The Asymptotic Covering Density of Generalized Petersen Graphs

Remark on the paper “Minimum vertex covers in the generalized Petersen graphs \( P(n, 2) \)” by M. Behzad, P. Hatami, and E.S. Mahmoodian [1]

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Dedicated to Tomaž Pisanski on the occasion of his 60th birthday

Abstract
The covering density of a graph \( G = (V, E) \) is \( \delta(G) = \beta(G)/|V| \) where \( \beta(G) \), the covering number, is the minimum number of vertices that represent all edges of \( G \). The asymptotic covering density of the generalized Petersen graph is determined.

Keywords: Petersen graph (generalized), vertex cover, edge representation, covering number, covering density

Introduction
Let \( G = (V, E) \) be a finite graph on \( v(G) = |V| \) vertices. The covering number \( \beta(G) \) is the minimum number of vertices that cover (i.e., represent) all edges of \( G \), the covering density is \( \delta(G) = \beta(G)/v(G) \).

The generalized Petersen graph \( P(n, k) \) \( (n = 3, 4, 5 \ldots; \ k = 1, 2, \ldots, \lfloor n/2 \rfloor) \) has vertex set \( V = \{u_i, v_i \mid i = 1, 2, \ldots, n\} \) and edge set \( E = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid i = 1, 2, \ldots, n; \ \text{subscripts to be reduced mod } n\} \) (Figure 1); Petersen’s classic graph is \( P(5, 2) \).

In [1] it is shown that \( \beta(P(n, 2)) = n + \lfloor n/5 \rfloor \) and conjectured that
\[
\beta(P(n, k)) \leq n + \lfloor n/5 \rfloor, \text{i.e., } \delta(P(n, k)) \leq \frac{1}{2}(1 + \lfloor n/5 \rfloor)/n, \ k = 1, 2, \ldots, \lfloor n/2 \rfloor.
\]
Here we shall show that, from an asymptotic point of view \((n \to \infty)\), we can do better.

**The Result**

Let \(\delta(k)\) denote the asymptotic density of the sequence \(\{P(n,k) \mid n = 3, 4, 5, \ldots\}\) for \(n \to \infty\).

**Theorem.**

\(\text{Claim A: } \delta(k) = \frac{1}{2} \text{ if } k \text{ is odd.}\)

\(\text{Claim B: } \delta(k) = \frac{1}{2}(1 + \frac{1}{2k}) \text{ if } k \text{ is even, } k > 2.\)

\(\text{Claim C: } \delta(2) = \frac{1}{2}(1 + \frac{1}{5}).\)

**Proof.** Let the *Petersen strip* \(P(\infty,k)\) consist of \(k + 1\) infinite paths

\[
P_0 = \{\ldots, u_{-2}, u_{-1}, u_0, u_1, u_2, \ldots\},
\]

\[
P_{\kappa+1} = \{\ldots, v_{\kappa-2k}, v_{\kappa-k}, v_{\kappa}, v_{\kappa+k}, v_{\kappa+2k}, \ldots\} \quad (\kappa = 0, 1, \ldots, k - 1)
\]

and the edges (spokes) \(u_iv_i, \ i = 0, \pm 1, \pm 2, \ldots\) (Figure 2). An \((m,n)\)-section \(P_m(n,k)\) of \(P(\infty,k)\) is the part of \(P(\infty,k)\) induced by the vertex set

\[
\{u_m, v_m; u_{m+1}, v_{m+1}; \ldots; u_{m+n-1}, v_{m+n-1}\}
\]

with dangling half-edges at both ends; \(n\) is called the length of the section (Figure 3). Such a section is obtained from \(P(n,k)\) by performing a radial cut \(C\) between two consecutive spokes (Figure 1). Conversely, graph \(P(n,k)\) is retrieved from \(P_m(n,k)\) by gluing together
pairs of half-edges according to a suitable identification scheme (Figure 4). A cover of an 
\((m, n)\)-section is assumed to represent the edges, not the half-edges.

The covering density of \(P(\infty, k)\) is \(\delta(P(\infty, k)) = \lim_{n \to \infty} \delta(P_{-\lfloor n/2 \rfloor}(n, k))\); it is easy to see 
that this limit exists. Clearly,

\[
\delta(k) = \lim_{n \to \infty} \delta(P(n, k)) = \lim_{n \to \infty} \delta(P_{-\lfloor n/2 \rfloor}(n, k)) = \delta(P(\infty, k)).
\]

To determine this limit we distinguish three cases.

**Case a:** \(k\) is odd.

Note that, in this case, \(P(\infty, k)\) is bipartite.

**Case b:** \(k\) is even and greater than two.

**Case c:** \(k = 2\).

As at least one vertex of every spoke must be covered, \(\delta(k) \geq \frac{1}{2}\) for all \(k\).

**Cases b and c, \(k\) is even.** \(P(\infty, k)\) being cubic and containing (many) odd circuits (of 
length \(k + 3\), e.g.), it is easy to show that any cover creates some spokes both ends of 
which are covered.

