Funnel control with saturation: nonlinear SISO systems

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Abstract

Tracking – by the system output – of a reference signal (assumed bounded with essentially bounded derivative) is considered in the context of a class of nonlinear, single-input, single-output systems modelled by functional differential equations and subject to input saturation. Prespecified is a parameterized performance funnel within which the tracking error is required to evolve; transient and asymptotic behaviour of the tracking error is influenced through choice of parameter values which define the funnel. The control structure is a saturating error feedback with time-varying non-monotone gain designed to evolve in such a way as to preclude contact with the funnel boundary. A feasibility condition – formulated in bounds of the plant data, the saturation bound, the funnel data, the reference signal and the initial data – is presented under which the tracking objective is achieved, whilst maintaining boundedness of the state and gain function.

Keywords. Output feedback, input saturation, nonlinear systems, transient behaviour, tracking.

Nomenclature: $\mathbb{R}_+ := [0,\infty)$; $C(I,\mathbb{R}^\ell)$, $I \subset \mathbb{R}$, is the space of continuous functions $I \to \mathbb{R}^\ell$; $L^\infty(I,\mathbb{R}^\ell)$ is the space of measurable, essentially bounded functions $f\colon I \to \mathbb{R}^\ell$, with norm $\|f\|_\infty := \operatorname{ess\,sup}_{t\in I}\|y(t)\|$; the space of measurable, locally essentially bounded functions $f\colon I \to \mathbb{R}^\ell$ is denoted by $L^\infty_{\text{loc}}(I,\mathbb{R}^\ell)$; if $\ell = 1$, we simply write $L^\infty(I)$ and $L^\infty_{\text{loc}}(I)$; $W^{1,\infty}(\mathbb{R}_+)$ is the space of absolutely continuous functions $r\colon \mathbb{R}_+ \to \mathbb{R}$ with $r, \dot r \in L^\infty(\mathbb{R}_+)$. A function $\beta\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{K}$ function if it is continuous and strictly increasing, with $\beta(0) = 0$; the class of *unbounded* $\mathcal{K}$ functions is denoted by $\mathcal{K}_\infty$. A continuous function $\alpha\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{KL}$ function if $\alpha(\cdot,t) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$ and, for all $s \in \mathbb{R}_+$, $\alpha(s,\cdot)$ is non-increasing with $\alpha(s,t) \to 0$ as $t \to \infty$.

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1. INTRODUCTION

In common with its precursor [1], we investigate funnel control in the presence of input constraints. In contrast with [1], the systems to be controlled are nonlinear and are described by functional differential equations. We restrict attention to single-input, single-output systems. By way of motivation, consider a system of two interconnected nonlinear subsystems

$$\dot{y}(t) = f_1(d(t), y(t), z(t)) + \mathrm{sat}_{\hat{u}}(u(t)), \quad \dot{z}(t) = f_2(y(t), z(t)), \quad (y(0), z(0)) = (y^0, z^0) \tag{1.1}$$

where $f_1 \colon \mathbb{R}^p \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and $f_2 \colon \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are locally Lipschitz, $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is a disturbance, and $\mathrm{sat}_{\hat{u}}$ is the saturation function, parameterized by $\hat{u} > 0$, given by

$$\mathrm{sat}_{\hat{u}} \colon \mathbb{R} \to [-\hat{u}, \hat{u}], \qquad v \mapsto \mathrm{sat}_{\hat{u}}(v) := \begin{cases} +\hat{u}, & v \geq \hat{u} \\ v, & |v| < \hat{u} \\ -\hat{u}, & v \leq -\hat{u}. \end{cases} \tag{1.2}$$

Momentarily regarding the second subsystem in (1.1) as an independent system with (continuous) input y, let $\varphi(\cdot; z^0, y)$ denote the unique maximal solution of the initial-value problem

$$\dot{z}(t) = f_2(y(t), z(t)), \quad z(0) = z^0. \tag{1.3}$$

Now assume that this system is input-to-state stable (ISS) and so there exist functions $\alpha \in \mathcal{KL}$ and $\beta \in \mathcal{K}_\infty$ such that, for all $(z^0, y) \in \mathbb{R}^{n-1} \times C(\mathbb{R}_+)$, the unique maximal solution $z(\cdot) = \varphi(\cdot; z^0, y)$ is global (i.e., is defined on $\mathbb{R}_+$) and satisfies the ISS estimate

$$\|z(t)\| \leq \alpha(\|\zeta\|, t) + \text{ess-sup}_{s \in [0,t]} \beta(|y(s)|) \quad \forall\, t \geq 0. \tag{1.4}$$

Example 1.1: As a highly specialized example (to which we will return in the simulations in Section 4) of a system of the form (1.1), consider the following

$$\left.\begin{aligned} \dot{y}(t) &= d(t) + a|y(t)|^b y(t) + z(t) + \mathrm{sat}_{\hat{u}}(u(t)), & y(0) &= y^0 \in \mathbb{R} \\ \dot{z}(t) &= -z(t) - z(t)^3 + [1 + z(t)^2]\, y(t), & z(0) &= z^0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.5}$$

with real constants a and $b \geq 0$, and disturbance $d \in L^\infty(\mathbb{R}_+)$. To see that the second subsystem in (1.5) is ISS, consider the subsystem in isolation with input $y \in C(\mathbb{R}_+)$ and let $z(\cdot) = \varphi(\cdot; z^0, y)$ denote its unique solution on its maximal interval of existence $[0, \omega)$, $0 < \omega \leq \infty$. Writing $V(t) := z^2(t)/2$ for all $t \in [0, \omega)$, we have

$$\dot{V}(t) = -2V(t) - z^4(t) + z(t)\, y(t) + z^3(t)\, y(t) \leq -2V(t) + \frac{y^2(t)}{2} + \frac{y^4(t)}{4} \quad \forall\, t \in [0, \omega).$$

