On the Hull Number of Triangle-free Graphs

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Abstract. A set of vertices C in a graph is convex if it contains all vertices which lie on shortest paths between vertices in C. The convex hull of a set of vertices S is the smallest convex set containing S. The hull number $h(G)$ of a graph G is the smallest cardinality of a set of vertices whose convex hull is the vertex set of G.

For a connected triangle-free graph G of order n and diameter $d \geq 3$, we prove that $h(G) \leq (n-d+3)/3$, if G has minimum degree at least 3 and that $h(G) \leq 2(n-d+5)/7$, if G is cubic. Furthermore, for a connected graph G of order n, girth $g \geq 4$, minimum degree at least 2, and diameter d, we prove $h(G) \leq 2+(n-d-1)/\left\lceil \frac{g-1}{2} \right\rceil$. All bounds are best possible.

Keywords. Convex hull; convex set; geodetic number; graph; hull number

1 Introduction

We consider finite, simple, and undirected graphs G with vertex set V and edge set E. The degree of a vertex u in G is the number of neighbours of u in G. The graph G is cubic if every vertex has degree 3. The distance $\mathrm{dist}(u,v)$ between two vertices u and v in G is the minimum number of edges of a path in G between u and v or ∞, if no such path exists. The diameter of G is the largest distance between two vertices in G. The graph G is triangle-free if it does not contain three pairwise adjacent vertices. The girth of G is the minimum length of a cycle in G or ∞, if G has no cycle.

For two sets of vertices C and D let $I[C,D]$ denote the set of all vertices which lie on a path of length $\mathrm{dist}(C,D)=\min\{\mathrm{dist}(u,v) \mid u \in C \text{ and } v \in D\}$ between a vertex in

C and a vertex in D. Furthermore, let $I(C,D) = I[C,D] \setminus (C \cup D)$. We write $I[u,v]$, $I(u,v)$, $I[u,D]$, and $I(u,D)$ instead of $I[\{u\},\{v\}]$, $I(\{u\},\{v\})$, $I[\{u\},D]$, and $I(\{u\},D)$, respectively.

For a set of vertices S let $I[S] = \bigcup_{u,v \in S} I[u,v]$. The set S is *convex* if $I[S] = S$. The *convex hull* of S is the smallest convex set C with $S \subseteq C$ and is denoted by $H[S]$. Since the intersection of two convex sets is convex, the convex hull is well defined.

In [9] Everett and Seidman introduce the *hull number* $h(G)$ of a graph G as the minimum cardinality of a set of vertices whose convex hull contains all vertices of G. Similarly, Harary et al. [11] define the *geodetic number* $g(G)$ of a graph G as the minimum cardinality of a set of vertices S for which $I[S]$ contains all vertices of G.

It was observed in [5, 9] that, if u and v are two vertices at maximum distance in a connected graph G of order n and diameter d, then $I(u,v)$ contains at least $d-1$ elements and $I[V \setminus I(u,v)]$ contains all vertices of G which immediately implies

$$h(G) \leq g(G) \leq n - d + 1.$$

Everett and Seidman [9] strengthened this simple bound as follows.

Theorem 1 (Everett and Seidman [9]) *Let G be a connected graph of order n and diameter d. If there is no clique K in G such that every vertex of G is at distance at most $d-1$ from K, then*

$$h(G) \quad \leq \quad \frac{n-d+3}{2}. \tag{1}$$

The hypothesis of Theorem 1 is obviously rather restrictive. If a graph G of diameter d satisfies the hypothesis of Theorem 1, then its radius equals its diameter. Even more strongly, for every breadth first search tree T of G rooted at a vertex u, there are two non-adjacent neighbours of u such that both have descendants in T at depth d.

Our first contribution in Section 2 is to show that (1) still holds under a less restrictive hypothesis. In Section 3, we prove best possible upper bounds on the hull number of triangle-free graphs which are either of minimum degree at least two or three or cubic. Finally, in Section 4, we prove a best possible upper bound on the hull number of graphs of large girth and minimum degree at least two.

For further results concerning the hull number and convexity in graphs, we refer the reader to [1, 3, 4, 6–8, 10, 12, 13].

