An $\Omega(n \log n)$ lower bound for computing the sum of even-ranked elements

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Abstract

Given a sequence $A$ of $2n$ real numbers, the EvEnRANKSUm problem asks for the sum of the $n$ values that are at the even positions in the sorted order of the elements in $A$. We prove that, in the algebraic computation-tree model, this problem has time complexity $\Theta(n \log n)$. This solves an open problem posed by Michael Shamos at the Canadian Conference on Computational Geometry in 2008.

1 Introduction

Let $A = (a_1, a_2, \ldots, a_{2n})$ be a sequence of $2n$ real numbers. We define the even-rank-sum of $A$ to be the sum of the $n$ values that are at the even positions in the sorted order of the elements in $A$. Formally, let $\pi$ be a permutation of $\{1, 2, \ldots, 2n\}$ that sorts the sequence $A$ in non-decreasing order; thus, $a_{\pi(1)} \leq a_{\pi(2)} \leq \ldots \leq a_{\pi(2n)}$. Then the even-rank-sum of the sequence $A$ is the real number

$$a_{\pi(2)} + a_{\pi(4)} + a_{\pi(6)} + \ldots + a_{\pi(2n)}.$$

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Observe that any permutation $\pi$ that sorts the sequence $A$ in non-decreasing order gives rise to the same even-rank-sum. We consider the following problem:

**EvenRankSum:** Given a sequence $A$ of $2n$ real numbers, compute the even-rank-sum of $A$.

By using an $O(n \log n)$–time sorting algorithm, this problem can be solved in $O(n \log n)$ time. In the Open Problem Session at the Canadian Conference on Computational Geometry in 2008, Michael Shamos posed the problem of proving an $\Omega(n \log n)$ lower bound on the time complexity of **EvenRankSum** in the algebraic computation-tree model. (See [1, 2] for a description of this model.) In this paper, we present such a proof:

**Theorem 1** In the algebraic computation-tree model, the time complexity of **EvenRankSum** is $\Theta(n \log n)$.

We prove Theorem 1 by presenting an $O(n)$–time reduction of **MinGap** to **EvenRankSum**. The former problem is defined as follows. Let $X = (x_1, x_2, \ldots, x_n)$ be a sequence of $n$ real numbers, and let $\pi$ be a permutation of $\{1, 2, \ldots, n\}$ such that $x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots \leq x_{\pi(n)}$. For each $1 \leq i < n$, we define the difference $x_{\pi(i+1)} - x_{\pi(i)}$ to be a gap in the sequence $X$.

**MinGap:** Given a sequence $X = (x_1, x_2, \ldots, x_n)$ of $n$ real numbers and a real number $g > 0$, decide if each of the $n - 1$ gaps in $X$ is at least $g$.

Since in the algebraic computation-tree model, **MinGap** has an $\Omega(n \log n)$ lower bound (see [2, Section 8.4]), our reduction will prove Theorem 1.

## 2 The proof of Theorem 1

We now show how to reduce, in $O(n)$ time, **MinGap** to **EvenRankSum**.

Let $\mathcal{A}$ be an arbitrary algorithm that solves **EvenRankSum**. We show how to use algorithm $\mathcal{A}$ to solve **MinGap**. Let $n \geq 2$ be an integer and consider a sequence $X = (x_1, x_2, \ldots, x_n)$ of $n$ real numbers and a real number $g > 0$. The algorithm for solving **MinGap** makes the following three steps:

**Step 1:** Compute $S = \sum_{i=1}^{n} x_i$ and, for $i = 1, 2, \ldots, n$, compute $a_{2i-1} = x_i$ and $a_{2i} = x_i + g$.

**Step 2:** Run algorithm $\mathcal{A}$ on the sequence $(a_1, a_2, \ldots, a_{2n})$, and let $R$ be the output, i.e., $R$ is the even-rank-sum of this sequence.
Step 3: If $R = S + ng$, then return YES. Otherwise, return NO.

It is clear that the running time of this algorithm is $O(n)$ plus the running time of $A$. Thus, it remains to show that the algorithm correctly solves MinGap. That is, we have to show that the minimum gap $G$ of $X$ is at least $g$ if and only if $R = S + ng$. This is an immediate consequence of the following lemma:

**Lemma 1** Let $x_1, x_2, \ldots, x_n$ and $g$ be real numbers such that $x_1 \leq x_2 \leq \ldots \leq x_n$ and $g > 0$. Let $(a_1, a_2, \ldots, a_{2n}) = (x_1, x_1 + g, x_2, x_2 + g, \ldots, x_n, x_n + g)$ and let $\pi$ be a permutation of $\{1, \ldots, 2n\}$ such that $b_1 \leq b_2 \leq \ldots \leq b_{2n}$ with $b_i = a_{\pi(i)}$ for $1 \leq i \leq 2n$.

If $R = \sum_{i=1}^{n} b_{2i}$, $U = \sum_{i=1}^{n} b_{2i-1}$, and $G = \min\{x_{i+1} - x_i \mid 1 \leq i \leq n - 1\}$, then $R - U \leq ng$ with equality if and only if $G \geq g$.

**Proof.** Since $x_1, x_1 + g, x_2, x_2 + g, \ldots, x_i, x_i + g \leq x_i + g$, we have $x_i + g \geq b_{2i}$ for $1 \leq i \leq n$. Since $x_i, x_i + g, x_{i+1}, x_{i+1} + g, \ldots, x_n, x_n + g \geq x_i$, we have $x_i \leq b_{2i-1}$ for $1 \leq i \leq n$. Hence $b_{2i} - b_{2i-1} \leq (x_i + g) - x_i = g$ for $1 \leq i \leq n$ which implies $R - U \leq ng$.

If $G \geq g$, then clearly $R - U = ng$. Conversely, if $R - U = ng$, then $b_{2i} - b_{2i-1} = g$ for $1 \leq i \leq n$. In view of the above, this implies that $x_i + g = b_{2i}$ and $x_i = b_{2i-1}$ for $1 \leq i \leq n$. Since $x_{i+1} = b_{2i+1} \geq b_{2i} = x_i + g$ for $1 \leq i \leq n - 1$, we obtain $G \geq g$.

We complete the proof of Theorem 1 by observing that $R + U = 2S + ng$ and by Lemma 1 we have $G \geq g$ if and only if $R = U + ng = S + ng$.

**References**
