

Minimum Degree and Density of Binary Sequences

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Abstract

For $d, k \in \mathbb{N}$ with $k \leq 2d$, let $g(d, k)$ denote the infimum density of binary sequences $(x_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ which satisfy the minimum degree condition $\sum_{j=1}^d (x_{i+j} + x_{i-j}) \geq k$ for all $i \in \mathbb{Z}$ with $x_i = 1$. We reduce the problem to determine $g(d, k)$ to a combinatorial problem related to the generalized k -girth of a graph G which is defined as the minimum order of an induced subgraph of G of minimum degree at least k . Extending results of Kézdy and Markert, and of Bermond and Peyrat, we present a minimum mean cycle formulation which allows to determine $g(d, k)$ for small values of d and k . For odd values of k with $d + 1 \leq k \leq 2d$, we conjecture $g(d, k) = \frac{k^2 - 1}{2(dk - 1)}$ and show that this holds for $k \geq 2d - 3$.

Keywords: Minimum degree; density; binary sequence; girth; generalized girth; power of cycle

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1 Introduction

Let $d \in \mathbb{N}$ be fixed. For a two-way infinite binary sequence

$$X = (x_i)_{i \in \mathbb{Z}} = (\dots, x_{-1}, x_0, x_1, \dots) \in \{0, 1\}^{\mathbb{Z}},$$

we define the *minimum degree* $\delta(X)$ of X as

$$\delta(X) = \min \left\{ \sum_{j=1}^d (x_{i+j} + x_{i-j}) \mid i \in \mathbb{Z}, x_i = 1 \right\}.$$

If $x_i = 0$ for all $i \in \mathbb{Z}$, then we write $X = 0$ and call X *trivial*.

For $k \in \mathbb{N}$ with $k \leq 2d$, we consider the infimum density $g(d, k)$ of non-trivial binary sequences subject to a minimum degree condition defined as

$$g(d, k) = \inf \left\{ \liminf_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{i=-n}^n x_i \mid X = (x_i)_{i \in \mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}, X \neq 0, \delta(X) \geq k \right\}.$$

Considering the binary sequence $(x_i)_{i \in \mathbb{Z}}$ with $x_i = 1$ if and only if $1 \leq i \leq k+1$, it follows that $g(d, k) = 0$ for $k \leq d$. While for such values of k , the calculation of $g(d, k)$ is trivial, for $k \geq d+1$, the calculation of $g(d, k)$ leads to an interesting combinatorial problem.

We prove as our first result that we can restrict ourselves to periodic sequences whose period is bounded in terms of d . Note that $g(d, 2d) = 1$ for all $d \in \mathbb{N}$.

Theorem 1 *Let $d, k \in \mathbb{N}$ with $d \geq 2$ and $d+1 \leq k \leq 2d$. There is a non-trivial periodic binary sequence $X = (x_i)_{i \in \mathbb{Z}}$ whose period p is at most $d2^{2d+1}$ such that $\delta(X) \geq k$ and*

$$g(d, k) = \frac{1}{p} \sum_{j=1}^p x_j.$$

Proof: Let $0 < \epsilon < \frac{1}{3}$. Let $X = (x_i)_{i \in \mathbb{Z}}$ be a non-trivial binary sequence such that $\delta(X) \geq k$ and $\liminf_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n x_j \leq g(d, k) + \epsilon$. Since $x_i = 1$ for infinitely many $i \in \mathbb{Z}$, we have

$$\liminf_{n \rightarrow \infty} \frac{1}{2n+1} \sum_{j=-n}^n x_j \geq \frac{1}{2} \left(\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j + \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_{-j} \right).$$

By symmetry, we may assume that $\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n x_j \leq g(d, k) + \epsilon$.

Note that $\delta(X) \geq k \geq d+1$ implies that X does not contain d consecutive 0-entries.

We call some $n \in \mathbb{N}$ *good* if

- $\frac{1}{n} \sum_{j=1}^n x_j \leq g(d, k) + 2\epsilon$ and

- 27 • $(x_{j_1}, x_{j_1+1}, \dots, x_{j_1+2d-1}) = (x_{j_2}, x_{j_2+1}, \dots, x_{j_2+2d-1})$ for some $1 \leq j_1 \leq \lfloor \epsilon n \rfloor - 2d + 1$
 28 and $n - \lfloor \epsilon n \rfloor + 1 \leq j_2 \leq n - 2d + 1$.

29 **Claim** *There are infinitely many good $n \in \mathbb{N}$.*

30 *Proof of the claim:* Let $n_1, n_2, \dots, n_{2^{2d}} \in \mathbb{N}$ be such that $\frac{1}{n_i} \sum_{j=1}^{n_i} x_j \leq g(d, k) + 2\epsilon$ for $1 \leq i \leq$
 31 2^{2d} , $2d \leq \lfloor \epsilon n_1 \rfloor$, and $n_i \leq \lfloor \epsilon n_{i+1} \rfloor$ for $1 \leq i \leq 2^{2d} - 1$. Clearly, it suffices to prove that one
 32 of the n_i 's is good. For contradiction, we assume that all n_i 's are bad. Inductively, this
 33 implies that for $1 \leq i \leq 2^{2d}$, the sequence $(x_j)_{j \in \{1, 2, \dots, \lfloor \epsilon n_i \rfloor\}}$ contains i distinct subsequences
 34 of the form $(x_j, x_{j+1}, \dots, x_{j+2d-1})$ with $1 \leq j \leq \lfloor \epsilon n_i \rfloor - 2d + 1$ which are different from all
 35 subsequences of the form $(x_j, x_{j+1}, \dots, x_{j+2d-1})$ with $n_i - \lfloor \epsilon n_i \rfloor + 1 \leq j \leq n_i - 2d + 1$. Since
 36 there are exactly 2^{2d} distinct binary sequences of length d , this is impossible for $i = 2^{2d}$,
 37 which completes the proof of the claim. \square

38 Let $n \in \mathbb{N}$ be good. Let $(x_{j_1}, x_{j_1+1}, \dots, x_{j_1+2d-1}) = (x_{j_2}, x_{j_2+1}, \dots, x_{j_2+2d-1})$ for $1 \leq j_1 \leq$
 39 $\lfloor \epsilon n \rfloor - 2d + 1$ and $n - \lfloor \epsilon n \rfloor + 1 \leq j_2 \leq n - 2d + 1$.