By Gronwall's Lemma, it follows that

$$V(t) \le e^{-2t}V(0) + \frac{1}{4}\int_0^t \big(2y^2(s) + y^4(s)\big)e^{-2(t-s)}\,\mathrm{d}s \qquad \forall\, t \in [0,\omega)$$

from which we may infer that $\omega = \infty$ and

$$|z(t)| \le e^{-t}|z^0| + \frac{1}{2\sqrt{2}}\,\text{ess-sup}_{s\in[0,t]}\left(|y(s)|\sqrt{2+|y(s)|^2}\right) \quad \forall\, t \ge 0.$$

Therefore, the ISS estimate (1.4) holds with

$$\alpha\colon (\zeta,t) \mapsto e^{-t}\zeta \;\text{ and }\; \beta\colon \rho \mapsto \frac{1}{2\sqrt{2}}\,\rho\sqrt{2+\rho^2}. \tag{1.6}$$

$\diamond$

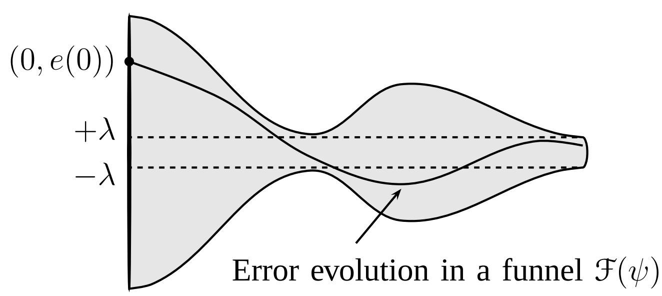


Fig. 1.1. Prescribed performance funnel $\mathcal{F}(\psi)$.

Returning to the prototype system (1.1), the control objective is formulated in terms of a *performance funnel*, see Figure 1.1,

$$\mathcal{F}(\psi) := \{(t,e) \in \mathbb{R}_+ \times \mathbb{R} \mid |e| < \psi(t)\}\ , \tag{1.7}$$

determined by a bounded function $\psi\colon \mathbb{R}_+ \to [\lambda,\infty)$ which is globally Lipschitz with Lipschitz constant $\Lambda > 0$ and is bounded away from 0 by $\lambda > 0$, that is, by a function of the class:

$$\mathcal{G}(\Lambda,\lambda) := \left\{\psi\colon \mathbb{R}_+ \to [\lambda,\infty) \,\middle|\, \psi \text{ bounded \& globally Lipschitz with Lipschitz constant } \Lambda\right\} \tag{1.8}$$

The *control objective* is as follows. Determine a feedback structure which ensures that, for a given reference signal $r \in W^{1,\infty}(\mathbb{R}_+)$, the output tracking error $e = y - r$ evolves within the funnel (i.e. graph$(e) \subset \mathcal{F}(\psi)$): transient and asymptotic behaviour of the tracking error is influenced through choice of the function ψ. The proposed control structure is an error feedback of the form $u(t) = -k(t)e(t)$ wherein the gain function $k\colon t \mapsto 1/(\psi(t) - |e(t)|)$ evolves so as to preclude contact with the funnel boundary. A feasibility condition (formulated in terms of the plant data, the funnel data, the reference signal r, the disturbance signal d, and the initial state y^0) is presented under which the tracking objective is achieved, whilst maintaining boundedness of all signals.

Given $\lambda > 0$ (arbitrarily small) and $\Lambda > 0$, a wide variety of funnels are possible. For example, choosing $a, b > 0$ such that $a > \lambda$ and $ab \le \Lambda$, then the function $t \mapsto \psi(t) := \max\{ae^{-bt}, \lambda\}$ is in $\mathcal{G}(\Lambda, \lambda)$ and evolution within the associated funnel ensures a prescribed exponential decay in the transient phase $[0, \ln(a/\lambda)/b]$ and tracking accuracy $\lambda > 0$ thereafter (we stress that λ may be taken arbitrarily small). Monotonicity of the funnel boundary is not required: in Section 4, we will provide an example of a non-monotone funnel. Non-monotone funnels may be advantageous in applications for which it is known a priori when perturbations or set-point changes may occur - in this sense, non-monotone funnels have the connotation of re-initialization of the control structure.

Example 1.2 (Example 1.1 revisited): In the highly specialized context of the exemplar (1.5), the main result of the paper translates into the following: for arbitrary $\Lambda, \lambda \ge 0$, $\psi \in \mathcal{G}(\Lambda, \lambda)$, and any absolutely continuous reference signal $r : \mathbb{R}_+ \to \mathbb{R}$ with essentially bounded derivative, the simple control strategy

$$u(t) = -\operatorname{sat}_{\hat{u}}(k(t)e(t)), \quad k(t) = \frac{1}{\psi(t) - |e(t)|}, \quad e(t) = y(t) - r(t),$$

applied to (1.5) ensures attainment of the tracking objective (and, moreover, the gain function k is bounded) provided that the initial data satisfy $|y^0 - r(0)| < \psi(0)$ and the following feasibility assumption is satisfied: $\widehat{u} > L + \Lambda + \|\dot{r}\|_\infty$, where

$$L := \|d\|_\infty + a(\|\psi\|_\infty + \|r\|_\infty)^{b+1} + \|z^0\| + \frac{1}{2\sqrt{2}}(\|\psi\|_\infty + \|r\|_\infty)\sqrt{2 + (\|\psi\|_\infty + \|r\|_\infty)^2}$$

◇

The concept of funnel control was introduced in [4]. For several generalizations and other aspects of this control strategy, see the survey article [3] and references therein. For experimental results on controlling the speed of electric devices using the funnel control methodology, see [5]. The control problem to be considered in the present paper is the analogue - in a context of nonlinear single-input, single-output systems - of the problem considered (in a context of linear multi-input, multi-output systems) in its precursor [1].

