The Asymptotic Covering Density of Generalized Petersen Graphs

Remark on the paper "Minimum vertex covers in the generalized Petersen graphs $P(n, 2)$" by M. Behzad, P. Hatami, and E.S. Mahmoodian [1]

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Dedicated to Tomaž Pisanski on the occasion of his 60th birthday

Abstract

The covering density of a graph $G = (V, E)$ is $\delta(G) = \beta(G)/|V|$ where $\beta(G)$, the covering number, is the minimum number of vertices that represent all edges of G. The asymptotic covering density of the generalized Petersen graph is determined.

Keywords: Petersen graph (generalized), vertex cover, edge representation, covering number, covering density

Introduction

Let $G = (V, E)$ be a finite graph on $v(G) = |V|$ vertices. The *covering number* $\beta(G)$ is the minimum number of vertices that cover (i.e., represent) all edges of G, the *covering density* is $\delta(G) = \beta(G)/v(G)$.

The generalized Petersen graph $P(n, k)$ $(n = 3, 4, 5\ldots;\ k = 1, 2, \ldots, \lfloor n/2 \rfloor)$ has vertex set $V = \{u_i, v_i \mid i = 1, 2, \ldots, n\}$ and edge set $E = \{u_i u_{i+1}, u_i v_i, v_i v_{i+k} \mid i = 1, 2, \ldots, n\,;$ subscripts to be reduced mod $n\}$ (Figure 1); Petersen's classic graph is $P(5, 2)$.

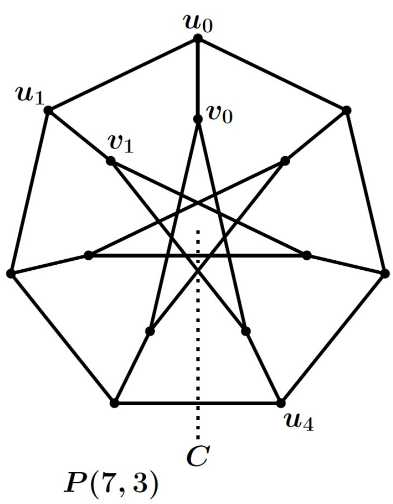
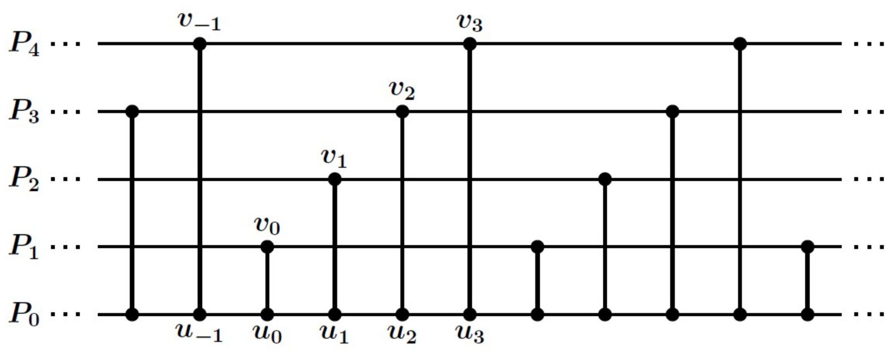


Figure 1

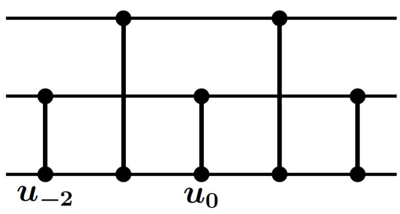
In [1] it is shown that $\beta(P(n, 2)) = n + \lceil n/5 \rceil$ and conjectured that

$$\beta(P(n, k)) \le n + \lceil n/5 \rceil, \text{i.e.}, \delta(P(n, k)) \le \frac{1}{2}(1 + \lceil n/5 \rceil/n), k = 1, 2, \ldots, \lfloor n/2 \rfloor.$$



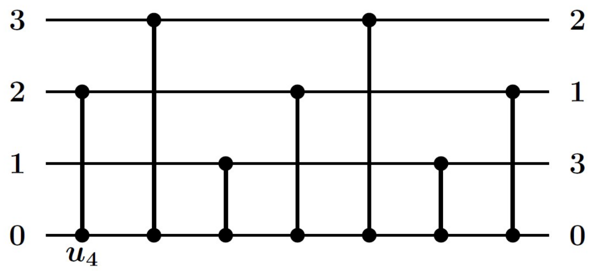
$P(\infty, 4)$

Figure 2



$P_{-2}(5, 2)$

Figure 3



$P_4(7, 3) \to P(7, 3)$

Figure 4

Here we shall show that, from an asymptotic point of view ($n \to \infty$), we can do better.

