Exponential stability of time-varying linear systems

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Abstract

This paper considers the stability of both continuous and discrete time-varying linear systems. Stability estimates are obtained in either case in terms of the Lipschitz constant for the governing matrices and the assumed uniform decay rate of the corresponding frozen time linear systems. The main techniques used in the analysis are comparison methods, scaling, and the application of continuous stability estimates to the discrete case. Counterexamples are presented to show the necessity of the stability hypotheses. The discrete results are applied to derive sufficient conditions for the stability of a Backward Euler approximation of a time-varying system, and a one-leg linear multistep approximation of a scalar system.

Keywords: Exponential stability, discrete time-varying linear systems, continuous time-varying linear systems, Backward Euler approximation, one-leg multistep approximation

Notation

$$\begin{aligned}
\lfloor x \rfloor &:= \max\{n \in \mathbb{Z} \mid n \le x\}, \quad x \in \mathbb{R}, \\
\operatorname{sign}[x] &:= \begin{cases} x/|x|, & x \in \mathbb{R} \setminus \{0\}, \\ 0, & x = 0, \end{cases} \\
\mathbb{R}_{\ge p} &:= [p, \infty), \ p \in \mathbb{R}, \\
\mathbb{R}_{> p} &:= (p, \infty), \ p \in \mathbb{R},
\end{aligned}$$

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$$
\begin{array}{rcl}
\mathcal{D} & := & \{(t,s)\in\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}|\ t\geq s\},\\
\Delta & := & \{(n,m)\in\mathbb{N}_0\times\mathbb{N}_0|\ n\geq m\},\\
M^{\mathbb{N}} & := & \text{the set of all mappings from } \mathbb{N} \text{ to a set } M,\\
\|x\| & & \text{denotes a norm of } x\in\mathbb{C}^N,\\
\|x\|_2 & := & \sqrt{x^*x}, \text{ Euclidean norm of } x\in\mathbb{C}^N,\\
\|A\| & := & \max\{\|Ax\| \mid \|x\|=1\}, \text{ induced operator of } A\in\mathbb{C}^{\ell\times q},\\
\operatorname{spec}(A) & := & \{\lambda\in\mathbb{C} \mid \det(\lambda I-A)=0\}, \text{ the spectrum of } A\in\mathbb{C}^{N\times N},\\
\mu(A) & := & \max\{|\lambda| \mid \lambda\in\operatorname{spec}(A)\}, \text{ spectral radius of } A\in\mathbb{C}^{N\times N},\\
C(J,\mathbb{R}^{\ell\times q}) & & \text{is the vector space of continuous functions } f: J\to\mathbb{R}^{\ell\times q},\\
 & & J\subset\mathbb{R} \text{ an interval, with sup norm},\\
\|f\|_\infty & := & \sup_{t\in J}\|f(t)\|.
\end{array}
$$

1 Introduction

We study exponential growth (stability) of time-varying linear systems in continuous time

$$\dot{u}(t) = A(t)\,u(t), \qquad t\geq 0, \tag{1.1}$$

where $A(\cdot):\mathbb{R}_{\geq 0}\to C^{N\times N}$ is a continuous function, and in discrete time

$$u_{n+1} = A_n\,u_n, \qquad n\in\mathbb{N}_0, \tag{1.2}$$

where $(A_n)_{n\in N_0}$ is a sequence of matrices with elements in $\mathbb{C}^{N\times N}$. Recall, see for example [18], that (1.1) is said to be *(uniformly) exponentially stable* if, and only if,

$$\exists\ (M,\eta)\in\mathbb{R}_{\geq 1}\times\mathbb{R}_{>0}\ \forall \text{ slns. } u \text{ of (1.1) } \forall\ (t,s)\in\mathcal{D}:\ \|u(t)\|\leq Me^{-\eta(t-s)}\|u(s)\|, \tag{1.3}$$

and (1.2) is said to be *(uniformly) exponentially stable* if, and only if,

$$\exists\ (M,\eta)\in\mathbb{R}_{\geq 1}\times\mathbb{R}_{>0}\ \forall \text{ slns. } u \text{ of (1.2) } \forall\ (n,m)\in\Delta:\ \|u_n\|\leq Me^{-\eta(n-m)}\|u_m\|. \tag{1.4}$$

To derive sufficient conditions for bounds of M and η, we assume that the frozen systems $\dot{u}(t) = A(\tau)\,u(t)$ and $u_{n+1} = A_m\,u_n$ are exponentially stable and A is globally Lipschitz. More precisely, for $(K,\omega,L)\in\mathbb{R}_{\geq 1}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$, we consider the classes of generators

$$\mathcal{S}_{K,\omega,L} := \left\{ A\in\mathcal{C}_{\text{pw}}\left(\mathbb{R}_{\geq 0},\mathbb{C}^{N\times N}\right) \,\middle|\, \begin{array}{lrcl} \forall\ t,\ s\in\mathbb{R}_{\geq 0}: & \|e^{A(t)s}\| & \leq & Ke^{-\omega s} \\ \forall\ t,\ s\in\mathbb{R}_{\geq 0}: & \|A(t)-A(s)\| & \leq & L\,|t-s| \end{array} \right\}$$

and

$$\Sigma_{K,\omega,L} \;:=\; \left\{ (A_n) \in \left(\mathbb{C}^{N\times N}\right)^{\mathbb{N}} \;\middle|\; \begin{array}{lrcl} \forall\, n,\, m \in \mathbb{N}_0 : & \|A_n^m\| & \le & K e^{-\omega m} \\ \forall\, n, m \in \mathbb{N}_0 : & \|A_n - A_m\| & \le & L\,|n-m| \end{array} \right\}.$$

It may be worth knowing that, although we consider time-varying systems, due to the special system classes $\mathcal{S}_{K,\omega,L}$ and $\Sigma_{K,\omega,L}$, the decay rate η and the constant M prescribing the exponential stability in (1.3) and (1.4), respectively, hold for every initial value if, and only if, they hold for the initial value at time $t = 0$; to be more precise, see Lemma 5.1 and Lemma 5.10, respectively.

A nice textbook on time-varying systems, continuous as well as discrete time, is [18], see also the references therein. Bounds on the exponential growth of continuous/discrete time-varying systems have been suggested by numerous authors: [4, 7, 12, 13, 14, 17, 19, 20]/[1, 8].

A good description of the stability analysis available for numerical methods is given in [10], which summarises earlier work for linear multistep, Runge-Kutta and general linear methods by [15, 5, 2, 3]. These results relate to the propagation of errors made in the approximation of the problem $\dot{u} = f(t,\, u)$, where f is assumed to satisfy a structural assumption ensuring the stability of solutions u. More recently, articles [9] and [16] have investigated the stability of linear multistep methods approximating time-varying systems of the form $\dot{u}(t) = A(t)u$ in a Hilbert space setting.

In the continuous case, we use the variation of constants formula and the properties of the set $\mathcal{S}_{K,\omega,L}$ to obtain an integral inequality for $\|u(t)\|$. The main idea revived in this paper is to use scaling to eliminate as many apparently independent parameters as possible in the integral inequality. Once the inequality is simplified in this way, it is then relatively simple to bound $\|u(t)\|$ sharply in terms of the solution of a comparison equation.

In the continuous case, a natural time-scale $\sqrt{KL}$ arises out of the scaling process. For the discrete problem, a similar time-scale $\beta := \sqrt{KLe^{\omega}}$ battles with the intrinsic unit time-scale of the discrete process. When β is small, estimates from the continuous problem are also sharp for the corresponding discrete case. For larger β, a direct discrete approach yields better bounds.

The paper is organized as follows. In Section 2 the two main results give sufficient conditions for the uniform decay of solutions of continuous and discrete time systems. Counterexamples are also presented in cases where the hypotheses do not hold. The proof of the two theorems are given in Section 5. Applications of the main discrete result are given in Sections 3 and 4: In Section 3 sufficient conditions are given for the exponential stability of a Backward Euler approximation of a general continuous time-varying system of the type considered in Section 2. In Section 4, the results of Section 2 are used to prove the stability of a one-leg multistep approximation of a time-varying scalar differential equation.

2 Exponential Stability

2.1 Continuous time systems

Theorem 2.1. *Suppose that* $A \in \mathcal{S}_{K,\omega,L}$ *for some* $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. *Then every solution* u *of* (1.1) *satisfies*

$$\text{(i)} \quad \|u(t)\| \leq K \exp\left\{ \left(\tfrac{KL(t-s)}{4} - \omega \right)(t-s) \right\} \|u(s)\|, \qquad \forall\ (t,s) \in \mathcal{D},$$

$$\text{(ii)} \quad \|u(t)\| \leq K \exp\left\{ \left(\sqrt{KL \log(\min\{2,\ K\})} - \omega \right)(t-s) \right\} \|u(s)\|, \qquad \forall\ (t,s) \in \mathcal{D}.$$

The above theorem is proved in Subsection 5.1.