2. The System Class

We proceed to make precise the class of systems. In particular, we consider single-input, single-output systems described by a functional differential equation of the form

$$\dot{y}(t) = f\big(d(t), (Ty)(t)\big) + g\big(u(t)\big), \qquad y|_{[-h,0]} = y^0 \in C[-h,0], \tag{2.1}$$

wherein $h \geq 0$, the functions $f\colon \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous, $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ is a disturbance, and the operator T satisfies the following.

(T) $T\colon C[-h,\infty) \to L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^q)$ is a causal operator with the properties:

(TB) there exists $\eta \in C(\mathbb{R}_+)$ such that, for all $c_1, \omega > 0$ and all $y \in C[-h,\omega)$,

$$\sup_{t\in[-h,\omega)} |y(t)| \leq c_1 \implies \sup_{t\in[0,\omega)} |(Ty)(t)| \leq \eta(c_1)$$

(TL) for all $t \geq 0$ and all $w \in C[-h,t]$, there exist $\tau > t$ and $\delta, c_0 > 0$ such that

$$\text{ess-sup}_{s\in[t,\tau]} \|(Ty)(s) - (Tz)(s)\| \leq \; c_0 \max_{s\in[t,\tau]} |y(s) - z(s)| \quad \forall\, y, z \in \mathcal{C}(w; h, t, \tau, \delta)$$

where

$$\mathcal{C}(w; h, t, \tau, \delta) := \big\{ v \in C[-h,\tau] \;\big|\; v|_{[-h,t]} = w,\; |v(s) - w(t)| \leq \delta \;\; \forall\, s \in [t,\tau] \big\},$$

i.e., the space of all continuous extensions v of $w \in C[-h,t]$ to the interval $[-h,\tau]$ such that that $|v(s) - w(t)| \leq \delta$ for all $s \in [t,\tau]$.

The continuous input nonlinearity $g\colon \mathbb{R} \to \mathbb{R}$ is assumed to satisfy the following condition.

(g) g is non-decreasing with $g(0) = 0$ and

$$\gamma^+ := \sup_{v\geq 0} g(v) \in (0,\infty] \qquad \text{and} \qquad \gamma^- := -\inf_{v\leq 0} g(v) \in (0,\infty].$$

Some remarks on the above assumptions are warranted.

Remark 2.1 (On the operator T):

(i) The parameter $h \geq 0$ in the definition of T quantifies the memory in the system and permits the incorporation of delay elements.

(ii) Property (TL) is a technical assumption of local Lipschitz type which is required for well-posedness of the closed-loop system (by appealing to Theorem 7.1 of [3]).

(iii) To interpret (TB) and (TL) correctly, we need to give meaning to Ty, for a function $y \in C(I)$ on a bounded interval I of the form $[-h,\rho)$ or $[-h,\rho]$, where $0 < \rho < \infty$. This we do by showing that T "localizes", in a natural way, to an operator $\tilde{T}\colon C(I) \to L^\infty_{\rm loc}(J,\mathbb{R}^q)$, where $J := I \setminus [-h,0)$. Let $y \in C(I)$. For each $\sigma \in J$, define $y_\sigma \in C[-h,\infty)$ by

$$y_\sigma(t) := \begin{cases} y(t), & t \in [-h,\sigma], \\ y(\sigma), & t > \sigma\,. \end{cases}$$

By causality, we may define $\tilde{T}y \in L^\infty_{\rm loc}(J,\mathbb{R}^q)$ by the property $\tilde{T}y|_{[0,\sigma]} = Ty_\sigma|_{[0,\sigma]}$ for all $\sigma \in J$. Henceforth, we will not distinguish notationally an operator T and its "localisation" $\tilde{T}$: the correct interpretation being clear from context.

(iv) Property (TB) is a bounded-input, bounded-output assumption on T. In the context of the prototype system in Section 1, this property is a consequence of the ISS assumption. $\diamond$

Remark 2.2 (Examples for the input nonlinearity g): The prototype input nonlinearity is $g = \mathrm{sat}_{\hat{u}}$ as in (1.2), with $\hat{u} > 0$, in which case $\gamma^+ = \hat{u} = \gamma^-$. Note that the general form of g allows for saturation nonlinearities with non-symmetric bounds. $\diamond$

Example 2.3 (System (1.1) *revisited):* We show that the prototype system (1.1) can be written in the form (2.1) and that the conditions (T) and (g) are satisfied, assuming that the ISS estimate (1.4) holds. That condition (g) holds is an immediate consequence of Remark 2.2. Again, temporarily regarding the first subsystem in (1.1) in isolation with input $y \in C(\mathbb{R}_+)$, let $\varphi(\cdot\,; z^0, y)$ denote the unique maximal solution of the initial-value problem (1.3): in view of (1.4), we know that this solution is global (i.e. exists on $\mathbb{R}_+$). For each $z^0 \in \mathbb{R}^{n-1}$, we may define a causal operator $T_{z^0}\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+,\mathbb{R}^{n-1})$ by $\big(T_{z^0}(y)\big)(t) := \varphi(t; z^0, y)$ for all $t \in \mathbb{R}_+$. Introducing the causal operator

$$T\colon C(\mathbb{R}_+) \to C(\mathbb{R}_+,\mathbb{R}^n), \quad y \mapsto \big(y, \big(T_{z^0}(y)\big)\big),$$

and the function

$$f\colon \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}, \quad (\delta,\xi) = \big(\delta,(\theta,\zeta)\big) \mapsto f(\delta,\xi) := f_1(\delta,\theta,\zeta),$$

then original initial-value problem (1.1) may be expressed in the form in the form of the functional differential equation (2.1). It remains to show that the operator T satisfies conditions (TB) and (TL) of Assumption (T). Clearly, (TB) holds since, by virtue of the ISS estimate (1.4), we may infer that,

$$\forall c_1 > 0\ \forall y \in C(\mathbb{R}_+): \quad \|y\|_\infty \le c_1 \implies \|Ty\|_\infty \le \eta(c_1) := c_1 + \alpha(|z^0|, 0) + \beta(c_1). \tag{2.2}$$