2 A weaker hypothesis for Theorem 1

A natural strategy already adopted by Everett and Seidman [9], to construct a small set of vertices in a graph whose convex hull equals the entire vertex set, is to start with a small initial set having a large convex hull and to extend this set iteratively while its convex hull still does not contain all vertices. A reasonable choice for the elements of the initial set are two vertices at maximum distance. To extend the set, one can iteratively add vertices greedily trying to maximize the cardinality increase of the convex hull divided by the number of added vertices.

As a first and simple illustration of this strategy, we prove that the bound in Theorem 1 still holds under a less restrictive hypothesis. We say that a graph G has property $\mathcal{P}$ if there are no three sets of vertices A, B and C such that

- A is the vertex set of a component of $G - C$,
- B is a clique and, for every vertex v in A, the set of neighbours of v in C is B, and
- C is convex.

To check that property $\mathcal{P}$ is more general than the hypothesis of Theorem 1, simply note that the existence of A, B and C as above for some graph G of diameter d implies that every vertex of G is at distance at most $d-1$ from B.

The following lemma is somewhat implicit in the proof of Theorem 9 in [9].

Lemma 2 *Let G be a connected graph and let S be a non-empty set of vertices whose convex hull does not contain all vertices of G.*

If G has property $\mathcal{P}$, then there is a vertex $u \in V \setminus S$ such that $|H[S \cup \{u\}]| \geq |H[S]| + 2$.

Proof: Let G and S be as in the statement of the lemma. For contradiction, we assume that a vertex with the desired property does not exist.

Let $C = H[S]$. Let A be the vertex set of a component of $G - C$.

If there is a vertex $u \in A$ with $\text{dist}(u, C) \geq 2$, then u has the desired property. Hence all vertices in A have a neighbour in C. If a vertex $u \in A$ has two neighbours $v, w \in C$ which are not adjacent, then $u \in I[v, w] \subseteq H[S]$ which is a contradiction. Hence the set of neighbours in C of every vertex in A is a clique. If there are two adjacent vertices $u, v \in A$ such that v has a neighbour $v' \in C$ which is not adjacent to u, then $v \in I[u, v']$ and u has the desired property. Hence the sets of neighbours in C of the vertices in A are all equal to some set B. In view of the sets A, B and C, the graph G does not satisfy property $\mathcal{P}$.

This final contradiction completes the proof. $\Box$

With Lemma 2 at hand, the proof of the following result is straightforward.

Theorem 3 *Let G be a connected graph of order n and diameter d.*

If G has property $\mathcal{P}$, then

$$h(G) \leq \frac{n - d + 3}{2}.$$

Proof: Since $n - d \geq 1$, we may assume that $h(G) \geq 3$. Let $S_0 = \{u, v\}$ for two vertices u and v with $\text{dist}(u, v) = d$. $H[S_0]$ contains at least $d + 1$ vertices.

By Lemma 2, there is a sequence $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ of sets of vertices such that for $i \geq 1$, $|S_i| = 2 + i$ and either $H[S_{i-1}] = V$ or $|H[S_i]| \geq (d + 1) + 2i$.

If $i^* \geq 1$ is maximum such that $H[S_{i^*-1}] \neq V$, then $(d + 1) + 2(i^* - 1) \leq n - 2$. Hence $i^* \leq \frac{n-d-1}{2}$ and $h(G) \leq |S_{i^*}| = 2 + i^* \leq \frac{n-d+3}{2}$ which completes the proof. $\Box$

3 Triangle-free graphs

Our first result in this section describes a natural and less abstract hypothesis which implies that a graph has property $\mathcal{P}$.

Corollary 4 *If G is a connected triangle-free graph of order n, minimum degree at least 2 and diameter d, then G has property $\mathcal{P}$ and — hence —*

$$h(G) \leq \frac{n-d+3}{2}.$$

Proof: In view of Theorem 3, we may assume that G does not have property $\mathcal{P}$.

Let the three sets of vertices A, B and C certify this. Since A is connected, B is a clique and G is triangle-free, we obtain that A and B both contain exactly one vertex. This clearly implies the contradiction that the vertex in A has degree 1 which completes the proof. $\square$

The graphs whose structure is illustrated in Figure 1 are connected, triangle-free, and of minimum degree at least 2. Their order n, diameter d, and hull number h satisfy $n = 5d-7$ and $h = 2d - 2$, i.e. $h = \frac{n-d+3}{2}$ and Theorem 3 and Corollary 4 are best possible.