Remark 2.2. Theorem 2.1 (ii) implies that $\dot{u}(t) = A(t)u(t)$ is exponentially stable if

$$KL \log(\min\{2,\ K\}) < \omega^2. \tag{2.1}$$

2.2 Discrete time systems

Theorem 2.3. *Suppose that* $A \in \Sigma_{K,\omega,L}$ *for some* $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. *Then, for* $\beta := \sqrt{KLe^{\omega}}$, *every solution* $u\colon \mathbb{N}_0 \to \mathbb{C}^N$ *of* $u_{n+1} = A_n u_n$ *satisfies*

$$\text{(i)} \quad \|u_n\| \leq K^{n-m} e^{-\omega(n-m)} \|u_m\|, \qquad \forall\ (n,m) \in \Delta,$$

$$\text{(ii)} \quad \|u_n\| \leq K \exp\left\{ \beta^2 (n-m)^2/4 - \omega(n-m) \right\} \|u_m\|, \qquad \forall\ (n,m) \in \Delta,$$

$$\text{(iii)} \quad \|u_n\| \leq K \exp\left\{ (\beta\sqrt{\log 2} - \omega)(n-m) \right\} \|u_m\|, \qquad \forall\ (n,m) \in \Delta,$$

$$\text{(iv)} \quad \|u_n\| \leq \tfrac{1}{2} \left\{ (1+\beta)^{n-m} + (1-\beta)^{n-m} \right\} K e^{-\omega(n-m)} \|u_m\|, \qquad \forall\ (n,m) \in \Delta.$$

The above theorem is proved in Subsection 5.2.

Remark 2.4. (Bound comparison and time–scales)
Bounds (ii) and (iii) are the respective small and long time analogs of the corresponding continuous results. Such bounds are particularly useful if $\beta \ll 1$, when it is harder to obtain sharp discrete bounds directly.

A direct discrete approach works better if $\beta \geq 1$. Comparing the growth rate of (iii) and (iv) for large $n-m$, we see that the direct discrete bound (iv) is sharper than (iii) if

$$\beta > \beta_0 \approx 0.43 \qquad \text{such that} \qquad \exp(\beta_0\sqrt{\log 2}) = 1 + \beta_0.$$

For some problems, the trivial bound (i) is best, particularly for small $n-m$ if a special norm is chosen so that K is close to, or equal to, 1.

Remark 2.5. (Exponential stability criteria)
By inequalities (i), (iii) and (iv), a system $A \in \Sigma_{K,\omega,L}$ for some $(K,\omega,L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ is exponentially stable if

$$\omega > \min\left\{\log(1+\beta),\, \beta\sqrt{\log 2},\, \log K\right\}. \tag{2.2}$$

2.3 Continuous and discrete generalisations of a counterexample of Hoppenstaedt

The following examples generalise a well known example of [11].

Example 2.6. Description of system: Define $A(\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{C}^{2\times 2}$ for $a > \theta > 0$ by

$$A(t) := Q(t)A_0Q^T(t); \quad Q(t) := \begin{bmatrix} \cos\theta t & \sin\theta t \\ -\sin\theta t & \cos\theta t \end{bmatrix}, \quad A_0 := \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}. \tag{2.3}$$

Properties of the frozen systems: Since $Q(\tau)$ is orthogonal,

$$\|e^{A(\tau)t}\|_2 = \|Q(\tau)e^{A_0t}Q^T(\tau)\|_2 = \|e^{A_0t}\|_2 = \left\|\begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix}\right\|_2 \leq 1 + at, \qquad \forall\, \tau, t \geq 0. \tag{2.4}$$

Lipschitzian properties of $A(\cdot)$: We observe that, for all t, $s \geq 0$,

$$\|A(t) - A(s)\|_2 = \|Q^T(t)A_0Q(t) - Q^T(s)A_0Q(s)\|_2 = \|Q^T(t-s)A_0Q(t-s) - A_0\|_2,$$

and also

$$\|Q^T(t)A_0Q(t) - A_0\|_2 = a\left\|\begin{bmatrix} \sin\theta t\cos\theta t & -\sin^2\theta t \\ -\sin^2\theta t & \sin\theta t\cos\theta t \end{bmatrix}\right\|_2 = a|\sin\theta t|, \qquad \forall\, t \geq 0.$$

Hence,

$$\|A(t) - A(s)\|_2 = a|\sin(\theta(t-s))| \leq a\theta|t-s|, \qquad \forall\, t,\, s \geq 0.$$

Properties of the time–varying system: If $u : \mathbb{R}_{\geq 0} \to \mathbb{C}^2$ is a solution of (1.1), define

$$x : \mathbb{R}_{\geq 0} \to \mathbb{C}^2, \quad t \mapsto x(t) := Q^T(t)u(t) \qquad \text{and} \qquad B := \begin{bmatrix} 0 & a-\theta \\ \theta & 0 \end{bmatrix}.$$

Since $\dot{Q}(t)^T Q(t) = \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$ for all $t \geq 0$, we obtain

$$\dot{x}(t) = \dot{Q}(t)^T u(t) + Q^T(t)\dot{u}(t) = (\dot{Q}(t)^T Q(t) + A_0)x(t) = Bx(t).$$

Since $B^2 = \theta(a-\theta)I$,

$$\begin{aligned} e^{Bt} &= \sum_{m=0}^{\infty} B^{2m} \frac{t^{2m}}{(2m)!}\left(I + B\frac{t}{2m+1}\right) = \sum_{m=0}^{\infty} \frac{[\theta(a-\theta)]^m t^{2m}}{(2m)!} \begin{bmatrix} 1 & \frac{(a-\theta)t}{2m+1} \\ \frac{\theta t}{2m+1} & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cosh[\theta(a-\theta)]^{1/2}t & \sqrt{\frac{a-\theta}{\theta}}\sinh[\theta(a-\theta)]^{1/2}t \\ \sqrt{\frac{\theta}{a-\theta}}\sinh[\theta(a-\theta)]^{1/2}t & \cosh[\theta(a-\theta)]^{1/2}t \end{bmatrix}. \end{aligned}$$

For all $t \in \mathbb{R}_{\geq 0}$, $\|u(t)\|_2 = \|Q(t)x(t)\|_2 = \|x(t)\|_2 = \|e^{Bt}x(0)\|_2$ and $x(0) = u(0)$. Hence,

$$\max_{u(0)\in\mathbb{C}^2\setminus\{0\}} \frac{\|u(t)\|_2}{\|u(0)\|_2} = \max_{x(0)\in\mathbb{C}^2\setminus\{0\}} \frac{\|e^{Bt}x(0)\|_2}{\|x(0)\|_2} = \|e^{Bt}\|_2 \geq \exp\left([\theta(a-\theta)]^{1/2}t\right). \tag{2.5}$$

Comments: We observe that whilst solutions of the frozen time systems only grow at most linearly with time, those of the time–varying system may grow exponentially. Choosing $\epsilon \in (0, \sqrt{\theta(a-\theta)})$ and replacing $A(t)$ by $A_\epsilon(t) := A(t) - \epsilon I$ has the effect of multiplying the RHS of (2.4) and (2.5) by $e^{-\epsilon t}$. This makes the frozen time systems exponentially stable, but leaves the time–varying system exponentially unstable, and does not affect the Lipschitzian properties of $A(\cdot)$.

Example 2.7. The Euler method for Example 2.6
The Euler method for (2.3) is

$$u_{n+1} = (I + hA(nh))u_n =: T_n(h)u_n, \qquad n \in \mathbb{N}_0, \tag{2.6}$$

where $h > 0$ is a time-step. Thus, for Q as in Example 2.6,

$$T_n(h) = Q(nh)T_0(h)Q(-nh), \qquad T_0(h) = \begin{bmatrix} 1 & ah \\ 0 & 1 \end{bmatrix}.$$

Properties of the frozen system: The approximation is exact here, so (2.4) implies that

$$\|T_n^m(h)\|_2 = \|e^{A(nh)mh}\|_2 \leq 1 + amh, \qquad \forall\,(n,\,m) \in \mathbb{N}_0^2. \tag{2.7}$$

Lipschitzian properties: We observe that

$$\begin{aligned} \|T_n(h) - T_m(h)\|_2 &= \|Q(nh)T_0(h)Q(-nh) - Q(mh)T_0(h)Q(-mh)\|_2 \\ &= \|Q((n-m)h)T_0(h)Q(-(n-m)h) - T_0(h)\|_2, \qquad \forall\,(n,\,m) \in \mathbb{N}_0^2. \end{aligned}$$

Also,

$$\|Q(nh)T_0(h)Q(-nh) - T_0(h)\|_2 = ah \left\| \begin{bmatrix} \sin\theta nh \cos\theta nh & -\sin^2\theta nh \\ -\sin^2\theta nh & \sin\theta nh\cos\theta nh \end{bmatrix} \right\|_2$$
$$= ah|\sin\theta nh|, \qquad \forall\, n \in \mathbb{N}_0.$$