The non-trivial periodic binary sequence $X' = (x'_i)_{i \in \mathbb{Z}}$ with $x'_i = x_i$ for $j_1 + 2d \leq i \leq$
 $j_2 + 2d - 1$ of period $p' = j_2 - j_1$ satisfies $\delta(X') \geq k$ and

$$\frac{1}{p'} \sum_{j=1}^{p'} x'_j \leq \frac{1}{1-2\epsilon} (g(d, k) + 2\epsilon).$$

40 If $p' > 2d2^{2d}$, then the pigeonhole principle implies the existence of indices $1 \leq j_1, j_2 \leq p'$
 41 with $(x'_{j_1}, x'_{j_1+1}, \dots, x'_{j_1+2d-1}) = (x'_{j_2}, x'_{j_2+1}, \dots, x'_{j_2+2d-1})$ and $j_1 + 2d \leq j_2 \leq j_1 + p' - 2d$.
 42 Let $X'' = (x''_i)_{i \in \mathbb{Z}}$ be the non-trivial p'' -periodic binary sequence with $x''_i = x'_i$ for $j_1 + 2d \leq$
 43 $i \leq j_2 + 2d - 1$ with $p'' = j_2 - j_1$. Similarly, let $X''' = (x'''_i)_{i \in \mathbb{Z}}$ be the non-trivial p''' -periodic
 44 binary sequence with $x'''_i = x'_i$ for $j_2 + 2d \leq i \leq j_1 + p' + 2d - 1$ with $p''' = j_1 + p' - j_2$. Clearly,
 45 $p'', p''' < p'$, $\delta(X''), \delta(X''') \geq k$, and either $\frac{1}{p''} \sum_{j=1}^{p''} x''_j \leq \frac{1}{1-2\epsilon} (g(d, k) + 2\epsilon)$ or $\frac{1}{p'''} \sum_{j=1}^{p'''} x'''_j \leq$
 46 $\frac{1}{1-2\epsilon} (g(d, k) + 2\epsilon)$. This implies that for every $0 < \epsilon < \frac{1}{3}$, there is a non-trivial periodic
 47 binary sequence $X = (x_i)_{i \in \mathbb{Z}}$ whose period p is at most $d2^{2d+1}$ such that $\delta(X) \geq k$ and
 48 $\frac{1}{p} \sum_{j=1}^p x_j \leq \frac{1}{1-2\epsilon} (g(d, k) + 2\epsilon)$. Since for every such sequence X , the quantity $\frac{1}{p} \sum_{j=1}^p x_j$ is a
 49 rational number whose denominator is bounded by $d2^{2d+1}$, the desired result follows. \square

For the further investigations, it is more convenient to consider a cyclic binary sequence

$$X = (x_0, x_1, \dots, x_{p-1}) = x_0 x_1 \dots x_{p-1}$$

of length p instead of a periodic binary sequence $(x_i)_{i \in \mathbb{Z}}$ with period p . As usual, we will
 consider indices modulo the length p . We say that an entry x_i of X *sees* another entry x_j of
 X if the cyclic distance of x_i and x_j is at least 1 and at most d . To avoid double-counting,

we define *the minimum degree* $\delta(X)$ of X as the minimum number of distinct 1-entries of X seen by a 1-entry of X . Furthermore, we define *the density* $\mu(X)$ of X as

$$\mu(X) = \frac{1}{p} \sum_{j=1}^p x_j.$$

50 With these notions, Theorem 1 implies that $g(d, k)$ equals the minimum density of a non-
51 trivial cyclic binary sequence X of length at most $d2^{2d+1}$ and minimum degree $\delta(X) \geq k$.

52 Our original motivation to study $g(d, k)$ comes from graph theory: For a finite, simple and
53 undirected graph $G = (V, E)$ and $k \in \mathbb{N}$, the *k-girth* $g_k(G)$ of G is the minimum order of an
54 induced subgraph of G of minimum degree at least k . The notion of k -girth was proposed
55 and studied by Erdős et al. [3–5] and Bollobás and Brightwell [2]. It generalizes the usual
56 girth, the length of a shortest cycle, which coincides with the 2-girth.

57 Kézdy and Markert studied bounds on this generalized girth [7, 8]. They conjectured
58 that the d -th power of the cycle of length $n \geq 2d + 1$, denoted by C_n^d , is the $2d$ -regular
59 graph with largest $(d + 1)$ -girth [8] (see also Chapter 5 of [7]). During the 1988 SIAM
60 conference, Kézdy [6] posed the problem to determine the exact value of the $(d + 1)$ -girth
61 of C_n^d . For odd values of d , this problem was solved by Bermond and Peyrat [1] who proved
62 that for $d + 1 \leq k \leq 2d$, the k -girth of C_n^d satisfies

$$\frac{g_k(C_n^d)}{n} \geq \frac{k}{2d}. \quad (1)$$

63 The bound (1) is best-possible whenever k is even in view of the induced subgraph of C_n^d
64 where n is a multiple of d and which alternately contains $\frac{k}{2}$ consecutive vertices of C_n^d and
65 does not contain the next $d - \frac{k}{2}$ consecutive vertices of C_n^d . For odd values of k , Bermond
66 and Peyrat mentioned results for some small values of d and k , and proved the best-possible
67 estimate $\frac{g_{2d-1}(C_n^d)}{n} \geq \frac{2d}{2d+1}$.

An induced subgraph G of C_n^d can be conveniently identified with a cyclic binary sequence $X = (x_0, x_1, \dots, x_{n-1})$ of length n where 1-entries correspond to vertices of C_n^d which belong to G and 0-entries correspond to vertices of C_n^d which do not belong to G . This correspondence implies that $g(d, k)$ equals the minimum k -girth of the d -th power of cycles, i.e.

$$g(d, k) = \min \left\{ \frac{g_k(C_n^d)}{n} \mid n \geq 3 \right\}.$$

The above-mentioned results of Bermond and Peyrat imply that $g(d, k) = \frac{k}{2d}$ for $d+1 \leq k \leq 2d$ with even k and that $g(d, 2d - 1) = \frac{2d}{2d+1}$. Kézdy and Markert determined $g(4, 5) = \frac{12}{19}$ and $g(6, 7) = \frac{24}{41}$ with the help of a computer. Bermond and Peyrat [1] claimed that $g(5, 7) = \frac{5}{7}$ which is not correct (see Section 2). Furthermore, they conjectured that

$$g(d, k) = \frac{d(2d + 3 - k)}{2(d^2 - (k - d - 2)d - (k - d))}$$

68 for $d + 1 \leq k \leq 2d$ with k odd. Since this expression is less than $\frac{k}{2d}$ if and only if
 69 $|k - \frac{3d}{2}| < \frac{d}{2}\sqrt{1 - \frac{4}{d+1}}$, this conjecture is obviously not correct in view of (1).