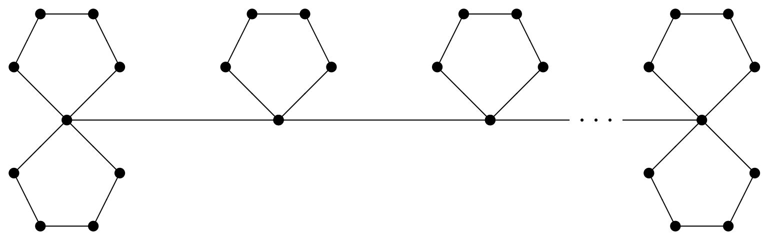

Figure 1 Extremal graphs for Corollary 4.

We proceed to graphs of minimum degree at least 3 and cubic graphs.

For these graphs, the simple choice for the initial set described at the beginning of Section 2 would not yield best possible results. Therefore, we first prove the existence of a better initial set.

Lemma 5 *If G is a connected triangle-free graph of minimum degree at least 3 and diameter $d \geq 3$, then there is a set of vertices S_0 such that $2 \leq |S_0| \leq 4$ and $|H[S_0]| \geq d+3|S_0|-3$.*

Proof: Let x and y be two vertices such that $\text{dist}(x, y) = d$. Clearly, $|H[\{x, y\}]| \geq d + 1$. Depending on properties of x and y, we define two disjoint sets X and Y such that $S_0 = X \cup Y$ has the desired properties. Since X and Y are defined analogously, we will only describe the definition of X in detail.

If x has at least two neighbours at distance $(d-1)$ from y, then let $X = \{x\}$. Note that $|H[X \cup \{y\}]| \geq (d + 1) + 1$. In what follows, we assume that x has exactly one neighbour x' at distance $(d - 1)$ from y.

Let u and w be two neighbours of x different from x'. Since $\text{dist}(u, y) = \text{dist}(w, y) = d$, we may assume, by symmetry with x, that each of u and w has exactly one neighbour u' and w' at distance $(d-1)$ from y.

If u' and w' are distinct, then let $X = \{u, w\}$. Note that $u', w' \in I[y, X]$, $x \in I[u, w]$, and $x' \in I[x, y]$ which implies that $|H[X \cup \{y\}]| \geq (d+1) + 4$. In what follows, we assume that $u' = w'$.

Let v be a neighbour of u different from x and u'. We may assume, by symmetry, that x' is the unique neighbour of v at distance $(d-1)$ from y. Let $X = \{u, w\}$. Note that $u' \in I[y, X]$, $x \in I[u, w]$, $x' \in I[x, y]$, and $v \in I[u, x']$ which implies that $|H[X \cup \{y\}]| \geq (d+1) + 4$.

By defining Y analogously, we obtain a set $S_0 = X \cup Y$ such that either $|S_0| = 2$ and $|H[S_0]| \geq (d+1) + 1 + 1$, or $|S_0| = 3$ and $|H[S_0]| \geq (d+1) + 4 + 1$, or $|S_0| = 4$ and $|H[S_0]| \geq (d+1) + 4 + 4$.

This completes the proof. $\Box$

As a further ingredient, we need a strengthened extension lemma.

Lemma 6 *If G is a connected triangle-free graph of minimum degree at least 3 and S is a non-empty set of vertices whose convex hull does not contain all vertices of G, then there is a set $T \subseteq V \setminus S$ such that $1 \leq |T| \leq 2$ and $|H[S \cup T]| \geq |H[S]| + 3|T|$.*

Proof: Let G and S be as in the statement of the lemma. For contradiction, we assume that a set T with the desired properties does not exist. Let $C = H[S]$.

If there is a vertex u with $\text{dist}(u, C) \geq 3$, then $T = \{u\}$ has the desired property. Hence $V \setminus C = D_1 \cup D_2$ for $D_i = \{u \in V \mid \text{dist}(u, C) = i\}$ for $1 \leq i \leq 2$.