Hence,

$$\|T_n(h) - T_m(h)\|_2 = ah|\sin(\theta(n-m)h)| \leq a\theta h^2|n-m|, \qquad \forall\,(n,\, m) \in \mathbb{N}_0^2. \tag{2.8}$$

Properties of time–varying system: If $u_n : \mathbb{N}_0 \to \mathbb{C}^2$ is a solution of (2.6), define

$$x : \mathbb{N}_0 \to \mathbb{C}^2, \qquad n \mapsto x_n := Q(-nh)u_n.$$

Hence,

$$x_{n+1} = Q(-(n+1)h)u_{n+1} = Q(-(n+1)h)Q(nh)T_0(h)Q(-nh)u_n$$
$$= Q(-h)T_0(h)x_n = \begin{bmatrix} \cos\theta h & -\sin\theta h + ah\cos\theta h \\ \sin\theta h & \cos\theta h + ah\sin\theta h \end{bmatrix} x_n =: B(h)x_n, \quad \forall\, n \in \mathbb{N}_0.$$

Since $x_0 = u_0$, and $Q(\cdot)$ is orthogonal,

$$\max_{u_0 \in \mathbb{C}^2\setminus\{0\}} \frac{\|u_n\|_2}{\|u_0\|_2} = \max_{x_0 \in \mathbb{C}^2\setminus\{0\}} \frac{\|x_n\|_2}{\|x_0\|_2} = \|B^n(h)\|_2 \geq \mu^n(B(h)), \qquad n \in \mathbb{N}_0. \tag{2.9}$$

As $a > \theta > 0$, there is a unique $h_0 \in (0, \pi/\theta)$ such that $ah_0 = 2\tan(\theta h_0/2)$ and $ah > 2\tan(\theta h/2)$, for all $h \in (0,\, h_0)$. Elementary trigonometry now implies that

$$\mu(B(h)) = \sigma(h) + \sqrt{\sigma^2(h) - 1} > 1, \qquad \sigma(h) := \cos\theta h + (ah/2)\sin\theta h > 1, \qquad \forall\, h \in (0,\, h_0).$$

Comments: We observe that whilst solutions of the frozen systems only grow at most linearly with time, those of the time–varying system may grow exponentially. Replacing $A(t)$ by $A_\epsilon(t) = A(t) - \epsilon I$ for $\epsilon \in (0,\, \sqrt{\theta(a-\theta)})$ multiplies the RHS of (2.7) by $(1-\epsilon h)^m$, for all sufficiently small $h > 0$, rendering the discrete frozen systems exponentially stable. This change has no effect on the Lipschitzian properties of the discrete system. A similar analysis to the above shows that $\mu(B_\epsilon(h)) > 1$ for all sufficiently small $h > 0$, implying that the discrete time-varying system is exponentially unstable in that parameter range.

Remark 2.8. The fact that the Euler method in Example 2.7 inherits both the stability of the frozen time systems and the instability of the time-varying system for all sufficiently small h is an expected consequence of consistency. This would be true for any consistent one-step method — even a very stable one, such as Backward Euler. Thus, to establish the stability of a discrete time-varying system for any consistent method, more is required than the stability of the frozen time systems combined with a Lipschitz condition of the form (2.8). In general, one also requires additional information about the underlying system, such as (2.1), or a corresponding bound on the discretisation, such as (2.2).

3 The stability of a Backward Euler approximation

Here, we consider the approximation of (1.1) by the Backward Euler method. For a time-step $h > 0$, the equation of the method is

$$u_{n+1} = u_n + hA((n+1)h)u_{n+1}, \qquad n \in \mathbb{N}_0. \tag{3.1}$$

Under suitable assumptions on $A(\cdot)$, Lemma 3.1 below shows that the sequence $(T_n(h))_{n\in:\mathbb{N}_0}$ of matrices in $\mathbb{C}^{N\times N}$ given by

$$T_n(h) := (I - hA((n+1)h))^{-1}, \qquad n \in \mathbb{N}_0, \tag{3.2}$$

is well defined. This allows us to rewrite (3.1) as the discrete time-varying system

$$u_{n+1} = T_n(h)u_n, \qquad n \in \mathbb{N}_0, \tag{3.3}$$

and to attempt to establish the stability properties of the Backward Euler method using Theorem 2.3.

Lemma 3.1. *Suppose that $h > 0$ and that $A(\cdot) \in \mathcal{S}_{K,\omega,L}$ for $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Then, for each $n \in \mathbb{N}_0$, $T_n(h)$ given by (3.2) is well defined, and*

$$\|T_n^m(h)\| \leq K(1 + h\omega)^{-m}, \qquad m \in \mathbb{N}_0. \tag{3.4}$$

Proof: For every $m \in \mathbb{N}$,

$$\begin{aligned} \|(I - hA(nh))^{-m}\| &= \left\| \int_0^\infty \frac{t^{m-1}}{(m-1)!} e^{-(I-hA(nh))t} \,\mathrm{d}t \right\| \\ &\leq \int_0^\infty \frac{t^{m-1}}{(m-1)!} K e^{-(1+\omega h)t} \,\mathrm{d}t = K(1+\omega h)^{-m}, \end{aligned}$$

where the boundedness of the RHS ensures the well–definedness of the LHS. The case $m = 0$ is clear, as $K \geq 1$. □

Lemma 3.2. *Suppose that $h > 0$ and that $A(\cdot) \in \mathcal{S}_{K,\omega,L}$ for $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Then,*

$$\|T_n(h) - T_m(h)\| \leq LK^2(1 + \omega h)^{-2}h^2|n - m|, \qquad \forall\,(n, m) \in \mathbb{N}_0^2. \tag{3.5}$$

Proof: As in the second resolvent identity,

$$(I - hA(nh))^{-1} - (I - hA(mh))^{-1} = h(I - hA(nh))^{-1}[A(nh) - A(mh)](I - hA(mh))^{-1}.$$

Thus, by (3.4) and the Lipschitzian properties of $A(\cdot)$,

$$\begin{aligned} \|T_n(h) - T_m(h)\| &\leq \|(I - hA(nh))^{-1}\|\|(I - hA(mh))^{-1}\|h\|A(nh) - A(mh)\| \\ &\leq K^2(1 + \omega h)^{-2}hL|nh - mh| \leq LK^2(1 + \omega h)^{-2}h^2|n - m|. \end{aligned}$$

□

Lemma 3.3. *Suppose that $h > 0$ and that $A(\cdot) \in \mathcal{S}_{K,\omega,L}$ for $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Then, for $\delta := \sqrt{K^3 L}$, every solution $u\colon \mathbb{N}_0 \to \mathbb{C}^N$ of $u_{n+1} = T_n(h)\, u_n$ satisfies*

(i) $\|u_n\| \leq K^{n-m}(1+\omega h)^{-(n-m)}\|u_m\|, \qquad \forall\ (n,m) \in \Delta,$

(ii) $\|u_n\| \leq K(1+\omega h)^{-(n-m)} \exp\{\delta^2(n-m)^2h^2/4\}\,\|u_m\|, \qquad \forall\ (n,m) \in \Delta,$

(iii) $\|u_n\| \leq K(1+\omega h)^{-(n-m)} \exp\left\{(\delta\sqrt{\log 2})(n-m)h\right\}\|u_m\|, \qquad \forall\ (n,m) \in \Delta,$

(iv) $\|u_n\| \leq \frac{1}{2}\{(1+\delta h)^{n-m} + (1-\delta h)^{n-m}\}\, K(1+\omega h)^{-(n-m)}\|u_m\|, \quad \forall\ (n,m) \in \Delta.$

Proof: By Lemmas 3.1 and 3.2,

$$(T_n(h))_{n\in\mathbb{N}_0} \in \Sigma_{\widehat{K},\widehat{\omega},\widehat{L}}, \qquad \text{for} \qquad \widehat{K} := K,\ \widehat{L} := K^2Lh^2(1+\omega h)^{-1},\ \widehat{\omega} := \log(1+\omega h),$$

(where $\widehat{L}$ has been increased by a factor of $(1+\omega h)$ for convenience). Applying the conclusions of Theorem 2.3 with

$$\widehat{\beta} := \sqrt{\widehat{K}\widehat{L}e^{\widehat{\omega}}} = \sqrt{K^3Lh^2} = \delta h \qquad \text{and} \qquad e^{-\widehat{\omega}} = (1+\omega h)^{-1},$$

we obtain (i)–(iv). □

The following is clear from bounds (iii) and (iv) in Lemma (3.3):

Theorem 3.4. *Suppose that $A(\cdot) \in \mathcal{S}_{K,\omega,L}$ for $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Then,* (3.1) *is exponentially stable if*

$$\omega > \min\left\{\sqrt{K^3L},\ \frac{\exp(h\sqrt{K^3L\log 2}) - 1}{h}\right\}. \tag{3.6}$$

Remark 3.5. (Stability criteria)
Bounds (i) and (ii) in Lemma 3.3 may be the sharpest for small $n-m$. Inequalities (ii)–(iv) all give rise to bounds of the form

$$\forall\, \tau > 0\ \forall\, (h,\, n) \in \mathbb{R}_{>0} \times \mathbb{N}_0 \ \text{ such that }\ nh \in [0,\, \tau]\ :\quad \|u_n\| \leq M\|u_0\|.$$

The condition for exponential stability of the method, given by (3.6), is good in the sense that it is essentially h–independent, (with a mild improvement as $h \to 0+$). However, ω is required to be approximately a factor of K larger than in (2.1). This factor arises in the Lipschitz analysis of $T_n(h)$. In the non-stiff case, where $h\|A(\cdot)\|_\infty \ll 1$, approximation could be used to bound solutions of the method indirectly.

