geq 2$, and, if $l_i = 2$, then $(l_{i,1}, l_{i,2}, l_{i,3}, l_{i,4})$ belongs to

$$\{(1,-1,0,0),(-1,1,0,0),(0,0,1,-1),(0,0,-1,1)\}. \tag{6}$$

Note that for each of these four possibilities, the contribution of $\sum_{j=1}^{4} l_{i,j}x_{i,j}$ to the second digit of (4) written as a d-ary number is either x_i or $-x_i$.

By a simple inductive argument, we obtain $l_{i,1}+l_{i,3}=1$ and $l_i>0$ for all $i\in[1,k]$. Since $l_1+l_2+\ldots+l_{k+1}\leq l=2(k+1)$, this implies $l_i=2$ for all $i\in[1,k]$. Therefore, since the second digit of the sum in (4) written as a d-ary number is 0, the $l_{i,j}$ yield a solution for the PARTITION instance.

"$\Leftarrow$" If the PARTITION instance is a "yes"-instance, then setting

$$(l_{k+1,1},l_{k+1,2})=(1,-1)$$

and suitably selecting $(l_{i,1},l_{i,2},l_{i,3},l_{i,4})$ from the four possibilites given in (6) according to some solution of the PARTITION instance, yields integers $l_{i,j}$ which satisfy (4) and (5). In view of Claim A, this completes the proof of the claim. □

This completes the proof. □

Note that for the instances of SHORT PATH IN P_n^D constructed in the proof of Theorem 5, the length bound l is polynomially bounded in terms of the encoding length of D. Since a path in P_n^D of length at most l can be encoded using $O(l\log(|D|))$ many bits, the restriction of SHORT PATH IN P_n^D to such instances actually yields an NP-complete problem.

As a final remark, we note that the existence of a monotonic path between two vertices of P_n^D is equivalent to the feasibility of an integer linear program in $|D|$ dimensions which can be decided in polynomial time for bounded $|D|$ [15].

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