The Result

Let $\delta(k)$ denote the asymptotic density of the sequence $\{P(n, k) \mid n = 3, 4, 5, \ldots\}$ for $n \longrightarrow \infty$.

Theorem.

Claim A : $\delta(k) = \frac{1}{2}$ *if k is odd.*

Claim B : $\delta(k) = \frac{1}{2}(1 + \frac{1}{2k})$ *if k is even, $k > 2$.*

Claim C : $\delta(2) = \frac{1}{2}(1 + \frac{1}{5})$.

Proof. Let the *Petersen strip* $P(\infty, k)$ consist of $k + 1$ infinite paths

$$\begin{aligned} P_0 &= \{\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots\}, \\ P_{\varkappa+1} &= \{\ldots, v_{\varkappa-2k}, v_{\varkappa-k}, v_{\varkappa}, v_{\varkappa+k}, v_{\varkappa+2k}, \ldots\} \quad (\varkappa = 0, 1, \ldots, k-1) \end{aligned}$$

and the edges (*spokes*) $u_i v_i$, $i = 0, \pm 1, \pm 2, \ldots$ (Figure 2). An (m, n)-*section* $P_m(n, k)$ of $P(\infty, k)$ is the part of $P(\infty, k)$ induced by the vertex set

$$\{u_m, v_m; u_{m+1}, v_{m+1}; \ldots; u_{m+n-1}, v_{m+n-1}\}$$

with dangling half-edges at both ends; n is called the *length* of the section (Figure 3). Such a section is obtained from $P(n, k)$ by performing a radial cut C between two consecutive spokes (Figure 1). Conversely, graph $P(n, k)$ is retrieved from $P_m(n, k)$ by gluing together pairs of half-edges according to a suitable identification scheme (Figure 4). A cover of an (m, n)-section is assumed to represent the edges, not the half-edges.

The covering density of $P(\infty, k)$ is $\delta(P(\infty, k)) = \lim_{n\to\infty} \delta(P_{-\lfloor n/2\rfloor}(n, k))$; it is easy to see that this limit exists. Clearly,

$$\delta(k) = \lim_{n\to\infty} \delta(P(n, k)) = \lim_{n\to\infty} \delta(P_{-\lfloor n/2\rfloor}(n, k)) = \delta(P(\infty, k)).$$

To determine this limit we distinguish three cases.

Case a: k is odd.

Note that, in this case, $P(\infty, k)$ is bipartite.

Case b: k is even and greater than two.

Case c: $k = 2$.

As at least one vertex of every spoke must be covered, $\delta(k) \geq \frac{1}{2}$ for all k.

Case a, k is odd. Colour all vertices u_{2i}, v_{2i+1} of $P(\infty, k)$ white and the rest black. $P(\infty, k)$ being bipartite with bipartition white/black, the white vertices form a cover of density $\frac{1}{2}$ which is best possible.

This proves Claim A.

Cases b and c, k is even. $P(\infty, k)$ being cubic and containing (many) odd circuits (of length $k + 3$, e.g.), it is easy to show that any cover creates some spokes both ends of which are covered.