It remains to show that T satisfies (TL). Let $w \in C(\mathbb{R}_+)$ and fix $t \ge 0$, $\delta > 0$ and $\tau > 0$ arbitrarily, and, for notational convenience, set $\Delta := \delta + \sup_{s\in[0,t]} |w(s)|$. Define the compact set

$$K := \left\{ (\rho, \theta) \in \mathbb{R} \times \mathbb{R}^{n-1} \,\middle|\, |\rho| \le \Delta,\ \|\theta\| \le \alpha(|z^0|, 0) + \beta(\Delta) \right\}.$$

By the local Lipschitz property of f_2, there exists $c > 0$ such that

$$\|f_2(\rho, \theta) - f_2(\varrho, \vartheta)\| \le c\big[|\rho - \theta| + \|\varrho - \vartheta\|\big] \qquad \forall\, (\rho, \theta), (\varrho, \vartheta) \in K.$$

Let $y_1, y_2 \in C(\mathbb{R}_+)$ be such that $y_1(s) = w(s) = y_2(s)$ for all $s \in [0, t]$, and $|y_1(s)|, |y_2(s)| \le \delta$ for all $s \in [t, t+\tau]$. Then

$$\begin{aligned}\|(T_{z^0} y_1)(s) - (T_{z^0} y_2)(s)\| &\le \int_0^s \|f_2(y_1(\sigma), z(\sigma; z^0, y_1)) - f_2(y_2(\sigma), z(\sigma; z^0, y_2))\| \mathrm{d}\sigma \\ &\le c \int_t^s \big[|y_1(\sigma) - y_2(\sigma)| + \|(T_{z^0} y_1)(\sigma) - (T_{z^0} y_2)(\sigma)\|\big] \mathrm{d}\sigma \quad \forall\, s \in [t, t+\tau].\end{aligned}$$

By Gronwall's Lemma, it follows that

$$\|(T_{z^0} y_1)(s) - (T_{z^0} y_2)(s)\| \le c \int_t^s e^{L(s-\sigma)} |y_1(\sigma) - y_2(\sigma)| \mathrm{d}\sigma \quad \forall\, s \in [t, t+\tau]$$

whence

$$\sup_{s\in[t,t+\tau]} \|(T_{z^0} y_1)(s) - (T_{z^0} y_2)(s)\| \le e^{c\tau} \sup_{s\in[t,t+\tau]} |y_1(s) - y_2(s)|.$$

We may now conclude that property (TL) holds with $c_0 = 1 + e^{c\tau}$. ⋄

Example 2.4 (Examples for (2.1)*):*

(i) Linear systems, as investigated in the precursor [1], are encompassed by (2.1) if they are single-input, single-output.

(ii) In [2, Ex. 2.3], we have shown that a certain class of linear retarded minimum-phase systems with relative degree one can be written in the form (2.1) and satisfy (TB) and (TL).

(iii) In [3, Sec. 6.3], we have shown that systems with hysteresis can be written in the form (2.1) which satisfies (TB) and (TL).

(iv) In [4, Sec. 4] we have shown that infinite-dimensional regular linear systems and nonlinear delay systems can be written in the form (2.1) which satisfies (TB) and (TL). ⋄

3. The Main Result

We now arrive at the main result, the proof of which may be found in the Appendix.

Theorem 3.1: Let $\Lambda > 0$, $\lambda > 0$ and $\psi \in \mathcal{G}(\Lambda, \lambda)$ define the performance funnel $\mathcal{F}(\psi) = \{(t, e) \in \mathbb{R}_+ \times \mathbb{R} | \ |e| < \psi(t)\}$. Let $h \geq 0$, $y^0 \in C([-h, 0])$, $d \in L^\infty(\mathbb{R}_+, \mathbb{R}^p)$ and $r \in W^{1,\infty}(\mathbb{R}_+)$. Consider a system of form (2.1) satisfying (T), (TB) (with associated function $\eta \in C(\mathbb{R}_+)$), (TL) and (g). Define

$$L := \sup\left\{ |f(\rho, \sigma)| \,\middle|\, (\rho, \sigma) \in \mathbb{R}^p \times \mathbb{R}^q,\ \|\rho\| \leq \|d\|_\infty,\ |\sigma| \leq \eta(\|y^0\|_\infty + \|\psi\|_\infty + \|r\|_\infty) \right\}, \quad (3.1)$$

If the initial data y^0 and the reference signal r are such that

$$|y^0(0) - r(0)| < \psi(0), \quad (3.2)$$

and the feasibility condition

$$\gamma := \max\{\gamma^-, \gamma^+\} > L + \Lambda + \|\dot{r}\|_\infty =: \Gamma, \quad (3.3)$$

holds, then application of the feedback strategy

$$u(t) = -k(t)\, e(t), \qquad k(t) = \frac{1}{\psi(t) - |e(t)|}, \qquad e(t) = y(t) - r(t) \quad (3.4)$$

to (2.1) yields a closed-loop initial-value problem with the following properties.