70 Our results are as follows. In Section 2, we explain how for fixed values of d and k , the
 71 problem to determine $g(d, k)$ can be reduced to a minimum mean cycle problem on a
 72 suitably defined directed graph with arc costs. This allows to determine $g(d, k)$ and also
 73 the structure of optimal subgraphs of C_n^d for many small values of d and k and motivates a
 74 corresponding conjecture explained in Section 3. Moreover, in Section 4, we prove as our
 75 main result that our conjecture is true for $k = 2d - 3$, i.e. we determine $g(d, 2d - 3)$.

76 2 Minimum Mean Cycle Formulation

Given a directed graph $D = (V, A)$ and a cost function $c : A \rightarrow \mathbb{R}$, a *minimum mean cycle*
 is a directed cycle

$$C : v_1 v_2 \dots v_n v_1$$

in D for which

$$\bar{c}(A(C)) = \frac{1}{n} \sum_{a \in A(C)} c(a)$$

77 is minimum. Karp [9] observed that a minimum mean cycle can be found efficiently using
 78 shortest path methods.

For $d \in \mathbb{N}$ and $d + 1 \leq k \leq 2d$, let $D = (V, A)$ be the directed graph whose vertex set
 V consists of all binary sequences

$$(x_{-d}, \dots, x_{-1}, x_0, x_1, \dots, x_d)$$

of length $2d + 1$ with $x_0 = 1$ and $\sum_{i=1}^d (x_i + x_{-i}) \geq k$ and which contains a directed arc (x, y)
 of cost $c((x, y)) = -i^*$ from a vertex $x = (x_{-d}, \dots, x_d)$ to a vertex $y = (y_{-d}, \dots, y_d)$ exactly
 if

$$(x_{i^*-d}, \dots, x_0, \dots, x_{i^*}, \dots, x_d) = (y_{-d}, \dots, y_{-i^*}, \dots, y_0, \dots, y_{d-i^*})$$

79 for $i^* = \min\{i \mid 1 \leq i \leq d, x_i = 1\}$. Note that i^* is well-defined and that the last condition
 80 implies that x and y can be suitably overlaid, i.e. there is a binary sequence z of length
 81 $2d + 1 + i^*$ such that x corresponds to the first $2d + 1$ entries of z and y corresponds to the
 82 last $2d + 1$ entries of z . See Figure 1 for an illustration.

Theorem 2 *If D and c are as above and C is a minimum mean cycle of D , then*

$$g(d, k) = -\frac{1}{\bar{c}(A(C))}.$$

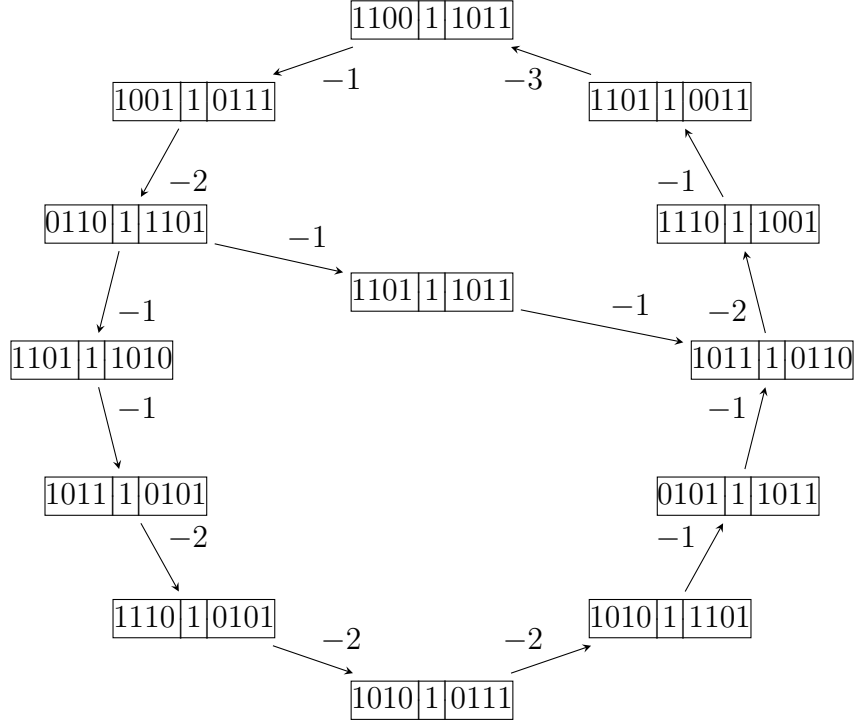


Figure 1: Induced subgraph of the directed graph D for $d = 4$ and $k = 5$.

83 *Proof:* Clearly, for every directed cycle $C : v_1v_2 \dots v_nv_1$ in D , suitably overlaying the
84 sequences v_1, v_2, \dots, v_n — as x and y above — results in a cyclic binary sequence X with
85 $\delta(X) \geq k$. Since the number of 1-entries of X equals n and the length of X equals
86 $-\sum_{a \in A(C)} c(a)$, we obtain $\mu(X) = -\frac{1}{\bar{c}(A(C))}$.

87 Conversely, if X is a cyclic binary sequence with $\delta(X) \geq k$, the sequences of length
88 $2d + 1$ centered at the consecutive 1-entries of X define a directed closed walk W in D .
89 By Euler's theorem, W contains a directed cycle C with $\bar{c}(A(C)) \leq \bar{c}(A(W))$. Since the
90 length of W equals the number of 1-entries of X and the length of X is $-\sum_{a \in A(C)} c(a)$, we
91 obtain $\bar{c}(A(C)) \leq \bar{c}(A(W)) = -\frac{1}{\mu(X)}$.

92 These two observations clearly imply the desired result. \square

Table 1 summarizes some explicit values of $g(d, k)$ obtained by this approach together with realizing cyclic binary sequences. In fact, we determined optimal sequences for all values of d and $d + 1 \leq k \leq 2d$ with $d \leq 13$, $k \geq 2d - 7$, and k odd. For $(d, k) = (5, 7)$ for example, we obtained $g(5, 7) = \frac{24}{34}$, and a realizing cyclic binary sequence is

$$(1, 1, 1, 0, 1, 1, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 0, 0),$$

93 which we write shortly as $1^3 01^4 0101^2 01^4 01^2 0101^4 01^3 0^2$.