4 Stability for a one–leg multistep approximation of a scalar problem

Here, we show how the general observations made above apply to the approximation of a simple time–varying problem.

We consider the approximation of the time–varying scalar equations of the form

$$\dot{u}(t) = \lambda(t)u(t), \qquad t \geq 0, \tag{4.1}$$

where $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{C}$ is assumed to be Lipschitz continuous; i.e.

$$\exists\, L_\lambda > 0 \;\forall\, t, s \geq 0 \;:\quad |\lambda(t) - \lambda(s)| \leq L_\lambda\, |t - s|. \tag{4.2}$$

A q–step one–leg method approximating (4.1) can be taken to be of the form

$$\sum_{j=0}^{q} \alpha_j y_{n+j} = h\lambda(nh) \sum_{j=0}^{q} \beta_j y_{n+j}, \qquad n \in \mathbb{N}_0, \tag{4.3}$$

where $h > 0$ is the time–step, and the coefficients α_0, $\beta_0 \ldots, \alpha_q, \beta_q \in \mathbb{R}$ are chosen so that the corresponding linear multistep method is irreducible and has order $p \in \mathbb{N}$. It is assumed that numerical initial data y_0, y_1, …, y_{q-1} is generated by some other method. It is known, see e.g. [10, Ch. V], that the stability properties of the linear multistep method may be studied in terms of those of the one–leg method.

Let $\overline{\mathbb{C}}$ denote $\mathbb{C} \cup \{\infty\}$. We define the companion matrix $C : \overline{\mathbb{C}} \to \overline{\mathbb{C}}^{q \times q}$ by

$$C(z) := \left[\begin{array}{c|cccc} 0 & 1 & & & \\ \vdots & & \ddots & & \\ 0 & & & & 1 \\ \hline c_0(z) & c_1(z) & \cdots & & c_{q-1}(z) \end{array}\right], \qquad c_j(z) := -\frac{\alpha_j - z\beta_j}{\alpha_q - z\beta_q}, \quad 0 \leq j \leq q-1, \tag{4.4}$$

where we follow the convention that $c_j(\infty) = \lim_{z\to\infty} c_j(z)$, $0 \leq j \leq q-1$.

Setting $z_n := h\lambda(nh)$ for $n \in \mathbb{N}_0$, we define

$$A_n := C(z_n), \qquad u_n := [y_n, \ldots, y_{n+q-1}]^T \in \mathbb{C}^q, \qquad n \in \mathbb{N}_0, \tag{4.5}$$

for a solution (y_n) to (4.3). We now observe that (u_n) satisfies the system

$$u_{n+1} = A_n u_n, \qquad n \in \mathbb{N}_0, \tag{4.6}$$

which is of the form (1.2).

A matrix $B \in C^{N\times N}$ is called *power–bounded* if, and only if,

$$\exists\, K > 0\ \forall\, n \in \mathbb{N}_0\ :\ \|B^n\| \le K.$$

The *linear stability region* for the linear multistep method corresponding to (4.3) is

$$\mathcal{S} := \{ z \in \overline{\mathbb{C}} \mid C(z) \text{ is power–bounded } \}. \tag{4.7}$$

Example 4.1. ***(Numerical instability for rapidly varying*** λ***)*** Consider (4.1) for

$$\lambda : \mathbb{R}_{\ge 0} \to \mathbb{C}, \qquad t \mapsto \lambda(t) = -\frac{81 + 79\cos(\pi t)}{8}. \tag{4.8}$$

Suppose that the scalar system (4.1) is approximated by the BDF4 one–leg method. In this case, the method is given by (4.3) for $q = 4$, and coefficients

$$[\alpha_0,\, \alpha_1,\, \alpha_2,\, \alpha_3,\, \alpha_4] = \left[\frac{1}{4},\, -\frac{4}{3},\, 3,\, -4,\, \frac{25}{12}\right], \qquad [\beta_0,\, \beta_1,\, \beta_2,\, \beta_3,\, \beta_4] = [0,\, 0,\, 0,\, 0,\, 1]\,.$$

The companion matrix corresponding to BDF4: $C : \overline{\mathbb{C}} \to \overline{\mathbb{C}}^{4\times 4}$, is defined by (4.4) with $q = 4$. If the time–step is $h = 1$, then

$$z_n = h\lambda(nh) = \begin{cases} -20, & n \text{ even}, \\ -1/4, & n \text{ odd}. \end{cases}$$

As in (4.5), let $A_n := C(z_n)$, $n \in \mathbb{N}_0$. Hence, for $n \in \mathbb{N}_0$,

$$A_{2n} = C(-20) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{265} & \frac{16}{265} & -\frac{36}{265} & \frac{48}{265} \end{bmatrix}, \quad A_{2n+1} = C(-1/4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{3}{28} & \frac{16}{28} & -\frac{36}{28} & \frac{48}{28} \end{bmatrix}.$$

Straightforward calculations show that $\{-1/4,\, -20\} \subset \mathcal{S}$:

$$\mu(A_{2n}) = \mu(C(-20)) = 0.42, \ \text{(2 d.p.)}, \qquad \mu(A_{2n+1}) = \mu(C(-1/4)) = 0.78, \ \text{(2 d.p.)}$$

Now $u_{2n} = A_{2n-1}A_{2n-2}\dots A_1 A_0 u_0$, where the product

$$A_{2n-1}\dots A_1 A_0 = (A_1 A_0)^n = (C(-1/4)C(-20))^n, \qquad n \in \mathbb{N}_0.$$

The stability of the system is therefore determined by

$$\mu(C(-1/4)C(-20)) = 1.02, \ \text{(2 d.p.)}$$

Since $\mu(C(-1/4)C(-20)) > 1$, we deduce that there are initial conditions $u_0 \in \mathbb{R}$ such that $\|u_n\|$ grows exponentially with n; i.e. the BDF4 method with $h = 1$ is unstable for this differential equation.

Theorem 4.2. *Consider the scalar time–varying equation* (4.1)*, such that* $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{C}$ *satisfies* (4.2)*. Suppose that* (4.1) *is approximated by a one–leg method* (4.3) *with corresponding linear stability region* $\mathcal{S} \subset \overline{\mathbb{C}}$*, as in* (4.7)*. Assume also that* $D \subseteq \mathcal{S}$ *is closed and that*

$$\forall\, h > 0\ \forall\, n \in \mathbb{N}_0\ :\quad h\lambda(nh) \in D. \tag{4.9}$$

Then, there exist $(K,\, \omega,\, \widehat{L}) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ *such that, for all* $h > 0$ *and* $\widehat{\beta} := \sqrt{K\widehat{L}e^{\omega}}$*, any solution* u *of* (4.6) *satisfies*

$$\text{(a)}\quad \|u_n\| \leq K \exp\Big\{[(\log 2)^{1/2}\widehat{\beta}h - \omega]n\Big\}\, \|u_0\|, \qquad \forall\, n \in \mathbb{N}_0,$$

$$\text{(b)}\quad \|u_n\| \leq \tfrac{1}{2}\Big\{(1+\widehat{\beta}h)^n + (1-\widehat{\beta}h)^n\Big\}\, Ke^{-\omega n}\|u_0\|, \quad \forall\, n \in \mathbb{N}_0.$$

Remark 4.3. Under slightly stronger conditions than the hypotheses of Theorem 4.2, the conclusion of Prop 4.4 below may be strengthened to

$$\exists\, K, \widetilde{\omega},\, h_0 > 0\ \ \forall\, n \in \mathbb{N}_0\ \ \forall\, h\lambda(nh) \in D\ :\quad \|C^n(h\lambda(nh))\| \leq Ke^{-n\widetilde{\omega}h}.$$

In this case, the results of Theorem 4.2 can be improved to

$$\text{(a}'\text{)}\quad \|u_n\| \leq K \exp\Big\{[(\log 2)^{1/2}\widehat{\beta} - \widetilde{\omega}]nh\Big\}\, \|u_0\|, \qquad \forall\, n \in \mathbb{N}_0,$$

$$\text{(b}'\text{)}\quad \|u_n\| \leq \tfrac{1}{2}\Big\{(1+\widehat{\beta}h)^n + (1-\widehat{\beta}h)^n\Big\}\, Ke^{-\widetilde{\omega}nh}\|u_0\|, \quad \forall\, n \in \mathbb{N}_0,$$

where $\widehat{\beta} := \sqrt{K\widehat{L}e^{\widetilde{\omega}h_0}}$. These bounds are similar to those already encountered in Sections 2 and 3. Considering (a′), for example, the condition for the exponent $(\log 2)^{1/2}\widehat{\beta} - \widetilde{\omega}$ to be negative is qualitatively similar to conditions found in Theorem 2.1 for the continuous problem to be exponentially stable: we observe that $\widehat{L}$ depends linearly on the Lipschitz constant for $\lambda(\cdot)$, (see the proof of Theorem 4.2 below,) and $\widetilde{\omega}$ is closely related to the exponential decay rate for the frozen time continuous systems.