Consider an arbitrary cover and assume w.l.o.g. that spoke \(u_0v_0\) is double-covered. Let 
\(r\) be the least positive integer such that (at least) one of the spokes \(u_1v_1, u_2v_2, \ldots, u_rv_r\) is 
double-covered. We shall determine the number 
\(R = R(k) = \max r\) where the maximum 
is taken over all covers of \(P(\infty, k)\): then 

\[
\delta(k) \geq \frac{R + 1}{2R} = \frac{1}{2}(1 + \frac{1}{R}).
\]

Consider any cover and colour the vertices that are covered white and the rest black (such a 
colouring will be called *feasible*). By hypothesis, \(u_0\) and \(v_0\) are white. Assume that in 
the sequence \(\{v_1, v_2, \ldots, v_{k-1}\}\) two consecutive vertices \(v_i, v_{i+1}\) have the same colour \(c\). If 
c is white then \(u_i, u_{i+1}\) cannot both be black (because this would leave the edge \(u_iu_{i+1}\) 
uncovered), thus \(r \leq i + 1 \leq k - 1\). If \(c\) is black then both \(v_{i+k}, v_{i+1+k}\) are white, thus, by 
the same argument, \(r \leq i + k + 1 \leq 2k - 1\).

Next assume that, in the sequence \(\{v_1, v_2, \ldots, v_{k-1}\}\), the colours alternate.

We distinguish two subcases.

**Subcase 1:** \(v_1\) is black; then \(v_{k-1}\) is black and \(v_{k+1}\), as a neighbour of \(v_1\), is white. If 
\(v_k\) is white, too, then \(r \leq k + 1\). Let \(v_k\) be black: then both \(v_{2k-1}\) and \(v_{2k}\) are white, thus 
\(r \leq 2k\).

Note that if \(r = 2k\) then, under the hypothesis that \(v_1\) is black, \(r\) is maximum and the 
colouring of \(P_0(2k, k)\) is unique (see Figure 6).
versions of \( P \) white and the remaining vertices black (Figure 6). This colouring can be isomorphic to obtain a fair section \( P \). There are feasible colourings of \( P \) that are again unique and reproduce the same initial conditions for the next section (compare Figure 5), the longest possible fair section following \( P \) the initial conditions for creating a fair section \( P \) beyond the last spoke \( v \) must be white. If \( v \) is white, too, then \( r \leq k + 2 \). If \( v \) is black then both \( v_{2k} \) and \( v_{2k+1} \) are white, and we conclude that \( r \leq 2k + 1 \). Note that if \( r = 2k + 1 \) then \( r \) is maximum and the colouring of \( P_0(2k + 1, k) \) is unique.

In any case, \( r \leq 2k + 1 \), thus \( R \leq 2k + 1 \) implying \( \delta(k) \geq \frac{1}{2}(1 + \frac{1}{2k+1}) \).

**Case b:** \( k \) is even, \( k > 2 \). Every feasible colouring induces a partition of \( P(\infty, k) \) into sections \( P_m^f(n, k) \) (which we shall call *fair* indicated by the superscript \( f \) ) of, in general, variable length \( n \) such that both the end vertices of the first spoke, \( u_m \) and \( v_m \), are white and no other spoke in \( P_m^f(n, k) \) has this property, and length \( n \) is maximum under this condition, i.e., both \( u_{m+n+1}, v_{m+n+1} \) are white. It is easy to check that a fair section \( P_0^f(r, k) \) of length \( r = 2k + 1 \) (which by Subcase 2 is maximum) is uniquely realized by the following colouring: vertices

\[
 u_0, v_0; u_1, v_1, u_2, v_2, \ldots, u_{k-1}, v_{k-1}, u_k, v_k, u_{k+2}, v_{k+2}, u_{k+3}, v_{k+3}, \ldots, u_{2k-1}, v_{2k}
\]

are white, the remaining vertices are black (Figure 5). If we try to extend this colouring beyond the last spoke \( u_{2k}v_{2k} \) we observe that, by the unique colouring of \( P_0^f(2k + 1, k) \), the initial conditions for creating a fair section \( P_{2k+1}^f(\cdot, k) \) of length \( 2k + 1 \) are not satisfied (Figure 5), the longest possible fair section following \( P_0^f(2k + 1, k) \) is \( P_{2k+1}^f(2k, k) \) with a colouring that is again unique and reproduces the same initial conditions for the next fair section (compare Figure 6). The analogue is true for any extension of the colouring of \( P_0^f(2k + 1, k) \) to the left hand side: we conclude that between any two fair sections of length \( 2k + 1 \) there lies a fair section of length less than \( 2k \) implying \( \delta(k) \geq \frac{1}{2}(1 + \frac{1}{2k}) \). There are feasible colourings of \( P(\infty, k) \) with covering density \( \frac{1}{2}(1 + \frac{1}{2k}) \), e.g., the following: to obtain a fair section \( P_0^f(2k, k) \) colour vertices

\[
 u_0, v_0; u_1, v_1, u_2, v_2, \ldots, u_{k-2}, v_{k-2}, u_k, v_k, u_{k+1}, u_{k+2}, v_{k+3}, \ldots, u_{2k-2}, v_{2k-1}
\]

white and the remaining vertices black (Figure 6). This colouring can by isomorphic versions of \( P_0^f(2k, k) \) be repeated arbitrarily often to both sides of \( P_0^f(2k, k) \).

This proves Claim B.
Case c: $k = 2$. Here the situation is somewhat different: the colouring of a fair section of (maximum) length $2k + 1 = 5$ can be extended to both sides of this section such that all fair sections of $P(\infty, 2)$ have length 5, see Figure 7. We conclude that, in accordance with Theorem 1 of [1], $\delta(2) = \frac{1}{2}(1 + \frac{1}{5})$.

This proves Claim C. □

Reference