Since C is convex and G is triangle-free, every vertex in D_1 has exactly one neighbour in C.

If D_2 is empty, then there are three vertices $u, v, w \in D_1$ with $uv, vw \in E$. If $u', w' \in C$ are such that $uu', ww' \in E$, then $u \in I[v, u']$ and $w \in I[v, w']$ which implies that $T = \{v\}$ has the desired property. Hence D_2 is not empty.

If a vertex $u \in D_2$ has two neighbours in D_1, then $T = \{u\}$ has the desired property. Hence every vertex in D_2 has exactly one neighbour in D_1.

Let $u, v, w \in D_2$ and $u', v', w' \in D_1$ be such that $uv, vw, uu', vv', ww' \in E$.

If $u' \neq w'$, then $v \in I[u, w]$, $u' \in I[u, C]$, $v' \in I[v, C]$, and $w' \in I[w, C]$ which implies that $T = \{u, w\}$ has the desired property. Hence $u' = w'$.

Since G has minimum degree at least 3 and every vertex in D_2 has exactly one neighbour in D_1, there is a vertex $x \in D_2$ different from v with $wx \in E$. Since G is triangle-free, $vx \notin E$. In view of the previous observation, we obtain, by symmetry, that v' is the unique neighbour of x in D_1. Now $v \in I[u, w]$, $u' \in I[u, C]$, $v' \in I[v, C]$, and $x \in I[w, v']$ which implies that $T = \{u, w\}$ has the desired property.

This final contradiction completes the proof. $\Box$

As before, with Lemma 5 and Lemma 6 at hand, the proof of the following bound for graphs of minimum degree at least 3 is straightforward.

Theorem 7 *If G is a connected triangle-free graph of order n, minimum degree at least 3, and diameter $d \geq 3$, then*

$$h(G) \leq \frac{n-d+3}{3}.$$

Proof: Since $n - d \geq 5$, we may assume that $h(G) \geq 3$.

By Lemma 5, there is a set $S_0 \subseteq V$ such that $2 \leq |S_0| \leq 4$ and $|H[S_0]| \geq d + 3|S_0| - 3$.

If $H[S_0] = V$, then

$$\frac{n-d+3}{3} = \frac{|H[S_0]|-d+3}{3} \geq \frac{d+3|S_0|-3-d+3}{3} = |S_0|$$

and the desired result follows. Hence we may assume that $H[S_0] \neq V$.

By Lemma 6, there is a sequence $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ of sets of vertices such that for $i \geq 1$, either $H[S_{i-1}] = V$ or $1 \leq |S_i| - |S_{i-1}| \leq 2$ and

$$\begin{aligned} |H[S_i]| &\geq |H[S_0]| + 3(|S_i| - |S_0|) \\ &\geq (d + 3|S_0| - 3) + 3(|S_i| - |S_0|) \\ &= d - 3 + 3|S_i|. \end{aligned}$$

Let $i^* \geq 1$ be maximum such that $H[S_{i^*-1}] \neq V$.

If $|S_{i^*}| - |S_{i^*-1}| = 1$, then $d - 3 + 3|S_{i^*-1}| \leq n - 3$ which implies

$$h(G) \leq |S_{i^*}| = |S_{i^*-1}| + 1 \leq \frac{n-d+3}{3}.$$

If $|S_{i^*}| - |S_{i^*-1}| = 2$, then $d - 3 + 3|S_{i^*-1}| \leq n - 6$ which implies

$$h(G) \leq |S_{i^*}| = |S_{i^*-1}| + 2 \leq \frac{n-d+3}{3}.$$

This completes the proof. $\square$

The graphs whose structure is illustrated in Figure 2 are connected, triangle-free, and of minimum degree at least 3. Their order n, diameter d, and hull number h satisfy $n = 7d-9$ and $h = 2d - 2$, i.e. $h = \frac{n-d+3}{3}$ and Theorem 7 is best possible.