Quantitatively, as observed for the Backward Euler method in Remark 3.5, the exponent $(\log 2)^{1/2}\widehat{\beta}-\widetilde{\omega}$ also depends on the method. For (a′), $\widehat{L}$ also depends on the Lipschitz constant for the companion matrix, (see the proof of Theorem 4.2). As for Backward Euler, bounds (a′) and (b′) may be sharpened in the non–stiff régime, $h\|\lambda(\cdot)\|_\infty \ll 1$, by first bounding the continuous problem, and then bounding the method using an approximation argument.

In the remainder of this section we prove Theorem 4.2. To this end we quote some results from the literature and prove two lemmas.

Proposition 4.4. ***[6, Theorem 3], [10, Lemmas V.7.3, V.7.4]***
Consider $C(\cdot)$ as defined in (4.4)*. If $D \subseteq \mathbb{S}$ is closed in $\overline{\mathbb{C}}$, then*

$$\exists\, K > 0 \;\; \forall\, n \in \mathbb{N}_0 \;\; \forall\, z \in D \;:\quad \|C^n(z)\| \le K. \tag{4.10}$$

If $D \subset \operatorname{int}[\mathbb{S}]$ is closed in $\overline{\mathbb{C}}$, then

$$\exists\, K, \omega > 0 \;\; \forall\, n \in \mathbb{N}_0 \;\; \forall\, z \in D \;:\quad \|C^n(z)\| \le K e^{-\omega n}. \tag{4.11}$$

Remark 4.5. Without loss of generality it may be assumed that $K \ge 1$ in (4.10) and (4.11).

Lemma 4.6. ***(A uniformly bounded companion matrix is uniformly Lipschitz)*** *Consider C as defined in* (4.4)*. Suppose that D is closed in $\overline{\mathbb{C}}$, and that*

$$\exists\, K > 0 \;\; \forall\, z \in D \;:\quad \|C(z)\| \le K. \tag{4.12}$$

Then,

$$\exists\, L_C > 0 \;\; \forall\, z,\, w \in D \;:\quad \|C(z) - C(w)\| \le L_C |z - w|. \tag{4.13}$$

Proof: From (4.12), we deduce that

$$\exists\, K_0 > 0 \;\; \forall\, z \in D \; \forall\, j \in \{0,\, \ldots,\, q-1\} \;:\quad |c_j(z)| \le K_0. \tag{4.14}$$

Since the method is irreducible, there is no $z \in \mathbb{C}$ such that $\alpha_j = \beta_j z$ for all $j \in \{0,\, 1,\, \ldots,\, q\}$. Consequently, (4.14) implies that

$$\exists\, K_1 > 0 \;\; \forall\, z \in D \;:\quad \frac{1}{|\alpha_q - z\beta_q|} \le K_1.$$

Hence,

$$\exists\, K_2 > 0 \;\; \forall\, z \in D \; \forall\, j \in \{0,\, \ldots,\, q-1\} \;:\quad |c_j'(z)| = \frac{|\alpha_q \beta_j - \beta_q \alpha_j|}{|\alpha_q - z\beta_q|^2} \le K_2.$$

Thus, we obtain (4.13). □

Lemma 4.7. *Consider* (4.3) *and suppose that $\lambda : \mathbb{R}_{\ge 0} \to \mathbb{C}$ satisfies* (4.2)*, $D \subseteq \mathbb{S}$ is closed and* (4.9) *is satisfied. Then, in the notation* (4.4), (4.5)*,*

$$\exists\, \widehat{L} > 0 \;\; \forall\, h > 0 \; \forall\, (n,\, m) \in \mathcal{D} \;:\quad \|A_n - A_m\| \le \widehat{L} h^2 |n - m|. \tag{4.15}$$

Proof: By (4.9), (4.13) and (4.2),

$$\|A_n - A_m\| = \|C(h\lambda(nh)) - C(h\lambda(mh))\| \le L_C h |\lambda(nh) - \lambda(mh)| \le L_C L_\lambda h^2 |n - m|.$$ □

Proof of Theorem 4.2:
The hypotheses of Theorem 2.3 are implied, firstly by the assumption on D and Proposition 4.4, and secondly by the assumption on (4.9) together with Lemma 4.7. Noting that $L = \widehat{L} h^2$ and $\beta = \widehat{\beta} h$, we obtain (a) and (b) from parts (iii) and (iv) of Theorem 2.3. □

5 Proofs

5.1 Continuous time systems

Lemma 5.1. *For* $(K,\omega,L) \in \mathbb{R}_{\geq 1}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$ *and* $f:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$*, the following statements are equivalent:*

$$\text{(i)}\quad \forall A \in \mathcal{S}_{K,\omega,L}\ \forall\ \textit{slns. } u \textit{ of } (1.1)\ \forall\ (t,s)\in\mathcal{D}:\quad \|u(t)\| \leq f(t-s)\|u(s)\|,$$

$$\text{(ii)}\quad \forall A \in \mathcal{S}_{K,\omega,L}\ \forall\ \textit{slns. } u \textit{ of } (1.1)\ \forall\ t\geq 0:\quad \|u(t)\| \leq f(t)\ \|u(0)\|.$$

Proof: It suffices to prove "(ii) ⇒ (i)".
Let u of be a solution of $\dot u(t) = A(t)u(t)$ satisfying (ii). For arbitrary but fixed $s \geq 0$ we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}u(t+s) = A(t+s)u(t+s), \quad \text{and} \quad A(\cdot+s)\in\mathcal{S}_{K,\omega,L}.$$

Thus, (ii) yields

$$\forall\ t\geq 0:\ \|u(t+s)\| \leq f(t)\ \|u(0+s)\|.$$

Setting $\tau := t+s$ impies that

$$\forall\ \tau \geq s:\ \|u(\tau)\| \leq f(\tau-s)\ \|u(s)\|,$$

and (i) follows since s is arbitrary. □

In the context of exponential stability, the above lemma is of particular interest for

$$f(t) := Me^{-\eta t}, \qquad \text{for some} \quad (M,\eta)\in\mathbb{R}_{\geq 1}\times\mathbb{R}_{\geq 0}.$$

Lemma 5.2. ***(Primary integral inequality)***
Suppose that $A \in \mathcal{S}_{K,\omega,L}$ *for some* $(K,\omega,L)\in\mathbb{R}_{\geq 1}\times\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$.

Then every solution u *of* (1.1) *satisfies*

$$\forall\, t\geq 0 \quad \forall\, \rho \geq 0\ :\ \|u(t)\| \leq Ke^{-\omega t}\|u(0)\| + KL\int_0^t |s-\rho|e^{-\omega(t-s)}\|u(s)\|\ \mathrm{d}s. \tag{5.1}$$

Proof: Given $\rho \geq 0$, then every solution u of (1.1) satisfies

$$\dot u(t) = A(\rho)u(t) + [A(t)-A(\rho)]u(t), \qquad t\geq 0.$$

By the variation of constants formula,

$$u(t) = e^{A(\rho)t}u(0) + \int_0^t e^{A(\rho)(t-s)}[A(s)-A(\rho)]u(s)\ \mathrm{d}s, \qquad t\geq 0.$$

Taking norms, and invoking the properties of class $\mathcal{S}_{K,\omega,L}$,

$$\|u(t)\| \leq Ke^{-\omega t}\|u(0)\| + \int_0^t Ke^{-\omega(t-s)}L|s-\rho|\|u(s)\| \, \mathrm{d}s, \qquad t \geq 0,$$

we obtain (5.1). □

The number of independent parameters is reduced by the following scaling lemma.