Consider an arbitrary cover and assume w.l.o.g. that spoke u_0v_0 is double-covered. Let r be the least positive integer such that (at least) one of the spokes $u_1v_1, u_2v_2, \ldots, u_rv_r$ is double-covered . We shall determine the number $R = R(k) = \max r$ where the maximum is taken over all covers of $P(\infty, k)$: then

$$\delta(k) \geq \frac{R+1}{2R} = \frac{1}{2}(1 + \frac{1}{R}).$$

Consider any cover and colour the vertices that are covered white and the rest black (such a colouring will be called *feasible*). By hypothesis, u_0 and v_0 are white. Assume that in the sequence $\{v_1, v_2, \ldots, v_{k-1}\}$ two consecutive vertices v_i, v_{i+1} have the same colour c. If c is white then u_i, u_{i+1} cannot both be black (because this would leave the edge u_iu_{i+1} uncovered), thus $r \leq i + 1 \leq k - 1$. If c is black then both v_{i+k}, v_{i+1+k} are white, thus, by the same argument, $r \leq i + k + 1 \leq 2k - 1$.

Next assume that, in the sequence $\{v_1, v_2, \ldots, v_{k-1}\}$, the colours alternate.

We distinguish two subcases.

Subcase 1: v_1 is black; then v_{k-1} is black and v_{k+1}, as a neighbour of v_1, is white. If v_k is white, too, then $r \leq k + 1$. Let v_k be black: then both v_{2k-1} and v_{2k} are white, thus $r \leq 2k$.

Note that if $r = 2k$ then, under the hypothesis that v_1 is black, r is maximum and the colouring of $P_0(2k, k)$ is unique (see Figure 6).

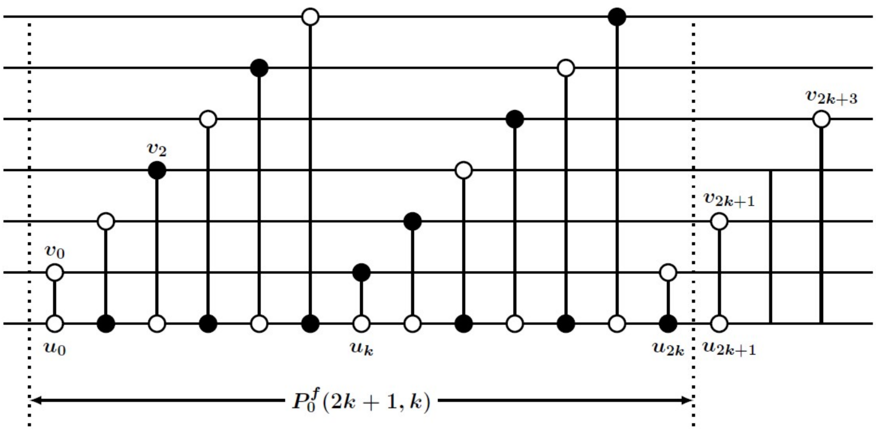


Figure 5: The fact that v_{2k+3} must be white prevents the length of the fair section following $P_0^f(2k+1,k)$ from exceeding $2k$.

Subcase 2: v_1 is white; then v_2 is black and v_{k-1} is white. If u_{k-1} is white then $r \leq k-1$. If u_{k-1} is black then u_k must be white. If v_k is white then $r \leq k$. Let v_k be black. v_{k+2}, as a neighbour of v_2, must be white. If v_{k+1} is white, too, then $r \leq k+2$. If v_{k+1} is black then both v_{2k} and v_{2k+1} are white, and we conclude that $r \leq 2k+1$.
Note that if $r = 2k+1$ then r is maximum and the colouring of $P_0(2k+1,k)$ is unique.

$$\text{In any case, } r \leq 2k+1, \text{ thus } R \leq 2k+1 \text{ implying } \quad \delta(k) \geq \frac{1}{2}(1+\frac{1}{2k+1}).$$

Case b: k is even, $k > 2$. Every feasible colouring induces a partition of $P(\infty,k)$ into sections $P_m^f(n,k)$ (which we shall call *fair* indicated by the superscript f) of, in general, variable length n such that both the end vertices of the first spoke, u_m and v_m, are white and no other spoke in $P_m^f(n,k)$ has this property, and length n is maximum under this condition, i.e., both u_{m+n+1}, v_{m+n+1} are white. It is easy to check that a fair section $P_0^f(r,k)$ of length $r = 2k+1$ (which by Subcase 2 is maximum) is uniquely realized by the following colouring: vertices

$$u_0, v_0; v_1, u_2, v_3, u_4, \ldots, v_{k-1}, u_k; u_{k+1}, v_{k+2}, u_{k+3}, v_{k+4}, \ldots, u_{2k-1}, v_{2k}$$