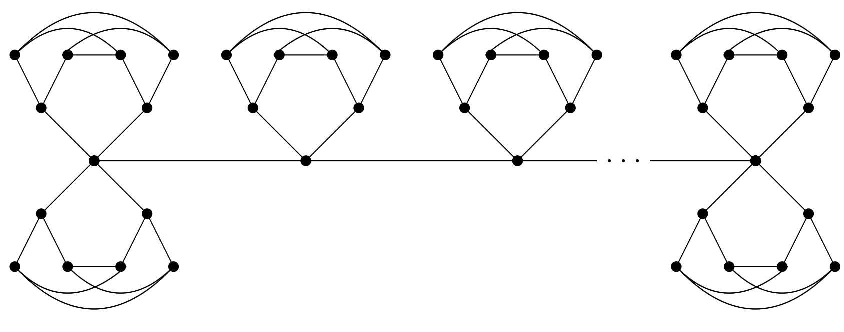

Figure 2 Extremal graphs for Theorem 7.

For triangle-free graphs G of order n, minimum degree at least 3, and diameter 2, applying the reasoning from the proof of Theorem 7 to a set S_0 containing two vertices at distance 2 easily implies $h(G) \leq \frac{n+3}{3}$.

For our next result concerning cubic graphs, it suffices to strengthen the extension lemma.

Lemma 8 *If G is a connected cubic triangle-free graph and S is a set of vertices of order at least 2 whose convex hull does not contain all vertices of G, then there is a set $S' \subseteq V$ such that*

(i) either $|S'| = |S| + 1$ and $|H[S']| \geq |H[S]| + 4$,

(ii) or $|S'| = |S| + 2$ and $|H[S']| \geq |H[S]| + 7$,

(iii) or $|S'| = |S|$ and $|H[S']| > |H[S]|$.

Proof: Let G and S be as in the statement of the lemma. For contradiction, we assume that a set S' with one of the desired properties does not exist. Let $C = H[S]$.

If there is a vertex u with $\text{dist}(u, C) \geq 4$, then $S' = S \cup \{u\}$ satisfies (i). Hence $V \setminus C = D_1 \cup D_2 \cup D_3$ for $D_i = \{u \in V \mid \text{dist}(u, C) = i\}$ for $1 \leq i \leq 3$.

Since G is connected and triangle-free, every vertex in D_1 has exactly one neighbour in C. Furthermore, since G is cubic and C is a convex set of order at least 2, no vertex in C has three neighbours in D_1.

First, we assume that D_3 is not empty. Let $v \in D_3$.

If $|I(v, C)| \geq 3$, then $S' = S \cup \{v\}$ satisfies (i). Hence $I(v, C)$ contains exactly two vertices $v' \in D_2$ and $v'' \in D_1$ such that $vv', v'v'' \in E$. Furthermore, there are vertices $u, w \in D_3$ such that $uv, vw \in E$.

If $|I(\{u, v, w\}, C)| \geq 4$, then $v \in I[u, w]$ and $S' = S \cup \{u, w\}$ satisfies (ii). Hence $I(\{u, v, w\}, C)| \leq 3$. Let $u' \in D_2$ be a neighbour of u. Since $u' \neq v'$, we obtain $I(\{u, v, w\}, C) = \{u', v', v''\}$ and $wu', u'v'' \in E$.

Since, by symmetry, $|I(w, C)| \leq 2$, there is a vertex $x \in D_3$ different from v with $wx \in E$. Since, by symmetry, $I|(\{v, w, x\}, C)| \leq 3$, we obtain $xv' \in E$, $w \in I[v, x]$, $u', v', v'' \in I[\{v, w\}, C]$, $u \in I[v, u']$, and $S' = S \cup \{v, x\}$ satisfies (ii).

This contradiction implies that we may assume that D_3 is empty.

Next, we assume that D_2 is not empty. Let $v \in D_2$.

If $|I(v, C)| \geq 3$, then $S' = S \cup \{v\}$ satisfies (i). Hence $|I(v, C)| \leq 2$. Let $u \in D_2$ be such that $uv \in E$. Let $u', v' \in D_1$ and $u'', v'' \in C$ be such that $uu', u'u'', vv', v'v'' \in E$.

If $u'' \neq v''$, then $u \in I[v, u'']$ and $S' = S \cup \{v\}$ satisfies (i). Hence $u'' = v''$.