(a) The closed-loop initial-value problem (2.1), (3.4) has a solution $y : [-h, \omega) \to \mathbb{R}$ and every solution can be extended to a global solution, i.e. $\omega = \infty$.

(b) There exists $\varepsilon > 0$ such that every global solution y satisfies

$$|y(t) - r(t)| \leq \psi(t) - \varepsilon \quad \forall\, t \geq 0.$$

(c) The function $u(\cdot) := -(y(\cdot) - r(\cdot))/(\psi(\cdot) - |y(\cdot) - r(\cdot)|)$ is bounded and the following hold:

  (i) $|g(u(\tau))| < \gamma$ for some $\tau \in \mathbb{R}_+$.

  (ii) $\big[\exists\, \tau \geq 0 : \ |g(u(\tau))| < \gamma\big] \implies \big[\ |g(u(t))| < \gamma \ \ \forall\, t \in [\tau, \infty)\big]$.

Remark 3.2 (Existence of solution): In view of the potential singularity in (3.4), some care is required in formulation the closed-loop initial-value problem (2.1), (3.4). Define

$$\mathcal{D} := \{(t, v) \in \mathbb{R}_+ \times \mathbb{R} | \ \ (t, v - r(t)) \in \mathcal{F}(\psi)\} \quad (3.5)$$

and define $F\colon \mathcal{D}\times\mathbb{R}^q \to \mathbb{R}$ by

$$F(t,v,w) := f(d(t),w) + g\big(-(v-r(t))/(\psi(t)-|v-r(t)|)\big), \tag{3.6}$$

in which case, the closed-loop, initial-value problem (2.1), (3.4) is formulated as

$$\dot y(t) = F(t,y(t),(Ty)(t)), \quad y|_{[-h,0]} = y^0\,. \tag{3.7}$$

By a *solution* of (3.7) we mean a function $y\in C[-h,\omega)$, $0<\omega\le\infty$, such that $y|_{[-h,0]} = y^0$, $y|_{[0,\omega)}$ is locally absolutely continuous, with $(t,y(t))\in\mathcal{D}$ for all $t\in[0,\omega)$ and $\dot y(t) = F(t,y(t),(Ty)(t))$ for almost all $t\in[0,\omega)$. A solution is said to be *maximal* if it has no proper right extension that is also a solution. A solution defined on $[-h,\infty)$ is said to be *global*.
That (3.7) has a solution, and that every solution can be extended to a maximal solution, is an immediate consequence of [3, Th. 7.1] which also implies that, if $y\in C[-h,\omega)$ is a maximal solution, then the closure of graph$\big(y|_{[0,\omega)}\big) = \{(t,y(t)) \mid t\in[0,\omega)\}\subset\mathcal{D}$ is not a compact subset of $\mathcal{D}$. $\diamond$

Remark 3.3 (Comments on Theorem 3.1):

(i) Assertion (b) is the essence of the result: it asserts that, if (3.2) and the feasibility condition (3.3) hold, then the funnel control (3.4) ensures achievement of the control objectives; in particular, the tracking error $e=y-r$ remains uniformly bounded away from the funnel boundary and the gain function k is bounded, with $\|k\|_\infty \le 1/\varepsilon$.

(ii) Assertion (c) has non-trivial content only in the case wherein γ is finite and either the supremum γ^+ or the infimum $-\gamma^-$ of g is attained, that is, the case wherein the input may saturate (the prototype being the saturation function $g=$ sat$_{\hat u}$). Assertion (c)(i) implies that the control input cannot remain saturated for all $t\ge 0$ and, when it becomes unsaturated, then Assertion (c)(ii) implies that the signal remains unsaturated thereafter. If the initial data is such that the signal $g(u(\cdot))$ is initially unsaturated, i.e. $|g(u(0))|<\gamma$, then the saturation bound is never attained (see Assertion (c)(ii)). If, on the other hand, the signal $g(u(\cdot))$ is initially saturated, i.e. $|g(u(0))|=\gamma$, then the conjunction of Assertions (c)(i) and (c)(ii) ensures that it remains so only on a finite interval $[0,\tau]$ and is unsaturated on (τ,∞).

(iii) The condition (3.2) is necessary for attainment of the control objective and is equivalent to the requirement that $(0,y^0(0))\in\mathcal{D}$.

The feasibility condition (3.3) is a sufficient condition for attainment of the control objective (of course, in the case $\gamma = \infty$, i.e. in the absence of saturation, (3.3) holds trivially). It quantifies and exhibits the interplay between the saturation bound γ (sufficiently large to ensure performance) and bounds of the plant data, funnel data, initial data, reference signal data and disturbance signal data. The nature of the dependence of the saturation bound on these data is not surprising. For example:

1) it is to be expected that tracking of "large and rapidly varying" reference signals r would require control inputs capable of taking sufficiently large values – this is reflected in the dependence of the saturation bound on both $\|r\|_\infty$ and $\|\dot{r}\|_\infty$;

2) transient and asymptotic behaviour of the tracking error is influenced by the choice of funnel $\mathcal{F}(\psi)$ determined by the globally Lipschitz function ψ – a stringent requirement that transient behaviour decays rapidly would be reflected in a large Lipschitz constant Λ which, not unexpectedly, appears in the feasibility condition;

3) it is to be expected that the saturation bound depends on the disturbance signal d – this is reflected in its dependence on $\|d\|_\infty$. ◇

4. SIMULATIONS

For purposes of illustration, consider the single-input, single-output system (1.5) subject to the saturation contant $\widehat{u} := 25$. Example 2.3 shows that the systems (1.5) can be written in the form (2.1) and that the conditions (T) and (g) are satisfied.