are white, the remaining vertices are black (Figure 5). If we try to extend this colouring beyond the last spoke $u_{2k}v_{2k}$ we observe that, by the unique colouring of $P_0^f(2k+1,k)$, the initial conditions for creating a fair section $P_{2k+1}^f(\cdot,k)$ of length $2k+1$ are not satisfied (Figure 5), the longest possible fair section following $P_0^f(2k+1,k)$ is $P_{2k+1}^f(2k,k)$ with a colouring that is again unique and reproduces the same initial conditions for the next fair section (compare Figure 6). The analogue is true for any extension of the colouring of $P_0^f(2k+1,k)$ to the left hand side: we conclude that between any two fair sections of length $2k+1$ there lies a fair section of length less than $2k$ implying $\delta(k) \geq \frac{1}{2}(1+\frac{1}{2k})$. There are feasible colourings of $P(\infty,k)$ with covering density $\frac{1}{2}(1+\frac{1}{2k})$, e.g., the following: to obtain a fair section $P_0^f(2k,k)$ colour vertices

$$u_0, v_0; u_1, v_2, u_3, v_4, \ldots, v_{k-2}, u_{k-1}; u_k, v_{k+1}, u_{k+2}, v_{k+3}, \ldots, u_{2k-2}, v_{2k-1}$$

white and the remaining vertices black (Figure 6). This colouring can by isomorphic versions of $P_0^f(2k,k)$ be repeated arbitrarily often to both sides of $P_0^f(2k,k)$.

This proves Claim B.

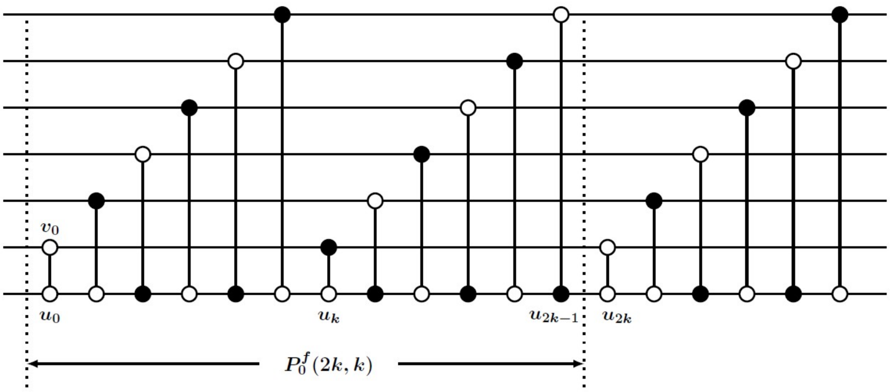


Figure 6

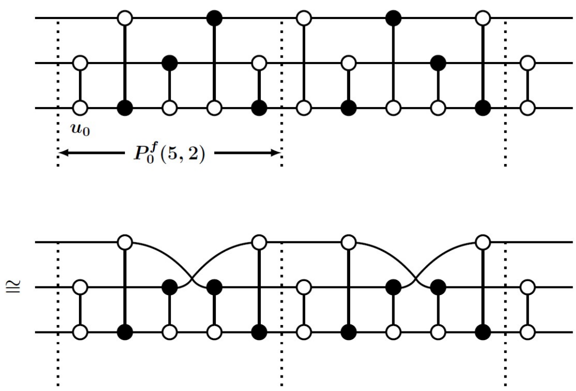


Figure 7

Case c: $k = 2$**.** Here the situation is somewhat different: the colouring of a fair section of (maximum) length $2k + 1 = 5$ *can* be extended to both sides of this section such that all fair sections of $P(\infty, 2)$ have length 5, see Figure 7. We conclude that, in accordance with Theorem 1 of [1], $\delta(2) = \frac{1}{2}(1 + \frac{1}{5})$.

This proves Claim C. $\square$

Reference

[1] M. Behzad, P. Hatami, and E.S. Mahmoodian, *Minimum vertex covers in the generalized Petersen graphs $P(n, 2)$.* Bulletin of the ICA **56** (2009), 98–102.