$2d-k$	(d, k)	$g(d, k)$	Optimal cyclic sequences, candidates for \mathbf{u} highlighted
3	(4, 5)	12/19	$\mathbf{1^201} 1^201 0 \mathbf{101^2} 101^2 0^2$
	(5, 7)	24/34	$\mathbf{1^301} 1^301 0 \mathbf{1^201^2} 1^201^2 0 \mathbf{101^3} 101^3 0^2$
	(6, 9)	40/53	$\mathbf{1^401} 1^401 0 \mathbf{1^301^2} 1^301^2 0 \mathbf{1^201^3} 1^201^3 0 \mathbf{101^4} 101^4 0^2$
	(7, 11)	60/76	$\mathbf{1^501} 1^501 0 \mathbf{1^401^2} 1^401^2 0 \mathbf{1^301^3} 1^301^3 0 \mathbf{1^201^4} 1^201^4 0 \mathbf{101^5} 101^5 0^2$
5	(6, 7)	24/41	$\mathbf{10^21^3} 10^21^3 0^3 \mathbf{1^30^21} 1^30^21 0 \mathbf{1^20^21^2} 1^20^21^2 0$
			$\mathbf{101^201} 101^201 0^2 \mathbf{1^20101} 1^20101 0 \mathbf{10101^2} 10101^2 0^2$
	(7, 9)	40/62	$\mathbf{1^40^21} 1^40^21 0 \mathbf{1^30^21^2} 1^30^21^2 0 \mathbf{1^20^21^3} 1^20^21^3 0 \mathbf{10^21^4} 10^21^4 0^3$
			$\mathbf{1^30101} 1^30101 0 \mathbf{1^20101^2} 1^20101^2 0 \mathbf{10101^3} 10101^3 0^2 \mathbf{101^301} 101^301 0^2$
		20/31	$\mathbf{101^201^2} 101^201^2 0^2 \mathbf{1^201^201} 1^201^201 0$
7	(8, 9)	40/71	$\mathbf{1^40^31} 1^40^31 0 \mathbf{1^30^31^2} 1^30^31^2 0 \mathbf{1^20^31^3} 1^20^31^3 0 \mathbf{10^31^4} 10^31^4 0^4$
			$\mathbf{1^20^21^201} 1^20^21^201 0 \mathbf{10^21^201^2} 10^21^201^2 0^3 \mathbf{1^201^20^21} 1^201^20^21 0 \dots$
			$\mathbf{101^30^21} 101^30^21 0^2 \mathbf{1^30^2101} 1^30^2101 0 \mathbf{1^20^2101^2} 1^20^2101^2 0 \dots$
			$\mathbf{101^20101} 101^20101 0^2 \mathbf{1^2010101} 1^2010101 0 \mathbf{1010101^2} 1010101^2 0^2 \dots$

Table 1

3 A Conjecture for $g(d, k)$

We have observed that all optimal sequences that we have computed can be obtained by applying a uniform construction rule.

Let U be the set of finite binary sequences starting and ending with a 1. For $\mathbf{u} \in U$ with $\mathbf{u} = 10^a\mathbf{v}$ for some $\mathbf{v} \in U$, the *shift operation* s applied to \mathbf{u} results in $s(\mathbf{u}) = \mathbf{v}0^a1$, i.e. it removes all entries of \mathbf{u} before the second 1 and appends them at the end in reverse order. For $\mathbf{u} = 11101$, for example, we obtain

$$s(\mathbf{u}) = 11011, \quad s^2(\mathbf{u}) = s(s(\mathbf{u})) = 10111, \quad \text{and} \quad s^3(\mathbf{u}) = 11101 = \mathbf{u}.$$

For $d, k \in \mathbb{N}$ with $d + 1 \leq k \leq 2d$ and k odd, let U_k^d be the set of those sequences in U with length d and exactly $l = \frac{k+1}{2}$ many 1-entries.

Note that for $\mathbf{u} \in U_k^d$, we have $s^{l-1}(\mathbf{u}) = \mathbf{u}$.

The *shifted sequence* for \mathbf{u} is the concatenation

$$X(\mathbf{u}) = \mathbf{uu}0^{a_1+1}s(\mathbf{u})s(\mathbf{u})0^{a_2+1} \dots 0^{a_{l-2}+1}s^{l-2}(\mathbf{u})s^{l-2}(\mathbf{u})0^{a_{l-1}+1},$$

where a_i is the number of 0s between the i -th and $(i + 1)$ -st 1-entry of \mathbf{u} , i.e. $\mathbf{u} = 10^{a_1}10^{a_2}1 \dots 10^{a_{l-1}}1$. For $\mathbf{u} = 11011 \in U_7^5$, we have

$$X(\mathbf{u}) = 11011 11011 0 10111 10111 00 11101 11101 0$$

which is a cyclic shift of the sequence for (5, 7) in Table 1.

A subsequence of consecutive entries of a cyclic binary sequence is called an *interval*.

Lemma 3 *Let $d, k \in \mathbb{N}$ be such that $d + 1 \leq k \leq 2d$ and k is odd. Let $\mathbf{u} \in U_k^d$.*

(i) $X(\mathbf{u})$ has length $dk - 1$,

106 (ii) $\mu(X(\mathbf{u})) = \frac{k^2-1}{2(dk-1)},$

107 (iii) $\delta(X(\mathbf{u})) = k,$ and

108 (iv) $g(d, k) \leq \mu(X(\mathbf{u})).$

Proof: Let $\mathbf{u} = 10^{a_1}10^{a_2}1 \dots 10^{a_{l-1}}1$. The length of $X(\mathbf{u})$ equals

$$(l-1)2d + \sum_{i=1}^{l-1} (a_i + 1) = (k-1)d + (d-1) = dk - 1.$$

109 Furthermore, $X(\mathbf{u})$ contains $(l-1)2l = \frac{k^2-1}{2}$ many 1-entries. This implies (i) and (ii).

110 Note that the shifted sequences for \mathbf{u} and for $s(\mathbf{u})$ are cyclic translates of each other.
 111 Furthermore, note that the reverse of a shifted sequence is also the cyclic translate of a
 112 shifted sequence. Therefore, in order to prove (iii), it suffices to consider the 1-entries
 113 within the first copy of $s(\mathbf{u})$ in $X(\mathbf{u})$.

By definition, the interval of $X(\mathbf{u})$ of length $2d+1$ centered at the first 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals (the central entry is highlighted)

$$10^{a_2}1 \dots 10^{a_{l-2}}10^{a_{l-1}}10^{a_1+1}\mathbf{1}0^{a_2}10^{a_3}1 \dots 10^{a_{l-1}}10^{a_1}11.$$

114 Hence this 1-entry sees $(l-1)$ 1-entries to the left and l 1-entries to the right, i.e. it sees
 115 $2l-1 = k$ 1-entries.