Lemma 5.3. ***(Scaling and the function*** r***)***
Suppose that $A \in \mathcal{S}_{K,\omega,L}$ *for some* $(K,\omega,L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$*, and* $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *is a bounded piecewise continuous function. For a solution* u *of* (1.1) *such that* $u(0) \neq 0$*, the function*

$$U : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \qquad U(t) := e^{\omega t/\alpha}\frac{\|u(t/\alpha)\|}{K\|u(0)\|}, \qquad \text{where} \quad \alpha := \sqrt{KL}, \tag{5.2}$$

satisfies the integral inequality

$$U(t) \leq 1 + \int_0^t |s - r(t)|\, U(s) \, \mathrm{d}s, \qquad \forall\, t \geq 0. \tag{5.3}$$

Proof: For the given function r, we may take $\rho = r(t)/\alpha$ in (5.1) to obtain

$$e^{\omega t}\|u(t)\| \leq K\|u(0)\| + \alpha^2 \int_0^t |s - r(t)/\alpha| e^{\omega s}\|u(s)\| \, \mathrm{d}s, \qquad \forall\, t \geq 0.$$

Dividing by $\|u(0)\| > 0$ and scaling time by $1/\alpha$, we obtain (5.3). □

Lemma 5.4. ***(A comparison inequality)***
Suppose that for some $t_0 > 0$*,* $r : [0,\, t_0] \to \mathbb{R}_{\geq 0}$ *and* $w : [0,\, t_0] \to \mathbb{R}$ *are bounded piecewise continuous functions satisfying*

$$w(t) \leq \int_0^t |s - r(t)| w(s) \, \mathrm{d}s, \qquad \forall\, t \in [0,\, t_0]. \tag{5.4}$$

Then,

$$w(t) \leq 0, \qquad \forall\, t \in [0,\, t_0]. \tag{5.5}$$

Proof: From (5.4),

$$w(t) \leq \int_0^t |s - r(t)| w(s) \, \mathrm{d}s \leq \int_0^t |s - r(t)| w_+(s) \, \mathrm{d}s, \qquad \forall\, t \in [0,\, t_0],$$

where $w_+ := \max\{w,\, 0\}$. Since the RHS is non–negative, and

$$\forall\, t \in [0,\, t_0]\ \forall\, s \in [0,\, t] \ : \quad |s - r(t)| \leq \max\{s,\, r(t)\} \leq \max\{t,\, r(t)\} =: R(t),$$

we conclude that

$$w_+(t) \le R(t) \int_0^t w_+(s)\ \mathrm{d}s, \qquad \forall\, t \in [0,\, t_0]. \tag{5.6}$$

Let $\{0 = \tau_0,\, \tau_1,\, \ldots,\, \tau_n = t_0\}$ be a partition of $[0,\, t_0]$, such that the restrictions of R and w_+ to each subinterval $(\tau_m,\, \tau_{m+1})$ are continuous. Then, (5.6) implies that, for $I(t) := \int_0^t w_+(s)\ \mathrm{d}s$,

$$\forall\, m \in \{0,\, 1,\, \ldots,\, n-1\}\ \forall\, t \in (\tau_m,\, \tau_{m+1})\ :\quad \frac{\mathrm{d}}{\mathrm{d}t} I(t) \le R(t)\, I(t).$$

The fundamental theorem of calculus now implies that

$$\forall\, m \in \{0,\, 1,\, \ldots,\, n-1\}\ \ \forall\, t \in [\tau_m,\, \tau_{m+1}]\ :\quad I(t) \exp\left(-\int_{\tau_m}^t R(s)\ \mathrm{d}s\right) \le I(\tau_m). \tag{5.7}$$

Taking $t = \tau_{m+1}$ in (5.7), and observing that $I(0) = 0$, it follows that $I(\tau_m) \le 0$, for $m = 0,\, 1,\, \ldots,\, n$, by induction. Inequality (5.7) then implies that $I(t) \le 0$, $t \in [0,\, t_0]$. Inequality (5.5) now follows from (5.6). □

Lemma 5.5. ***(Integral supersolutions yield upper bounds)***
Suppose that $A \in \mathcal{S}_{K,\omega,L}$ for some $(K, \omega, L) \in \mathbb{R}_{\ge 1} \times \mathbb{R}_{\ge 0} \times \mathbb{R}_{>0}$. Suppose also that $v : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $r : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ are bounded piecewise continuous functions satisfying

$$v(t) \ge 1 + \int_0^t |s - r(t)| v(s)\ \mathrm{d}s, \qquad \forall\, t \in [0,\, t_0], \tag{5.8}$$

for some $t_0 > 0$. Then, the function U defined by (5.2) satisfies

$$U(t) \le v(t), \qquad \forall\, t \in [0,\, t_0]. \tag{5.9}$$

Proof: By (5.3),

$$U(t) \le 1 + \int_0^t |s - r(t)| U(s)\ \mathrm{d}s, \qquad \forall\, t \in [0,\, t_0]. \tag{5.10}$$

The function $w : [0,\, t_0] \to \mathbb{R}$, defined by $w(t) := U(t) - v(t)$, is bounded piecewise continuous. Furthermore, subtracting (5.10) from (5.3), we deduce that

$$w(t) \le \int_0^t |s - r(t)| w(s)\ \mathrm{d}s, \qquad \forall\, t \in [0,\, t_0].$$

Inequality (5.9) now follows from Lemma 5.4. □

Lemma 5.6. ***(Sufficient conditions for a supersolution)***
Suppose that, for some $t_0 > 0$, $v : [0, t_0] \to \mathbb{R}_{\geq 0}$ and $r : [0, t_0] \to \mathbb{R}_{\geq 0}$ are continuous functions, with r differentiable on $[0, t_0]$ except at a finite number of points. Suppose that $v(0) \geq 1$, and that the following inequalities are satisfied for almost all $t \in [0, t_0]$:

$$t \geq r(t), \tag{5.11}$$
$$\dot{r}(t) \geq 0, \tag{5.12}$$
$$v(t) \geq 2\dot{r}(t)v(r(t)). \tag{5.13}$$
$$\dot{v}(t) \geq (t - r(t))v(t), \tag{5.14}$$

Then, v and r satisfy (5.8).

Proof: Integrating (5.13), we obtain

$$\int_0^t v(s)\ \mathrm{d}s \geq 2\int_0^t \dot{r}(s)v(r(s))\ \mathrm{d}s = 2\int_0^{r(t)} v(s)\ \mathrm{d}s, \qquad \forall\, t \in [0, t_0].$$

Hence, applying (5.11),

$$0 \geq \int_0^{r(t)} v(s)\ \mathrm{d}s - \int_{r(t)}^t v(s)\ \mathrm{d}s = \int_0^t \mathrm{sign}[r(t) - s]v(s)\ \mathrm{d}s, \qquad \forall\, t \in [0, t_0]. \tag{5.15}$$

For all but a finite number of $t \in [0, t_0]$, elementary calculus implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t |r(t) - s|\, v(s)\ \mathrm{d}s = (t - r(t))v(t) + \dot{r}(t)\int_0^t \mathrm{sign}[r(t) - s]v(s)\ \mathrm{d}s.$$

Hence, applying (5.12) and (5.14), we conclude that

$$\dot{v}(t) \geq (t - r(t))v(t) + \dot{r}(t)\int_0^t \mathrm{sign}[r(t) - s]v(s)\ \mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t |r(t) - s|\, v(s)\ \mathrm{d}s,$$

for all but a finite set of $t \in [0, t_0]$. Integration and the inequality $v(0) \geq 1$ imply that v satisfies (5.8). □

Lemma 5.7. ***(Supersolution suitable for small t)***
The functions v_1, $r_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, defined by

$$v_1(t) := e^{t^2/4}, \qquad r_1(t) := t/2, \qquad \forall\, t \geq 0, \tag{5.16}$$

satisfy

$$v_1(t) \geq 1 + \int_0^t |s - r_1(t)|v_1(s)\ \mathrm{d}s, \qquad \forall\, t \geq 0. \tag{5.17}$$

Proof: Verifying the hypotheses of Lemma 5.6, we observe that the conditions (5.11), (5.12) are satisfied by $r_1(t)$ for all $t \geq 0$, whilst $v_1(0) = 1$,

$$\dot{v}_1(t) = \frac{t}{2} e^{t^2/4} = (t - r_1(t))v_1(t), \qquad v_1(t) \geq v_1(t/2) = 2\dot{r}_1(t)v_1(r_1(t)), \qquad t \geq 0.$$

Hence, (5.8) is satisfied for all $t_0 > 0$, and so we obtain (5.17) by Lemma 5.6. □

Lemma 5.8. ***(Long time supersolution)***
Let v_2, $r_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by

$$\left.\begin{array}{rcl} r_2(t) & = & \max\{t - c,\, 0\}, \\ v_2(t) & = & \exp(ct), \end{array}\right\} \qquad t \geq 0; \qquad c := \sqrt{\log 2}. \tag{5.18}$$