If there is a vertex $y' \in D_1$ different from v' such that $vy' \in E$ and $y'' \in C$ is such that $y'y'' \in E$, then, because no vertex in C has three neighbours in D_1, we obtain $y'' \neq v''$ and $S' = S \cup \{u\}$ satisfies (i). Hence $I(v, C) = \{v'\}$ and there is a vertex $w \in D_2$ different from u such that $vw \in E$.

By symmetry, we obtain $wu' \in E$ and there is a vertex $x \in D_2$ different from v such that $ux \in E$, $xv' \in E$, and $I(\{u, v, w, x\}, C) = \{u', v'\}$.

If there is a vertex $y \in D_2$ different from u and w such that $xy \in E$, then there are vertices $y' \in D_1$ and $y'' \in C$ such that $yy', y'y'' \in E$, $y'' \neq v''$ and $S' = S \cup \{x\}$ satisfies (i). Hence $wx \in E$. Furthermore, v'' is a cutvertex of G and $G - \{v''\}$ has a component with vertex set $\{u, v, w, x, u', v'\}$. Since v'' has two neighbours in D_1 and C is convex, we obtain that $v'' \in S$. Since S contains at least two elements, this implies that $S' = (S \setminus \{v''\}) \cup \{v'\}$ satisfies (iii).

This contradiction implies that we may assume that D_2 is empty.

Since every vertex in D_1 has exactly one neighbour in C, there are vertices $u, v, w, x \in D_1$ such that $uv, vw, wx \in E$. If $v', w', x' \in C$ are such that $vv', ww', xx' \in E$, then $v \in I[u, v']$, $w \in I[v, w']$, $x \in I[w, x']$, and $S' = S \cup \{u\}$ satisfies (i).

This final contradiction completes the proof. $\Box$

We proceed to our next result.

Theorem 9 *If G is a connected cubic triangle-free graph of order n and diameter $d \geq 3$, then*

$$h(G) \leq \frac{2(n - d + 5)}{7}.$$

Proof: Since $n - d \geq 5$, we may assume that $h(G) \geq 3$.

By Lemma 5, there is a set $S_0 \subseteq V$ such that $2 \leq |S_0| \leq 4$ and $|H[S_0]| \geq d + 3|S_0| - 3$.

If $H[S_0] = V$, then

$$\frac{2(n - d + 5)}{7} = \frac{2(|H[S_0]| - d + 5)}{7} \geq \frac{2(d + 3|S_0| - 3 - d + 5)}{7} = \frac{6|S_0| + 4}{7} \geq |S_0|$$

and the desired result follows. Hence we may assume that $H[S_0] \neq V$.

By Lemma 8, there is a sequence $S_0, S_1, S_2, \ldots$ of sets of vertices such that for $i \geq 1$,

(i) either $H[S_{i-1}] = V$,

(ii) or $|S_i| = |S_{i-1}| + 1$ and $|H[S_i]| \geq |H[S_{i-1}]| + 4$,

(iii) or $|S_i| = |S_{i-1}| + 2$ and $|H[S_i]| \geq |H[S_{i-1}]| + 7$.

Note that this implies

$$\begin{aligned}
|H[S_i]| &\geq |H[S_0]| + \frac{7}{2}(|S_i| - |S_0|) \\
&\geq (d + 3|S_0| - 3) + \frac{7}{2}(|S_i| - |S_0|) \\
&= d - 3 + \frac{7}{2}|S_i| - \frac{1}{2}|S_0| \\
&\geq d - 5 + \frac{7}{2}|S_i|
\end{aligned}$$

for $i \geq 1$.

Let $i^* \geq 1$ be maximum such that $H[S_{i^*-1}] \neq V$.

If $|S_i| = |S_{i-1}| + 1$, then $d - 5 + \frac{7}{2}|S_{i^*-1}| \leq n - 4$ which implies

$$h(G) \leq |S_{i^*}| = |S_{i^*-1}| + 1 \leq \frac{2(n-d)+9}{7}.$$

If $|S_i| = |S_{i-1}| + 2$, then $d - 5 + \frac{7}{2}|S_{i^*-1}| \leq n - 7$ which implies

$$h(G) \leq |S_{i^*}| = |S_{i^*-1}| + 2 \leq \frac{2(n-d)+10}{7}.$$

This completes the proof. $\square$

The graphs whose structure is illustrated in Figure 3 are connected, cubic, and triangle-free. Their order n, diameter d, and hull number h satisfy $n = 8d - 26$ and $h = 2d - 6$, i.e. $h = \frac{2(n-d+5)}{7}$ and Theorem 9 is best possible.

