An $\Omega(n \log n)$ lower bound for computing the sum of even-ranked elements

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April 17, 2009

Abstract

Given a sequence A of $2n$ real numbers, the EvenRankSum problem asks for the sum of the n values that are at the even positions in the sorted order of the elements in A. We prove that, in the algebraic computation-tree model, this problem has time complexity $\Theta(n \log n)$. This solves an open problem posed by Michael Shamos at the Canadian Conference on Computational Geometry in 2008.

1 Introduction

Let $A = (a_1, a_2, \ldots, a_{2n})$ be a sequence of $2n$ real numbers. We define the *even-rank-sum* of A to be the sum of the n values that are at the even positions in the sorted order of the elements in A. Formally, let π be a permutation of $\{1, 2, \ldots, 2n\}$ that sorts the sequence A in non-decreasing order; thus, $a_{\pi(1)} \leq a_{\pi(2)} \leq \ldots \leq a_{\pi(2n)}$. Then the even-rank-sum of the sequence A is the real number

$$a_{\pi(2)} + a_{\pi(4)} + a_{\pi(6)} + \ldots + a_{\pi(2n)}.$$

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Observe that any permutation π that sorts the sequence A in non-decreasing order gives rise to the same even-rank-sum. We consider the following problem:

EVENRANKSUM: Given a sequence A of $2n$ real numbers, compute the even-rank-sum of A.

By using an $O(n \log n)$–time sorting algorithm, this problem can be solved in $O(n \log n)$ time. In the Open Problem Session at the Canadian Conference on Computational Geometry in 2008, Michael Shamos posed the problem of proving an $\Omega(n \log n)$ lower bound on the time complexity of EVENRANKSUM in the algebraic computation-tree model. (See [1, 2] for a description of this model.) In this paper, we present such a proof:

Theorem 1 *In the algebraic computation-tree model, the time complexity of* EVENRANKSUM *is* $\Theta(n \log n)$.

We prove Theorem 1 by presenting an $O(n)$–time reduction of MINGAP to EVENRANKSUM. The former problem is defined as follows. Let $X = (x_1, x_2, \ldots, x_n)$ be a sequence of n real numbers, and let π be a permutation of $\{1, 2, \ldots, n\}$ such that $x_{\pi(1)} \leq x_{\pi(2)} \leq \ldots \leq x_{\pi(n)}$. For each $1 \leq i < n$, we define the difference $x_{\pi(i+1)} - x_{\pi(i)}$ to be a *gap* in the sequence X.

MINGAP: Given a sequence $X = (x_1, x_2, \ldots, x_n)$ of n real numbers and a real number $g > 0$, decide if each of the $n - 1$ gaps in X is at least g.

Since in the algebraic computation-tree model, MINGAP has an $\Omega(n \log n)$ lower bound (see [2, Section 8.4]), our reduction will prove Theorem 1.

2 The proof of Theorem 1

We now show how to reduce, in $O(n)$ time, MINGAP to EVENRANKSUM.

Let $\mathcal{A}$ be an arbitrary algorithm that solves EVENRANKSUM. We show how to use algorithm $\mathcal{A}$ to solve MINGAP. Let $n \geq 2$ be an integer and consider a sequence $X = (x_1, x_2, \ldots, x_n)$ of n real numbers and a real number $g > 0$. The algorithm for solving MINGAP makes the following three steps:

Step 1: Compute $S = \sum_{i=1}^{n} x_i$ and, for $i = 1, 2, \ldots, n$, compute $a_{2i-1} = x_i$ and $a_{2i} = x_i + g$.

Step 2: Run algorithm $\mathcal{A}$ on the sequence $(a_1, a_2, \ldots, a_{2n})$, and let R be the output, i.e., R is the even-rank-sum of this sequence.

Step 3: If $R = S + ng$, then return YES. Otherwise, return NO.

It is clear that the running time of this algorithm is $O(n)$ plus the running time of $\mathcal{A}$. Thus, it remains to show that the algorithm correctly solves MinGap. That is, we have to show that the minimum gap G of X is at least g if and only if $R = S + ng$. This is an immediate consequence of the following lemma:

Lemma 1 *Let $x_1, x_2, \ldots, x_n$ and g be real numbers such that $x_1 \leq x_2 \leq \ldots \leq x_n$ and $g > 0$. Let $(a_1, a_2, \ldots, a_{2n}) = (x_1, x_1 + g, x_2, x_2 + g, \ldots, x_n, x_n + g)$ and let π be a permutation of $\{1, \ldots, 2n\}$ such that $b_1 \leq b_2 \leq \ldots \leq b_{2n}$ with $b_i = a_{\pi(i)}$ for $1 \leq i \leq 2n$.*

If $R = \sum_{i=1}^{n} b_{2i}$, $U = \sum_{i=1}^{n} b_{2i-1}$, and $G = \min\{x_{i+1} - x_i \mid 1 \leq i \leq n-1\}$, then $R - U \leq ng$ with equality if and only if $G \geq g$.

Proof. Since $x_1, x_1+g, x_2, x_2+g, \ldots, x_i, x_i+g \leq x_i+g$, we have $x_i+g \geq b_{2i}$ for $1 \leq i \leq n$. Since $x_i, x_i + g, x_{i+1}, x_{i+1} + g, \ldots, x_n, x_n + g \geq x_i$, we have $x_i \leq b_{2i-1}$ for $1 \leq i \leq n$. Hence $b_{2i} - b_{2i-1} \leq (x_i + g) - x_i = g$ for $1 \leq i \leq n$ which implies $R - U \leq ng$.

If $G \geq g$, then clearly $R - U = ng$. Conversely, if $R - U = ng$, then $b_{2i} - b_{2i-1} = g$ for $1 \leq i \leq n$. In view of the above, this implies that $x_i + g = b_{2i}$ and $x_i = b_{2i-1}$ for $1 \leq i \leq n$. Since $x_{i+1} = b_{2i+1} \geq b_{2i} = x_i + g$ for $1 \leq i \leq n-1$, we obtain $G \geq g$. ∎

We complete the proof of Theorem 1 by observing that $R+U = 2S+ng$ and by Lemma 1 we have $G \geq g$ if and only if $R = U + ng = S + ng$.

References

[1] M. Ben-Or. Lower bounds for algebraic computation trees. In *Proceedings of the 15th ACM Symposium on the Theory of Computing*, pages 8086, 1983.

[2] F. P. Preparata and M. I. Shamos. *Computational Geometry: An Introduction.* Springer-Verlag, Berlin, 1988.