As reference signal we choose $r(\cdot) = \xi_1(\cdot)$ the first component of the solution of the Lorentz system

$$\dot{\xi}_1 = \xi_2 - \xi_1, \qquad \dot{\xi}_2 = (28\xi_1/10) - (\xi_2/10) - \xi_1\xi_3, \qquad \dot{\xi}_3 = \xi_1\xi_2 - (8\xi_3/30),$$

with the initial values $(\xi_1(0), \xi_2(0), \xi_3(0)) = (1, 0, 3)$. It is shown in [6, App. C] that the solutions are chaotic and bounded. This yields a bounded r with bounded derivative. Note that $r(0) = 1$ and numerical simulations show $\|r\|_\infty \le 9/5$, and $\|\dot{r}\|_\infty \le 6/5$.

As disturbance signal we choose $d(\cdot) = -\xi_2(\cdot)$; again, numerical simulations show $\|d\|_\infty \le 2.4$.

Setting $\lambda = 0.1$, the funnel $\mathcal{F}(\psi)$ is determined by

$$\psi\colon \mathbb{R}_+ \to \mathbb{R}\,, \quad \psi(t) := \begin{cases} 2e^{-0.1\,t} & , \ t \in [0,\ 10\ln 20] \\ \max\{3/5\ \cos(t/3),\ \lambda\} & , \text{ else} \end{cases} \tag{4.1}$$

Then $\|\psi\|_\infty = 2$ and it prescribes an exponential (exponent 0.1) decay of the tracking error in the transient phase $[0,T]$, where $T = 10\ln 20 \approx 30$, and a tracking accuracy quantified by $\lambda = 0.1$ and $3/5\,\cos(t/3)$ thereafter; ψ is non-monotone with global Lipschitz constant $\Lambda = 0.2$.

For $a = b = 1$ and $z^0 = 1$ and L as in Example 1.2, we have

$$L + \|\dot r\|_\infty + \Lambda = 24.68 < 25 = \widehat{u}$$

and so the feasibility condition (3.3) is satisfied. The condition (3.2), i.e. $|e(0)| = |y^0 - r(0)| < 2$, implies $y^0 \in (-1,3)$. To illustrate the occurrence of saturation of the control input, we choose y^0 to be such that Assertion (c)(ii) fails to hold for $\tau = 0$ (in which case, there exists $\tau > 0$ such that the control u is saturated on $[0,\tau)$). Note that

$$|\text{sat}_{\widehat{u}}(u(0))| < \widehat{u} \qquad \Longleftrightarrow \qquad |y^0 - r(0)| < \frac{\psi(0)\widehat{u}}{1+\widehat{u}}$$

and so the input is saturated at the beginning if, and only if, $|e(0)| = |y^0 - r(0)| \geq 25/13$. Hence we choose $y^0 = -0.95$, and so $\varepsilon = \lambda/(2\widehat{u}) = 0.002$.

Figure 4.2 depicts the behaviour of the closed-loop system (1.5), (3.4). The simulations confirm the result of Theorem 3.1: the tracking error remains uniformly bounded away from the funnel boundary; moreover, the second picture suggests that the calculated bound $\varepsilon = 0.002$ is conservative. Non-monotonicity of gain function k is also evident: it increases when the error approaches the funnel boundary and decreases when the error recedes from the boundary. The third row of pictures confirms that the input is initially saturated: it remains so on an interval of short duration and remains unsaturated thereafter. The last picture shows the disturbance signal and the z-component of (1.5).

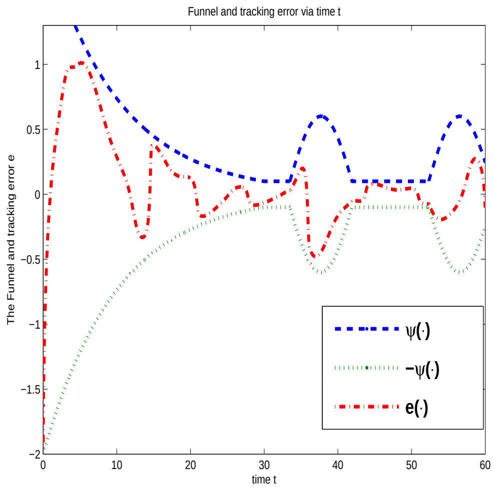
5. Appendix: Proof of Theorem 3.1

In view of Remark 3.2, we know that there exists a solution of the closed-loop system and every solution can be extended to a maximal solution. Let $y\colon [-h,\omega) \to \mathbb{R}$, $0 < \omega \leq \infty$, be any maximal solution of (2.1), (3.4), and set

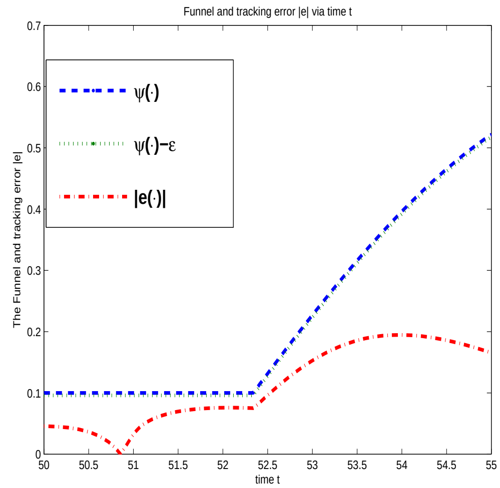
$$e(t) := y(t) - r(t), \quad k(t) := \frac{1}{\psi(t) - |e(t)|}, \quad u(t) = -k(t)e(t) \ \ \forall\, t \in [0,\omega).$$