For $2 \leq i \leq l-2$, the interval of $X(\mathbf{u})$ of length $2d+1$ centered at the i -th 1-entry of the first copy of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals

$$10^{a_{i+1}}1 \dots 10^{a_{l-1}}10^{a_1+1}10^{a_2}1 \dots 10^{a_i}\mathbf{1}0^{a_{i+1}}1 \dots 10^{a_{l-1}}10^{a_1}110^{a_2}1 \dots 10^{a_i}1.$$

116 Again this 1-entry sees $2l-1 = k$ 1-entries.

117 The interval of $X(\mathbf{u})$ of length $2d+1$ centered at the $(l-1)$ -th 1-entry of the first copy
 118 of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals $10^{a_1+1}10^{a_2}1 \dots 10^{a_{l-1}}\mathbf{1}0^{a_1}110^{a_2}1 \dots 10^{a_{l-1}}1$. Again this 1-entry
 119 sees $2l-1 = k$ 1-entries.

120 The interval of $X(\mathbf{u})$ of length $2d+1$ centered at the l -th 1-entry of the first copy
 121 of $s(\mathbf{u})$ within $X(\mathbf{u})$ equals $010^{a_2}1 \dots 10^{a_{l-1}}10^{a_1}\mathbf{1}s(\mathbf{u})$. Again this 1-entry sees $2l-1 = k$
 122 1-entries.

123 (iv) follows immediately from (ii) and (iii). \square

124 We pose the following conjecture.

Conjecture 4 *If $d \in \mathbb{N}$ and $d+1 \leq k \leq 2d$ are such that k is odd, then*

$$g(d, k) = \frac{k^2 - 1}{2(dk - 1)}.$$

125 *Furthermore, a cyclic binary sequence X with $\delta(X) \geq k$ has density $g(d, k)$ if and only if*
 126 *X is the concatenation of copies of a shifted sequence $X(\mathbf{u})$ for some $\mathbf{u} \in U_k^d$.*

127 The case $k = 2d - 1$ of Conjecture 4 follows from the results and arguments in [1]. In this
 128 case U_{2d-1}^d contains only the element $\mathbf{u} = 1^d$ and $X(\mathbf{u}) = 1^{2d}01^{2d}0 \dots 1^{2d}0$.

129 Since we will prove Conjecture 4 for $k = 2d - 3$, it is useful to consider the structure of
 130 $X(\mathbf{u})$ for $\mathbf{u} \in U_{2d-3}^d$. In this case, \mathbf{u} is a sequence of length d containing $(d - 1)$ 1-entries.
 131 If $\mathbf{u}^* = 10^{a_1}10^{a_2}1 \dots 10^{a_{l-1}}1$ with $a_1 = \dots = a_{l-2} = 0$ and $a_{l-1} = 1$, then $\mathbf{u}^* = 1^{d-2}01$ and

$$\begin{aligned} X(\mathbf{u}^*) &= 1^{d-2}011^{d-2}0101^{d-3}01^21^{d-3}01^20 \dots 101^{d-2}101^{d-2}0^2 \\ &= 1^{d-2}01^{d-1}0101^{d-3}01^{d-1}01^201^{d-4}01^{d-1}01^30 \dots 101^{d-1}01^{d-2}0^2 \end{aligned}$$

132 Since for every $\mathbf{u} \in U_{2d-3}^d$, there is some i with $s^i(\mathbf{u}^*) = \mathbf{u}$, every shifted sequence $X(\mathbf{u})$
 133 for $\mathbf{u} \in U_{2d-3}^d$ arises from $X(\mathbf{u}^*)$ by a cyclic shift. In this sense, the conjectured extremal
 134 sequences are unique.

135 4 The Value of $g(d, 2d - 3)$

136 Throughout this section let $d \geq 4$ and let \mathcal{X} be the set of cyclic binary sequences X with
 137 $\delta(X) \geq 2d - 3$. This section is devoted to the proof of Conjecture 4 for $k = 2d - 3$, i.e. we
 138 will prove the following result.

139 **Theorem 5** *Every $X \in \mathcal{X}$ satisfies $\mu(X) \geq \frac{(2d-3)^2-1}{2((2d-3)d-1)}$. Equality holds if and only if X
 140 is the concatenation of shifted sequences $X(\mathbf{u}^*)$ with $\mathbf{u}^* = 1^{d-2}01$.*

Before proving Theorem 5, we investigate structural properties of sequences in \mathcal{X} . Let

$$X = (x_0, x_1, \dots, x_{n-1}) = x_0x_1 \dots x_{n-1} \in \mathcal{X} \text{ with } n \geq 2d + 1.$$

141 Recall that an entry x_i of X sees another entry x_j of X , if x_j is in one of the intervals
 142 $x_{i-d}x_{i-d+1} \dots x_{i-1}$ or $x_{i+1}x_{i+2} \dots x_{i+d}$. We call x_i *regular* if it sees exactly $(2d - 3)$ 1-entries
 143 and hence exactly three 0-entries. We first show that all irregular entries see more than
 144 $(2d - 3)$ 1-entries and describe the local structure around regular 0-entries.

145 Lemma 6

146 (i) *All entries of X see at most three 0-entries.*

147 (ii) *For every regular 0-entry x_i , either $x_{i+1} = x_{i+d} = 0$, or $x_{i-1} = x_{i-d} = 0$, or $x_{i-d} =$
 148 $x_{i+d} = 0$.*

149 *Proof:* (i): By assumption, all 1-entries of X see at most three 0-entries. For contradiction,
 150 we assume that some 0-entry of X sees more than three 0-entries. This implies that X
 151 has an interval $X' = 10^a1$ such that some 0-entry of X' sees at least four 0-entries. Since
 152 $d \geq 4$ and each of the two 1-entries of X' see at most three 0-entries, we obtain $a \leq 3$.
 153 Moreover, the two 1-entries of X' together see at most $(6 - a)$ distinct 0-entries. If $a \geq 2$,
 154 then every 0-entry of X' sees at most three 0-entries, a contradiction. Hence $a = 1$. If x_i

155 is the 0-entry in X' , then each 1-entry of X' sees all but one entry seen by x_i . Thus it sees
 156 at least three 0-entries seen by x_i and the 0-entry x_i which is the final contradiction.