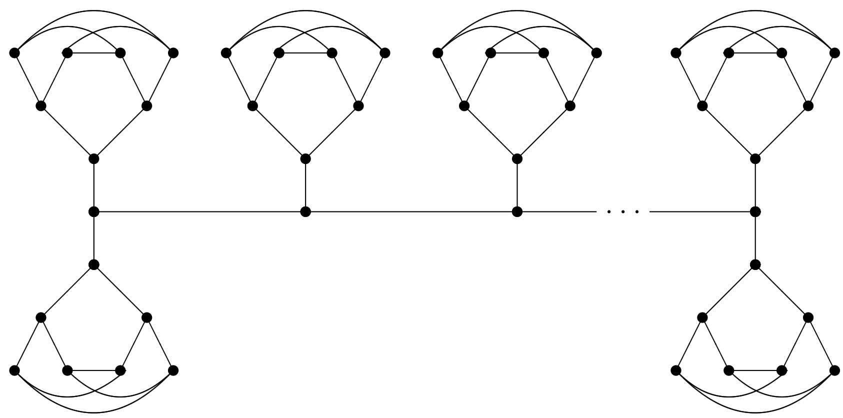

Figure 3 Extremal graphs for Theorem 9.

For the finitely many cubic triangle-free graphs G of order n and diameter 2, applying the reasoning from the proof of Theorem 9 to a set S_0 containing two vertices at distance 2 easily implies $h(G) \leq \frac{2(n+4)}{7}$.

For $r \geq 4$, let the graph $K^*_{r,r}$ arise from the complete bipartite graph $K_{r,r}$ by subdividing one edge. If a connected triangle-free graph G of order n, minimum degree at least r, and diameter d arises by suitably identifying the vertices of a sufficiently long path with the vertices of degree 2 in disjoint copies of $K^*_{r,r}$, then $h(G) = \frac{n-d}{r} + O(1)$. Note that the graphs in Figures 1 and 2 are obtained in this way and that the graphs in Figure 3 are obtained by a variation of this construction. We believe that this construction is essentially best possible and pose the following conjecture.

Conjecture 10 *If G is a connected triangle-free graph of order n, minimum degree δ, and diameter d, then*

$$h(G) \leq \frac{n-d}{\delta} + O(1).$$

Note that for triangle-free graphs, the 2-domination number is an upper bound on the geodetic number and hence also on the hull number. Therefore, Caro and Yuster's [2] results imply that the hull number of a connected triangle-free graph of order n and minimum degree δ is at most $(1+o_\delta(1))\frac{n\ln(\delta)}{\delta}$.

4 Graphs of large girth

All presented examples of graphs showing that our bounds are best possible contained short cycles. In this section, we show that a lower bound on the girth allows to improve the bounds on the hull number. Once again the key ingredient is an extension lemma.

Lemma 11 *If G is a connected graph of girth at least $g \geq 5$ and minimum degree at least 2 and S is a non-empty set of vertices whose convex hull does not contain all vertices of G, then there is a vertex $z \in V \setminus S$ such that $|H[S \cup \{z\}]| \geq |H[S]| + \left\lceil \frac{g-1}{2} \right\rceil$.*

Proof: Let G and S be as in the statement of the lemma. For contradiction, we assume that a vertex with the desired properties does not exist.

Let $C = H[S]$. Let $R = V \setminus C$.

Let $g_e = 2\left\lceil \frac{g}{2} \right\rceil$ and $g_o = 2\left\lceil \frac{g-1}{2} \right\rceil + 1$. Note that g_e and g_o are the smallest even and odd integers which are at least g, respectively.

Since G has minimum degree at least 2, there is a cycle which intersects R. We assume that K is a cycle in G which intersects R such that, if l_K denotes the length of K and d_K denotes $\text{dist}(V(K), C)$, then $l_K + 2d_K$ is smallest possible.