Step 1: We show that the tracking error e satisfies

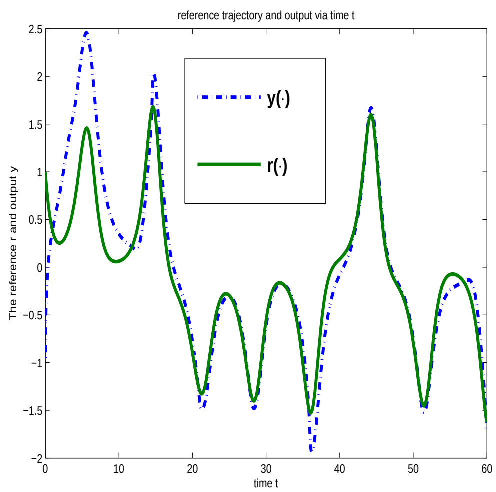
$$e(t)\dot e(t) \leq -|e(t)|\big(\Lambda - \Gamma + |g(u(t))|\big) \quad \text{for almost all } t \in [0,\omega). \tag{5.1}$$



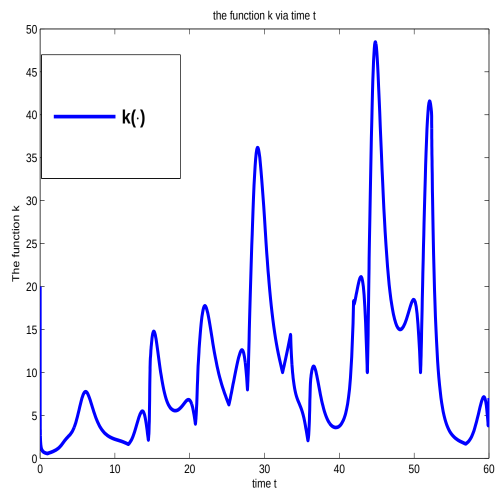
Funnel and tracking error e



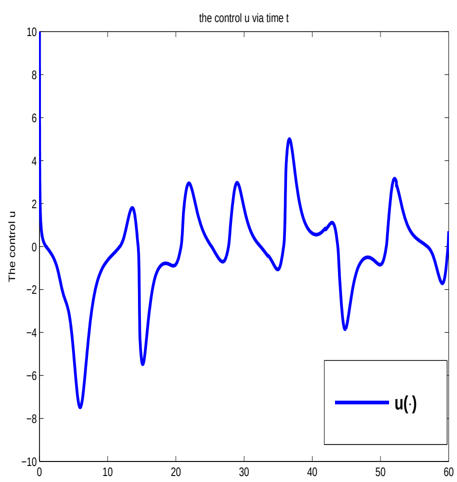
Funnel and tracking error $|e|$ - zoomed in



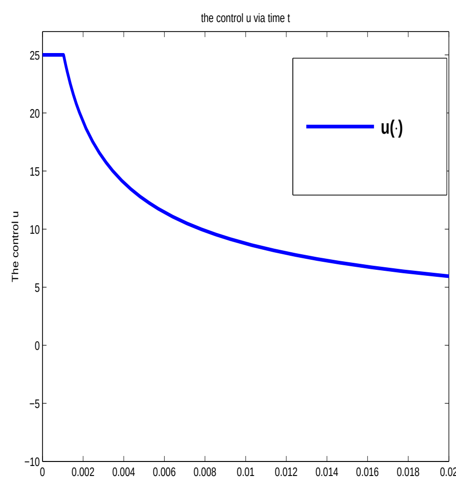
Reference signal r and output y



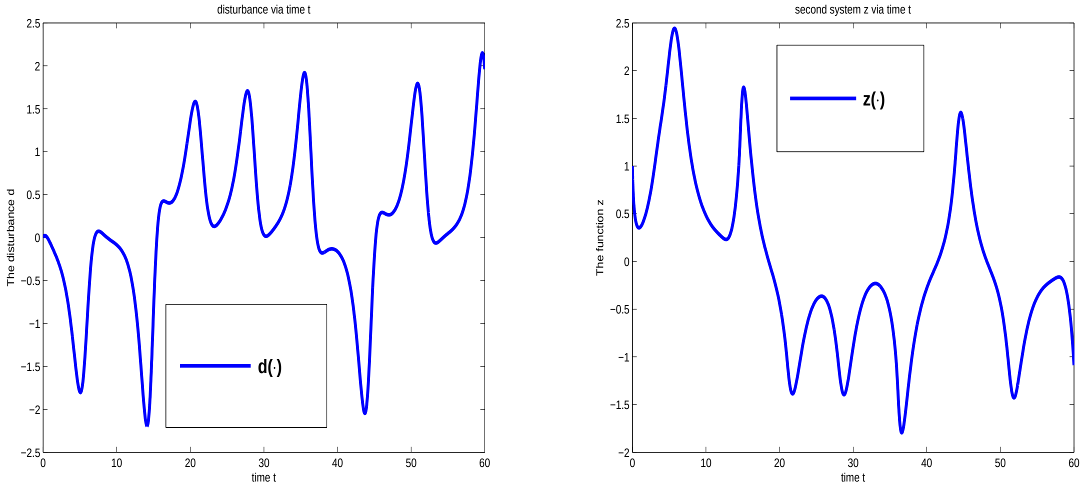
Gain function k



Control u



Control u - zoomed in



Disturbance signal d Function z

Fig. 4.2. Behaviour of the closed-loop system (1.5), (3.4)

Observe that

$$|y(t)| \le |e(t)| + |r(t)| \le \|\psi\|_\infty + \|r\|_\infty \qquad \forall\, t \in [0,\omega),$$

and so

$$|y(t)| \le c_1 := \|y^0\|_\infty + \|\psi\|_\infty + \|r\|_\infty \qquad \forall\, t \in [-h,\omega).$$

By Property (TB), we may infer that $\|(Ty)(t)\| \le \eta(c_1)$ for almost all $t \in [0,\omega)$, and so (3.1) yields

$$|f\big(d(t),(Ty)(t)\big)| \le L \quad \text{for almost all } t \in [0,\omega).$$

Therefore, we have, for almost all $t \in [0,\omega)$,

$$e(t)\dot e(t) \;=\; e(t)\left(f\big(d(t),(Ty)(t)\big) + g(u(t)) - \dot r(t)\right) \;\le\; |e(t)|\big(L + \|\dot r\|_\infty - |g(u(t))|\big),$$

and, since $L + \|\dot r\|_\infty = \Gamma - \Lambda$, (5.1) follows.