157 (ii): Again, the interval X' of the form 10^a1 of X containing the regular 0-entry x_i satisfies
 158 $a \leq 3$. If $a = 3$, then one of the two 1-entries of X' sees x_i and all three 0-entries seen by
 159 x_i which is a contradiction. If $a = 2$, then, by symmetry, we may assume that x_i is the
 160 first 0-entry of X' . Since the 1-entry x_{i-1} does not see one of the 0-entries seen by x_i , we
 161 have $x_{i+1} = x_{i+d} = 0$. Finally, if $a = 1$, then each of the 1-entries x_{i-1} and x_{i+1} does not
 162 see one of the 0-entries seen by x_i which implies $x_{i+d} = x_{i-d} = 0$ and completes the proof
 163 of (ii). \square

164 Let n_1 denote the number of 1-entries of X . Moreover, let n^+ denote the number of
 165 irregular entries of X .

166 We can relate the density of X to the number of irregular entries of X .

Lemma 7

$$\mu(X) = \frac{n_1}{n} \geq \frac{2d-3}{2d} + \frac{n^+}{2dn}.$$

167 *Proof:* By Lemma 6 (i), double-counting the pairs (x_i, x_j) where $x_i = 1$ and x_i sees x_j
 168 yields $(2d-3)(n-n^+) + (2d-2)n^+ \leq 2dn_1$ which implies $\mu(X) = \frac{n_1}{n} \geq \frac{2d-3}{2d} + \frac{n^+}{2d}$. \square

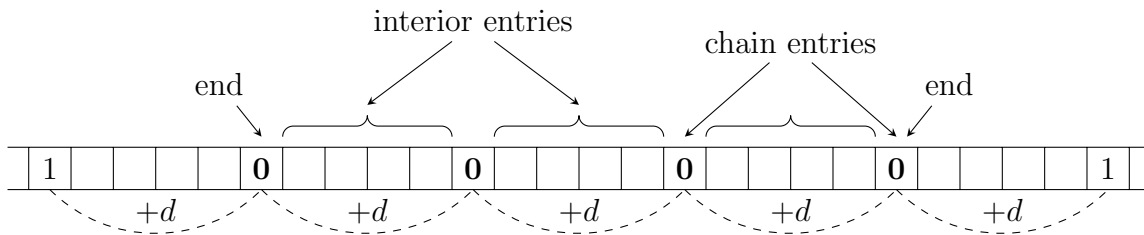


Figure 2: A chain of length 4 for $d = 5$.

A *chain* of X is a maximal subsequence

$$C = (x_i, x_{i+d}, \dots, x_{i+kd})$$

169 of distinct 0-entries of X such that $k \geq 1$. A chain may be *cyclic* in which case $i \equiv i+(k+1)d$
 170 (mod n). Otherwise C has two distinct *ends* x_i and x_{i+kd} where $x_{i-d} = 1 = x_{i+(k+1)d}$.
 171 Associated with the chain C are the *interior* entries of C , which are those entries that belong
 172 to one of the intervals $x_{i+jd+1}x_{i+jd+2} \dots x_{i+jd+d-1}$, $0 \leq j \leq k-1$, between consecutive chain
 173 entries x_{i+jd} and $x_{i+(j+1)d}$ of C . We say that two chains *overlap*, if a chain entry of one
 174 chain is an interior entry of the second chain. Clearly, in this case, also a chain entry of the
 175 second chain is an interior entry of the first chain. Note that a chain may overlap itself.

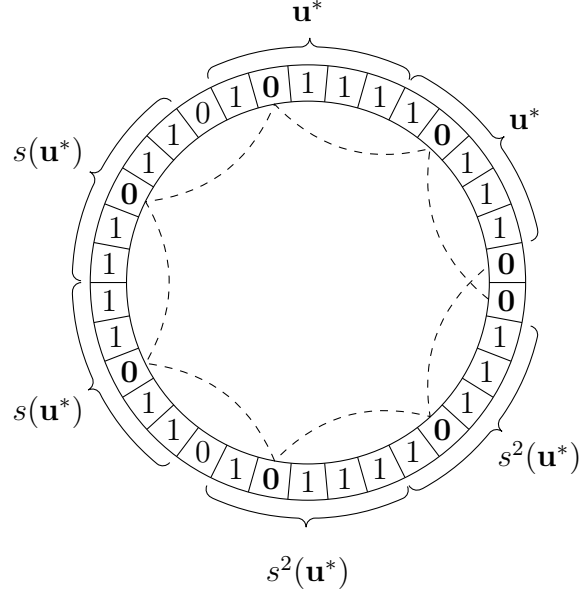


Figure 3: The example $X'(\mathbf{u}^*)$ for $d = 5$, i.e. with $\mathbf{u}^* = 11101$.

176 For example, the sequence $X'(\mathbf{u}^*) = x'_0 x'_1 \dots x'_{n-1}$ which arises from the shifted sequence
 177 $X(\mathbf{u}^*)$ for $\mathbf{u}^* = 1^{d-2}01$ by moving the final 0-entry to the beginning

$$\begin{aligned} X'(\mathbf{u}^*) &= x'_0 x'_1 \dots x'_{n-1} \\ &= \mathbf{01}^{d-2} \mathbf{01}^{d-1} \mathbf{0101}^{d-3} \mathbf{01}^{d-1} \mathbf{01}^2 \mathbf{01}^{d-4} \mathbf{01}^{d-1} \mathbf{01}^3 \mathbf{0} \dots \mathbf{101}^{d-1} \mathbf{01}^{d-2} \mathbf{0} \end{aligned} \quad (2)$$

178 has the single chain $C = (x'_{n-1}, x'_{d-1}, x'_{2d-1}, \dots, x'_{n-d}, x'_0)$ whose ends x'_{n-1} and x'_0 are both
 179 interior entries as well as chain entries of C . See Figures 2 and 3 for an illustration.

180 We will show that chains may overlap only in their respective ends. More precisely, in
 181 Lemma 8 (ii) below we show that if x_i is a chain entry of C which is an interior entry of
 182 chain C' and x_{i-d} is another chain entry of C , then x_i is an end of C and x_{i-1} is an end of
 183 $C' = (x_{i-1}, x_{i+d-1}, \dots)$. If this occurs, we call the interval $x_{i-1}x_i = 0^2$ a pair of *overlapping*
 184 *chain ends*.

185 **Lemma 8**

- 186 (i) Every regular 0-entry of X belongs to some chain of X .
- 187 (ii) If a chain entry of C is an interior entry of the (not necessarily distinct) chain C' ,
 188 then it belongs to a pair of overlapping chain ends.
- 189 (iii) Let $x_{i-1}x_i$ be a pair of overlapping chain ends. The intervals of length $2d$ ending and
 190 starting in $x_{i-1}x_i = 0^2$ have the form $1^{d-1}01^{d-2}0^2$ and $0^21^{d-2}01^{d-1}$, respectively.
- 191 (iv) An end of a chain is regular in X if and only if it belongs to a pair of overlapping
 192 chain ends.