First, we assume that K contains at least two vertices of C. By the choice of K and the convexity of C, we obtain that there are two vertices x and y in C such that K consists of two paths P_C and P_R between x and y, all internal vertices of P_C belong to C, and all internal vertices of P_R belong to R. Furthermore, since C is convex, the length l_C of P_C is less than the length l_R of P_R. Let z be the internal vertex of P_R such that the distance with respect to P_R between x and z is $\left\lfloor \frac{l_R}{2} \right\rfloor$. By the choice of K, we obtain that $\text{dist}(x, z) = \left\lfloor \frac{l_R}{2} \right\rfloor$ and $\text{dist}(y, z) = \left\lceil \frac{l_R}{2} \right\rceil$. If l_K is even, then $|H[S \cup \{z\}]| - |H[S]| \geq l_R - 1 \geq \frac{l_K}{2} \geq \frac{g_e}{2}$. If l_K is odd, then $|H[S \cup \{z\}]| - |H[S]| \geq l_R - 1 \geq \frac{l_K-1}{2} \geq \frac{g_o-1}{2}$. Since $\min\left\{\frac{g_e}{2}, \frac{g_o-1}{2}\right\} = \frac{g_o-1}{2} = \left\lceil \frac{g-1}{2} \right\rceil$, we obtain a contradiction. Hence K contains at most one vertex of C.

Let P_K be a path of length d_K between a vertex x in $V(K)$ and a vertex y in C. (Note that $x = y$ is possible for $d_K = 0$.) Let z be a vertex of K such that the distance with respect to K between x and z is $\left\lfloor \frac{l_K}{2} \right\rfloor$. (Note that z is unique if l_K is even.) By the choice of K, we obtain that $\text{dist}(z, C) = \left\lfloor \frac{l_K}{2} \right\rfloor + d_K$. If l_K is even, then $|H[S \cup \{z\}]| - |H[S]| \geq (l_K - 1) + d_K \geq g_e - 1$. If l_K is odd, then $|H[S \cup \{z\}]| - |H[S]| \geq \frac{l_K-1}{2} + d_K \geq \frac{g_o-1}{2}$. Since $\min\left\{g_e - 1, \frac{g_o-1}{2}\right\} = \frac{g_o-1}{2} = \left\lceil \frac{g-1}{2} \right\rceil$, we obtain a contradiction.

This final contradiction completes the proof. $\square$

We proceed to our final result.

Theorem 12 *If G is a connected graph of order n, girth $g \geq 5$, minimum degree at least 2, and diameter d, then*

$$h(G) \leq 2 + \frac{n-d-1}{\left\lceil \frac{g-1}{2} \right\rceil}.$$

Proof: Since $n - d \geq 1$, we may assume that $h(G) \geq 3$. Let $S_0 = \{u, v\}$ for two vertices u and v with $\text{dist}(u, v) = d$. $H[S_0]$ contains at least $d + 1$ vertices.

By Lemma 11, there is a sequence $S_0 \subseteq S_1 \subseteq S_2 \subseteq \ldots$ of sets of vertices such that for $i \geq 1$, $|S_i| = 2 + i$ and either $H[S_{i-1}] = V$ or $|H[S_i]| \geq (d+1) + \left\lceil \frac{g-1}{2} \right\rceil i$.

If $i^* \geq 1$ is maximum such that $H[S_{i^*-1}] \neq V$, then $(d+1) + \left\lceil \frac{g-1}{2} \right\rceil (i^* - 1) \leq n - \left\lceil \frac{g-1}{2} \right\rceil$. Hence $i^* \leq \frac{n-d-1}{\left\lceil \frac{g-1}{2} \right\rceil}$ and $h(G) \leq |S_{i^*}| = 2 + i^* \leq 2 + \frac{n-d-1}{\left\lceil \frac{g-1}{2} \right\rceil}$ which completes the proof. $\square$

As before it is easy to construct examples of graphs — similar to the graphs in Figure 1 — which show that Theorem 12 is best possible. For instance, the graphs in Figure 1 are connected, of girth 5, and minimum degree at least 2. Recall that their order n, diameter d, and hull number h satisfy $n = 5d - 7$ and $h = 2d - 2$, i.e. $h = 2 + (n - d - 1)/\left\lceil \frac{g-1}{2} \right\rceil$.

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