Then,

$$v_2(t) \geq 1 + \int_0^t |s - r_2(t)| v_2(s)\, \mathrm{d}s, \qquad \forall\, t \geq 0. \tag{5.19}$$

Proof: Verifying the hypotheses of Lemma 5.6, we see that conditions (5.11), (5.12) on r_2 are satisfied, except at $t = c$. Also, $v_2(0) = 1$ and

$$\dot{v}_2(t) = ce^{ct} = (t - (t - c))e^{ct} \geq (t - \max\{t - c,\, 0\})e^{ct} = (t - r_2(t))v_2(t), \quad \forall\, t \geq 0.$$

Since $\dot{r}_2(t) = 0$ for $t \in [0,\, c)$, (5.13) also holds for $t \in [0,\, c)$. Since $\dot{r}_2(t) = 1$ for $t > c$,

$$2\dot{r}_2(t)v_2(r_2(t)) = 2e^{c(t-c)} = e^{ct} = v_2(t), \qquad \forall\, t \geq c,$$

and so (5.13) holds for all $t \in \mathbb{R}_{\geq 0} \setminus \{c\}$. Applying Lemma 5.6, we deduce that (5.8) is satisfied for all $t_0 > 0$. Hence, (5.19) follows from Lemma 5.6. □

Lemma 5.9. ***(Short time bound implies a long time bound)***
Suppose that for $(K,\, \omega,\, \gamma) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$, $\widehat{u} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfies

$$\widehat{u}(t) \leq K \exp\left\{\gamma^2(t - s)^2/4 - \omega(t - s)\right\} \widehat{u}(s), \qquad \forall\, (t,\, s) \in \mathcal{D}. \tag{5.20}$$

Then,

$$\widehat{u}(t) \leq K \exp\left\{(\gamma\sqrt{\log K} - \omega)t\right\} \widehat{u}(0), \qquad \forall\, t \geq 0. \tag{5.21}$$

Proof: Set $t_1 := 2\gamma^{-1}\sqrt{\log K}$. Then, (5.20) implies that

$$\widehat{u}((n+1)t_1) \leq \exp\left(\left(\gamma\sqrt{\log K} - \omega\right) t_1\right) \widehat{u}(nt_1), \qquad \forall\, n \in \mathbb{N}_0.$$

Hence,

$$\widehat{u}(nt_1) \leq \exp\left(\left(\gamma\sqrt{\log K} - \omega\right) nt_1\right) \widehat{u}(0), \qquad \forall\, n \in \mathbb{N}_0. \tag{5.22}$$

For $n \in \mathbb{N}_0$ and $\tau \in [0,\, t_1)$, taking $t = nt_1 + \tau$ and $s = nt_1$ in (5.20) implies that

$$\widehat{u}(t) \leq K \exp\left(\gamma^2\tau^2/4 - \omega\tau\right) \widehat{u}(nt_1) \leq K \exp\left(\left(\gamma\sqrt{\log K}/2 - \omega\right)\tau\right) \widehat{u}(nt_1).$$

Combining with (5.22), we obtain (5.21). □

Proof of Theorem 2.1:
If $u(0) = 0$, then both (i) and (ii) are clear. Assume now that $u(0) \neq 0$.
Proof of (i): The functions $r_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $v_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined in Lemma 5.7 satisfy (5.8) and the other conditions of Lemma 5.5 for all $t_0 > 0$. Consequently, by (5.2),

$$e^{\omega t/\alpha} \frac{\|u(t/\alpha)\|}{K\|u(0)\|} \leq v_1(t) = e^{t^2/4}, \qquad \forall\, t \geq 0,$$

where $\alpha := \sqrt{KL}$. Taking the scaling $\widehat{t} = \alpha t$ and multiplying by $K\|u(0)\|e^{-\omega\widehat{t}}$, we obtain

$$\|u(t)\| \leq K \exp(\alpha^2 t^2/4 - \omega t)\|u(0)\|, \qquad \forall\, t \geq 0, \tag{5.23}$$

which is inequality (i) in the statement of Theorem 2.1 for $s = 0$. By Lemma 5.1, we deduce the general case for $(t,\, s) \in \mathcal{D}$.

Proof of (ii) when $K \in [1,\, 2]$: We observe that bound (i) implies that the hypotheses of Lemma 5.9 are satisfied for $\widehat{u} = \|u(\cdot)\|$ and $\gamma = \alpha$. Hence, we obtain bound (ii) for $s = 0$. The general case $(t,\, s) \in \mathcal{D}$ now follows from Lemma 5.1.

Proof of (ii) when $K > 2$: The functions $r_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $v_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by Lemma 5.8 satisfy (5.8) and the other conditions of Lemma 5.5 for all $t_0 > 0$. Consequently, by (5.2),

$$e^{\omega t/\alpha} \frac{\|u(t/\alpha)\|}{K\|u(0)\|} \leq v_2(t) = e^{ct}, \qquad \forall\, t \geq 0,$$

where $c := \sqrt{\log 2}$. Taking the scaling $\widehat{t} = \alpha t$ and multiplying by $K\|u(0)\|e^{-\omega\widehat{t}}$, we obtain

$$\|u(t)\| \leq K\|u_0\| \exp\left(\left(\sqrt{KL\log 2} - \omega\right) t\right), \qquad \forall\, t \geq 0.$$

Applying Lemma 5.1, we obtain statement (ii) of Theorem 2.1 for $K > 2$. □

5.2 Discrete time systems

Lemma 5.10. *For $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$ and $f : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ the following statements are equivalent:*

(i) $\forall A \in \Delta_{K,\omega,L}$ $\forall$ *slns.* u *of* (1.2) $\forall\, (n, m) \in \Delta :\quad \|u_n\| \leq f(n - m)\|u_m\|$,

(ii) $\forall A \in \Delta_{K,\omega,L}$ $\forall$ *slns.* u *of* (1.2) $\forall\, n \in \mathbb{N}_0 :\quad \|u_n\| \leq f(n)\|u_0\|$.

We omit the proof, since it is analogous to the proof of Lemma 5.1.

Lemma 5.11. ***(Primary summation inequality)***
Suppose that $A \in \Sigma_{K,\omega,L}$ for some $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Then every solution u of (1.2) *satisfies*

$$\|u_n\| \leq Ke^{-\omega n}\|u_0\| + \sum_{i=0}^{n-1} Ke^{-\omega(n-1-i)}L|i-\rho|\ \|u_i\|, \qquad \forall\, n \in \mathbb{N} \quad \forall\, \rho \geq 0. \tag{5.24}$$

Proof: Let $u : \mathbb{N}_0 \to \mathbb{C}^N$ be a solution of (1.2). Consider first the special case of $\rho \in \mathbb{N}_0$. Then,

$$u_{n+1} = A_\rho u_n + (A_n - A_\rho)u_n, \qquad \forall\, n \in \mathbb{N}_0.$$

By the discrete variation of constants formula,

$$u_n = A_\rho^n u_0 + \sum_{i=0}^{n-1} A_\rho^{n-1-i}(A_i - A_\rho)u_i \ \mathrm{d}s, \qquad \forall\, n \in \mathbb{N}_0.$$

Taking norms, and invoking the properties of class $\Delta_{K,\omega,L}$, (5.24) follows.

Suppose now that $\rho = \theta k + (1-\theta)(k+1)$ for $k \in \mathbb{N}_0$ and $\theta \in (0, 1)$. Since no integer i satisfies $k < i < k+1$,

$$\forall\, i \in \mathbb{N}_0 : \qquad \theta|i-k| + (1-\theta)|i-(k+1)| = |\theta(i-k) + (1-\theta)(i-(k+1))| = |i-\rho|.$$

Hence, (5.24) follows for ρ by taking a weighted average θ and $(1-\theta)$ of (5.24) for $\rho = k$ and $\rho = k+1$, respectively. Thus, we obtain (5.24). □

5.2.1 Connection to the continuous case

Lemma 5.12. ***(The summation inequality implies a scaled integral inequality)***
Suppose that $A \in \Sigma_{K,\omega,L}$ for some $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$, and that $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a bounded piecewise continuous function. For $u \colon \mathbb{N}_0 \to \mathbb{C}^N$ a solution of (1.2) *with $u_0 \neq 0$, the bounded piecewise continuous function*

$$\widehat{U} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto \widehat{U}(t) := \frac{e^{\omega n}\|u_n\|}{K\|u_0\|}, \qquad n = \lfloor t/\beta \rfloor, \quad t \geq 0, \quad \beta = \sqrt{KLe^{\omega}}, \tag{5.25}$$

satisfies the integral inequality (5.3),

$$\widehat{U}(t) \leq 1 + \int_0^t |s - r(t)|\widehat{U}(s)\ \mathrm{d}s, \qquad \forall\, t \geq 0.$$