193 *Proof:* (i): This follows immediately from Lemma 6 (ii).

194 (ii): Let x_i be a chain entry of C which is an interior entry of C' . Then there must be
 195 chain entries x_j, x_{j+d} with $i - d < j < i$ of C' . By symmetry, we may assume that x_{i-d} is
 196 another chain entry of C . If $j < i - 1$, then x_{i-1} sees at least four 0-entries, a contradiction.
 197 So $j = i - 1$. Moreover, $x_{j-d} = 1 = x_{i+d}$, otherwise x_{i-2} or x_{i+1} sees four 0-entries. So x_i
 198 is an end of C and x_{i-1} is an end of C' .

199 (iii): Since both x_{i-1} and x_i already see three of the four 0-entries $x_{i-d}, x_{i-1}, x_i, x_{i+d-1}$, we
 200 obtain that $x_{i-d+1} = 1 = x_{i+d-2}$. Since each of these two entries sees three of the four
 201 0-entries, too, all other entries seen by them must be 1, and the two intervals of X ending
 202 and starting in $x_{i-1}x_i$ have the required form.

203 (iv): It follows from (iii) that overlapping ends of chains are regular. Conversely, we assume
 204 that x_i is an end of a chain which is not an interior entry of any chain. By symmetry, we
 205 may assume that $x_{i-d} = 0$ and $x_{i+d} = 1$. If x_i is regular, then $x_{i-1} = 0$, otherwise x_{i-1}
 206 sees x_i and all the three 0-entries seen by x_i , a contradiction to Lemma 6 (i). But since
 207 x_{i-1} does not belong to a chain, it must be irregular by (i) and thus x_{i-1} sees only the two
 208 0-entries x_i and x_{i-d} . So x_i must be irregular as well. \square

209 **Lemma 9** *Let $I = x_{j-d}x_{j-d+1} \dots x_{j+d}$ be an interval of $2d + 1$ entries of X .*

210 (i) *If I contains no irregular entry, then I contains a regular end of a chain.*

211 (ii) *If I does not contain a regular chain end but contains an irregular chain end, then it*
 212 *contains at least two irregular entries.*

213 *Proof:* (i): Since the center x_j of I is regular, it sees exactly three 0-entries, all of which
 214 are regular. By the length of I , only two of them can belong to the same chain. So, by
 215 Lemma 8 (iv), the third must be a regular chain end belonging to a pair of overlapping
 216 chain ends.

217 (ii): For contradiction, we assume that I contains exactly one irregular entry, an irregular
 218 chain end. If the center x_j is not the irregular chain end itself, then it is regular. So
 219 it sees two further 0-entries apart from the irregular chain end. Since these are regular,
 220 they all belong to chains. Hence, by Lemma 8 (ii), one of them is a regular chain end,
 221 a contradiction. So let x_j be the irregular chain end. We may assume that $x_{j-d} = 0$. If
 222 x_j sees another 0-entry apart from x_{j-d} , then, by Lemma 8 (i) and (iv), this 0-entry is
 223 irregular. Otherwise, x_{j+1} is irregular, a contradiction. \square

224 **Lemma 10** *If X has a single chain whose ends overlap, then X has at least $d - 3$ irregular*
 225 *entries.*

226 *Proof:* Let $(x_0, x_d, x_{2d}, \dots, x_{n-d+1}, x_1)$ be the chain and let $2 \leq r \leq d - 2$. We prove that
 227 there is some irregular entry x_j with $2 \leq j \leq n - 2$ and $j \equiv r \pmod{d}$.

228 If an entry at such a position satisfies $x_j = 0$, then, by Lemma 8 (i) and (ii), x_j is
 229 irregular. Hence, we may assume that $x_j = 1$ for all $2 \leq j \leq n - 2$ with $j \equiv r \pmod{d}$.
 230 We choose a largest $s < r$ such that X has an entry $x_k = 0$ with $k \equiv s \pmod{d}$. Note
 231 that $x_1 = 0$ implies that s is well-defined and that $1 \leq s < r$. We claim that $x_{k-s+d+r}$ is
 232 irregular.

233 Note that every 1-entry in the interval $x_{k-s}x_{k-s+1}\dots x_{k-s+d}$ sees the three 0-entries
 234 x_{k-s}, x_k, x_{k-s+d} . Hence $x_{k-s+d-1} = 1$ and $k - s + d + r < n - d$. Moreover, all further
 235 entries seen by $x_{k-s+d-1}$ satisfy $x_{k-s+d+1} = x_{k-s+d+2} = \dots = x_{k-s+2d-1} = 1$. Furthermore,
 236 since x_{k+d} sees three 0 entries, $x_{k-s+2d+1} = \dots = x_{k+2d} = 1$. By the definition of s ,
 237 $x_{k+2d+1} = \dots = x_{k+2d+r-s-1} = 1$. So, indeed, $x_{k-s+d+r}$ sees only the two 0-entries x_{k-s+d}
 238 and x_{k-s+2d} and is irregular. \square

239 We are now prepared to prove Theorem 5.

240 *Proof of Theorem 5:*

241 Let $X^* = X'(\mathbf{u}^*)$ be as in (2). For contradiction, we assume that $X = (x_0, x_1, \dots, x_{n-1})$
 242 is a cyclic binary sequence in \mathcal{X} of smallest order n having minimum density $\mu(X) =$
 243 $g(d, 2d - 3)$, and that X is not the concatenation of copies of X^* . Clearly, $\mu(X) \leq$
 244 $\mu(X^*) = \frac{(2d-3)^2-1}{2((2d-3)d-1)}$. Since a 1-entry of X must see at least $2d - 3$ other 1-entries, we get
 245 for $n \leq 2d$ that $\mu(X) = \frac{n_1}{n} \geq \frac{2d-2}{n} \geq 1 - \frac{1}{d} > \mu(X^*)$, a contradiction. So we may assume
 246 that $n \geq 2d + 1$.

247 If X contains no pair of overlapping chain ends, then, by Lemma 9 (i), every interval I
 248 of length $2d + 1$ of X contains an irregular entry. Since every irregular entry contributes
 249 to $2d + 1$ such intervals, we get by double-counting

$$n \leq (2d + 1)n^+, \quad (3)$$

250 thus, by Lemma 7, $\mu(X) \geq \frac{2d-3}{2d} + \frac{1}{2d(2d+1)} > \mu(X^*)$ which is a contradiction.

251 Hence we may assume that X contains a pair of overlapping ends of chains.

First we assume that X contains more than one such pair. By cyclicity, we may assume
 that $x_{n-1}x_0$ and $x_{k-1}x_k$ are pairs of overlapping chain ends of X . Let

$$X' = x_0x_1\dots x_{k-1}$$

and

$$X'' = x_kx_{k+1}x_{k+2}\dots x_{n-1}.$$

252 By Lemma 8 (iii), X' and X'' , considered as cyclic sequences, are both in \mathcal{X} , because each
 253 entry sees the same entries as in X . Since X has minimum density $\mu(X)$ and $\mu(X)$ is a
 254 weighted average of the densities $\mu(X')$ and $\mu(X'')$, we obtain $\mu(X') = \mu(X'') = \mu(X)$.
 255 Since X' and X'' have smaller lengths than X , by our initial assumption, each of X' and
 256 X'' are the concatenation of copies of X^* . Hence X is the concatenation of copies of X^*
 257 which is a contradiction.

Therefore, X has exactly one pair of overlapping chain ends, say $x_{n-1}x_0$. Let \mathcal{J} be
 the set of intervals of length $2d + 1$ of X . Let $\mathcal{J}_0 \subseteq \mathcal{J}$ denote the set of those intervals

containing a regular chain end and let $\mathcal{J}_2 \subseteq \mathcal{J}$ denote the set of those intervals containing an irregular chain end. By Lemma 9, each interval in $\mathcal{J}_2 \setminus \mathcal{J}_0$ contains at least two irregular entries, while only the intervals in $\mathcal{J}_0 \setminus \mathcal{J}_2$ can contain no irregular entry. If X contains more than one chain, then X contains two different irregular chain ends, hence $|\mathcal{J}_2| \geq 2d + 2$ while $|\mathcal{J}_0| \leq 2d + 2$. Double-counting the incidences interval/irregular entry we obtain

$$n \leq n + |\mathcal{J}_2| - |\mathcal{J}_0| = n + |\mathcal{J}_2 \setminus \mathcal{J}_0| - |\mathcal{J}_0 \setminus \mathcal{J}_2| \leq (2d + 1)n^+,$$

258 as in (3), which again contradicts $\mu(X) \leq \mu(X^*)$.

So X has a single chain both ends of which overlap. By Lemma 10, X contains at least $d - 3$ irregular entries. Hence, by Lemma 7, $\mu(X) \geq \frac{2d-3}{2d} + \frac{d-3}{2dn}$. Since $\mu(X) \leq \mu(X^*) = \frac{(2d-3)^2-1}{2(d(2d-3)-1)}$, this implies that

$$n \geq d(2d - 3) - 1,$$

259 i.e. the length of X is at least the length of X^* . By Lemma 8 (iv), each of the $n - (2d + 2)$
 260 intervals of length $2d + 1$ in $\mathcal{J} \setminus \mathcal{J}_0$ contains at least one irregular entry. Hence $n^+ \geq \frac{n-(2d+2)}{2d+1}$
 261 and, by Lemma 7,

$$\begin{aligned} \mu(X) &\geq \frac{2d-3}{2d} + \frac{n-(2d+2)}{2d(2d+1)n} = \frac{2d-3}{2d} + \frac{1}{2d(2d+1)} - \frac{2d+2}{2d(2d+1)n} \\ &\geq \frac{(2d-3)^2-1}{2(d(2d-3)-1)} = \mu(X^*). \end{aligned}$$

262 Since $\mu(X) \leq \mu(X^*)$, we obtain $\mu(X) = \mu(X^*)$. Therefore, $n = d(2d - 3) - 1$, each irregular
 263 entry sees exactly $(2d - 2)$ 1-entries, and each of the $2d + 2$ intervals in \mathcal{J}_0 contains no
 264 irregular entry while all intervals in $\mathcal{J} \setminus \mathcal{J}_0$ contain exactly one irregular entry. Hence the
 265 irregular entries must be exactly $x_{2d+1}, x_{4d+2}, \dots, x_{(2d+1)(d-3)}$.

266 So the irregular entries of X and X^* , with the notation of (2), are located at the
 267 same positions and, by Lemma 8 (iii), the intervals $x_{n-2d+1} \dots x_{n-1} x_0 \dots x_{2d-2}$ of X and
 268 $x'_{n-2d+1} \dots x'_{n-1} x'_0 \dots x'_{2d-2}$ of X^* are equal.

269 We assume that for some $i \geq 2d - 2$, the intervals $x_{i-2d+1} \dots x_i$ of X and $x'_{i-2d+1} \dots x'_i$
 270 of X^* are equal. Now we show that $x_{i+1} = x'_{i+1}$. Indeed, since $x_{i-d+1} = x'_{i-d+1}$ has
 271 the same regularity status within X and X^* and sees the same entries in X and X^* ,
 272 respectively, except possibly at position $i + 1$, it follows that $x_{i+1} = x'_{i+1}$. Therefore,
 273 $X = X^*$ contradicting the assumption that X is a counterexample. This completes the
 274 proof. \square

If we define the quantity $\tilde{\delta}(X)$ for a cyclic binary sequence $X = (x_0, x_1, \dots, x_{n-1})$ as

$$\tilde{\delta}(X) = \min \left\{ \sum_{j=1}^d (x_{i+j} + x_{i-j}) \mid 0 \leq i \leq n - 1 \right\}$$

275 and $\tilde{g}(d, k)$ for $d, k \in \mathbb{N}$ with $k \leq 2d$ as the infimum density of a cyclic binary sequence X
 276 with $\tilde{\delta}(X) \geq k$, then $g(d, k) \leq \tilde{g}(d, k)$. A simple double-counting implies $\tilde{g}(d, k) \leq \frac{k}{2d}$.

277 The example described after (1) implies $g(d, k) = \tilde{g}(d, k)$ for $k \geq d + 1$ with k even.
 278 Furthermore, the comment after Conjecture 4 concerning $k = 2d - 1$ and Lemma 6 (i)
 279 imply $g(d, 2d - 1) = \tilde{g}(d, 2d - 1)$ and $g(d, 2d - 3) = \tilde{g}(d, 2d - 3)$, respectively. Finally, it is
 280 easy to check that $\tilde{\delta}(X(\mathbf{u})) \geq k$ for every shifted sequence $X(\mathbf{u})$ for every $\mathbf{u} \in U_k^d$ which
 281 does not contain two consecutive 0-entries.

282 Therefore, Conjecture 4 would - if true - imply that $g(d, k) = \tilde{g}(d, k)$ for all $d + 1 \leq$
 283 $k \leq 2d$.

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