Proof: We first observe that

$$\int_{m}^{m+1} |s-\rho|\,\mathrm{d}s = \begin{cases} |m-(\rho-1/2)|\,, & \rho \in \mathbb{R}_{\geq 0} \setminus (m,\, m+1), \\ \frac{(m-\rho)^2}{2} + \frac{(m+1-\rho)^2}{2} \geq |m-(\rho-1/2)|\,, & \rho \in (m,\, m+1). \end{cases} \tag{5.26}$$

We define the bounded piecewise continuous function,

$$\widetilde{U} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto \widetilde{U}(t) := \frac{e^{\omega n}\|u_n\|}{K\|u_0\|}, \qquad n = \lfloor t \rfloor\,, \quad t \geq 0.$$

Given $t \geq 1$, let $n = \lfloor t \rfloor$. Then, (5.26) with $\rho = r(\beta t)/\beta$ and Lemma 5.11 imply that

$$\begin{aligned} \widetilde{U}(t) := \frac{e^{\omega n}\|u_n\|}{K\|u_0\|} &\leq 1 + KLe^{\omega}\sum_{m=0}^{n-1} |m-(r(\beta t)/\beta - 1/2)|e^{\omega m}\frac{\|u_m\|}{K\|u_0\|} \\ &\leq 1 + \beta^2 \sum_{m=0}^{n-1}\int_m^{m+1} |s - r(\beta t)/\beta|\widetilde{U}(s)\,\mathrm{d}s \\ &\leq 1 + \beta^2 \int_0^t |s - r(\beta t)/\beta|\widetilde{U}(s)\,\mathrm{d}s, \quad \forall\, t \geq 1. \end{aligned}$$

Given $t \in [0,\, 1)$, $0 \leq \widetilde{U}(t) = 1/K \leq 1$.

Combining these two estimates, we deduce that

$$\widetilde{U}(t) \leq 1 + \beta^2 \int_0^t |s - r(\beta t)/\beta|\widetilde{U}(s)\,\mathrm{d}s, \quad \forall\, t \geq 0.$$

Noting that $\widehat{U}(t) = \widetilde{U}(t/\beta)$, $t \geq 0$, we deduce that $\widehat{U}$ satisfies (5.3). □

Lemma 5.13. ***(Comparison result for a continuous supersolution)***
Suppose that $A \in \Sigma_{K,\omega,L}$ for some $(K,\omega,L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$, and that $v : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are bounded piecewise continuous functions satisfying

$$v(t) \geq 1 + \int_0^t |s - r(t)|v(s)\,\mathrm{d}s, \qquad \forall\, t \geq 0.$$

Then, for $u\colon \mathbb{N}_0 \to \mathbb{C}^N$ a solution of (1.2),

$$\|u_n\| \leq Ke^{-\omega n}v(\beta n)\|u_0\|, \qquad n \in \mathbb{N}_0, \qquad \beta = \sqrt{KLe^{\omega}}. \tag{5.27}$$

Proof: If $u_0 = 0$, the result is trivial. Else, $u_0 \neq 0$, and by Lemma 5.12, the function $\widehat{U}$ defined by (5.25) satisfies (5.3). So, by Lemma 5.5,

$$\frac{e^{\omega n}\|u_n\|}{K\|u_0\|} = \widehat{U}(\beta n) \leq v(\beta n), \qquad n \in \mathbb{N}_0.$$

Hence, we obtain (5.27). □

5.2.2 A direct discrete approach

Lemma 5.14. ***(A scaled summation inequality)***
Suppose that $A \in \Sigma_{K,\omega,L}$ for some $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$, that u is a solution of (1.2) with $u_0 \neq 0$, and that $r : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Then, the function

$$\widehat{U} : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}, \qquad n \mapsto \widehat{U}_n := \frac{e^{\omega n}\|u_n\|}{K\|u_0\|}, \tag{5.28}$$

satisfies the inequality

$$\widehat{U}_n \leq 1 + \beta^2 \sum_{m=0}^{n-1} |m - r(n)|\widehat{U}_m, \qquad n \in \mathbb{N}, \qquad \beta = \sqrt{KLe^{\omega}}. \tag{5.29}$$

Proof: For $n \in \mathbb{N}$, we observe that we may choose $\rho = r(n)$ in Lemma 5.11 to obtain

$$\|u_n\| \leq Ke^{-\omega n}\|u_0\| + KLe^{\omega} \sum_{m=0}^{n-1} e^{-\omega(n-m)}|m - r(n)|\|u_m\|, \qquad n \in \mathbb{N}.$$

Dividing both sides by $Ke^{-\omega n}\|u_0\|$, we obtain (5.29). □

Remark 5.15. Unlike the continuous case, the factor β is not scaled out in (5.29).

Lemma 5.16. ***(Discrete supersolution and comparison result)***
Suppose that $A \in \Sigma_{K,\omega,L}$ for some $(K, \omega, L) \in \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}$. Suppose also that the sequence $(v_n)_{n\in\mathbb{N}_0}$ with real elements satisfies $v_0 \geq 1/K$ and

$$v_n \geq 1 + \beta^2 \sum_{m=0}^{n-1} |m - r(n)|v_m, \qquad n \in \mathbb{N}, \qquad \beta = \sqrt{KLe^{\omega}}, \tag{5.30}$$

for some function $r : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Then every solution u of (1.2) satisfies

$$\|u_n\| \leq Ke^{-\omega n}v_n\|u_0\|, \qquad n \in \mathbb{N}_0. \tag{5.31}$$

Proof: If $u_0 = 0$ the result is trivial. If $u_0 \neq 0$, consider the function $w : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$, $n \mapsto w_n := \widehat{U}_n - v_n$, where the sequence $(\widehat{U}_n)$ is as in (5.28). Subtracting (5.30) from (5.29), we obtain

$$w_n \leq \beta^2 \sum_{m=0}^{n-1} |m - r(n)|w_m, \qquad n \in \mathbb{N}.$$

Since $w_0 = \widehat{U}_0 - v_0 = 1/K - v_0 \leq 0$, the inequality $w_n \leq 0$ for $n \in \mathbb{N}_0$ follows by induction. Hence, we deduce (5.31). □

Lemma 5.17. ***(Discrete supersolution suitable for large n)***
Suppose that $\beta > 0$, and let the real sequence $(v_n)_{n\in\mathbb{N}_0}$ be defined by

$$v_n := \frac{1}{2}\left\{(1+\beta)^n + (1-\beta)^n\right\}, \qquad n \in \mathbb{N}_0. \tag{5.32}$$

Let $r : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ be defined by

$$r(n) := n - 1, \qquad n \in \mathbb{N}. \tag{5.33}$$

Then, $v_0 = 1$ and

$$v_n = 1 + \beta^2 \sum_{m=0}^{n-1} |m - r(n)| v_m, \qquad n \in \mathbb{N}. \tag{5.34}$$

Proof: Assume that $\gamma \in \mathbb{R} \setminus \{1\}$, and define $S_n(\gamma) := \sum_{m=0}^{n-1}(n-1-m)\gamma^m$, $n \in \mathbb{N}$. Then,

$$S_{n+1}(\gamma) - S_n(\gamma) = \sum_{m=0}^{n-1} \gamma^m = \frac{\gamma^n - 1}{\gamma - 1}, \qquad n \in \mathbb{N}.$$

Since $S_1(\gamma) = 0$,

$$S_n(\gamma) = \sum_{m=1}^{n-1}[S_{m+1}(\gamma) - S_m(\gamma)] = \sum_{m=1}^{n-1} \frac{\gamma^m - 1}{\gamma - 1} = \frac{\gamma^n - 1 - n(\gamma - 1)}{(\gamma - 1)^2}, \qquad n \in \mathbb{N}.$$

Thus, for (v_n) and r as in (5.32) and (5.33), respectively,

$$\begin{aligned}
1 + \beta^2 \sum_{m=0}^{n-1} |m - r(n)| v_m &= 1 + \beta^2 \sum_{m=0}^{n-1} (n - 1 - m) v_m \\
&= 1 + \frac{\beta^2}{2}\left\{S_n(1+\beta) + S_n(1-\beta)\right\} \\
&= 1 + \frac{1}{2}\left\{[(1+\beta)^n - 1 - n\beta] + [(1-\beta)^n - 1 + n\beta]\right\} = v_n, \qquad n \in \mathbb{N},
\end{aligned}$$

and $v_0 = 1$. □

Proof of Theorem 2.3: Bound (i) follows from the inequality,

$$\|u_{n+1}\| = \|A_n u_n\| \leq \|A_n\| \|u_n\| \leq K e^{-\omega n} \|u_n\|, \qquad n \in \mathbb{N}_0.$$

Bounds (ii) (iii) follow from the comparison result, Lemma 5.13, and the properties of the continuous supersolutions v_1 and v_2 shown in Lemmas 5.7 and 5.8, respectively.

Bound (iv) follows from the discrete comparison result, Lemma 5.16, and the properties of the supersolution (5.32) shown in Lemma 5.17. □

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