Step 2: Choose $\varepsilon > 0$ sufficiently small so that

$$\varepsilon \le \min\{\lambda/2\,,\, \psi(0) - |e(0)|\}, \quad g(\lambda/(2\varepsilon)) \ge \Gamma, \quad -g(-\lambda/(2\varepsilon)) \ge \Gamma. \tag{5.2}$$

We will show that

$$\psi(t) - |e(t)| \ge \varepsilon \qquad \forall\, t \in [0,\omega). \tag{5.3}$$

Seeking a contradiction, suppose that there exists $t_1 \in (0,\omega)$ such that $\psi(t_1) - |e(t_1)| < \varepsilon$. Since $\psi(0) - |e(0)| \ge \varepsilon$, the number $t_0 := \max\{t \in [0,t_1) \mid \; \psi(t) - |e(t)| = \varepsilon\} \in [0,t_1)$ is well defined.

It follows that $\psi(t) - |e(t)| \le \varepsilon$ for all $t \in [t_0, t_1]$ and so $|e(t)| \ge \psi(t) - \varepsilon \ge \lambda - \varepsilon \ge \lambda/2$ for all $t \in [t_0, t_1]$, whence

$$|u(t)| = k(t)\,|e(t)| \ge \frac{\lambda}{2\varepsilon} \qquad \forall\, t \in [t_0, t_1].$$

Therefore, by monotonicity of g and (5.2),

$$g(|u(t)|) \ge g(\lambda/(2\varepsilon)) \ge \Gamma \;\forall\, t \in [t_0, t_1] \quad \text{and} \quad -g(-|u(t)|) \ge -g(-\lambda/(2\varepsilon)) \ge \Gamma \;\; \forall\, t \in [t_0, t_1].$$

Noting that

$$|g(u(t))| = \begin{cases} g(|u(t)|), & \text{if } u(t) \ge 0 \\ -g(-|u(t)|), & \text{if } u(t) < 0\,, \end{cases} \tag{5.4}$$

and invoking (5.1), we may infer that

$$e(t)\dot{e}(t) \le -|e(t)|(\Lambda - \Gamma + \Gamma) = -\Lambda|e(t)| \;\; \forall\, t \in [t_0, t_1].$$

which, on integration, gives $|e(t_1)| - |e(t_0)| < -\Lambda\,|t_1 - t_0|$. We now arrive at the contradiction

$$\begin{aligned} 0 < \psi(t_0) - |e(t_0)| - (\psi(t_1) - |e(t_1)|) &\le |\psi(t_1) - \psi(t_0)| + |e(t_1)| - |e(t_0)| \\ &< \Lambda|t_1 - t_0| - \Lambda|t_1 - t_0| = 0, \end{aligned}$$

wherein we have invoked the global Lipschitz property of ψ. Therefore, (5.3) holds.

Step 3: We establish Assertions (a) and (b). In view of (5.2), boundedness of r implies boundedness of y. To establish Assertions (a), (b), it remains only to show that $\omega = \infty$. Seeking a contradiction, suppose that $\omega < \infty$. Then $\{(t,y) \in \mathcal{D} \mid \;\; t \in [0,\omega],\;\; \psi(t) - |y - r(t)| \ge \varepsilon\}$ is a compact subset of $\mathcal{D}$ and contains the graph of $y|_{[0,\omega)}$: this contradicts the fact that the closure of the graph is not a compact subset of $\mathcal{D}$. Therefore, $\omega = \infty$.

Step 4: We establish Assertion (c). Boundedness of u is an immediate consequence of Assertion (b). If $\gamma = \infty$, then Assertion (c) trivially holds. Assume $\gamma < \infty$.

Step 4a: First, we establish Assertion (c)(i). Seeking a contradiction, suppose $|g(u(t))| \ge \gamma$ for all $t \ge 0$. Recalling that $\gamma > \Gamma$ and invoking (5.1), we have $e(t)\dot{e}(t) \le -\Lambda|e(t)|$ for all $t \ge 0$, which, on integration, yields the contradiction:

$$0 \le |e(t)| \le |e(0)| - \Lambda t \;\; \forall\, t \ge 0.$$

Therefore, there exists $\tau \ge 0$ such that $|g(u(\tau))| < \gamma$. This establishes Assertion (c)(i).

Step 4b: Next, we show Assertion (c)(ii). Let $\tau \in \mathbb{R}_+$ be such that $|g(u(\tau))| < \gamma$. Again seeking a contradiction, suppose that $|g(u(t))| \geq \gamma$ for some $t \in [\tau, \infty)$. Define $t_1 := \min\{t \in [\tau, \infty) | \ |g(u(t))| = \gamma\}$. Choose $\rho \in [\Gamma, \gamma)$ such that $|g(u(\tau))| \leq \rho$ and define $t_0 := \max\{t \in [\tau, t_1) | \ |g(u(t))| = \rho\}$. Observe that

$$\tau \leq t_0 < t_1 \quad \text{and} \quad |g(u(t))| \geq \rho \geq \Gamma \ \ \forall\, t \in [t_0, t_1].$$

Therefore, $|u(t)| > 0$ for all $t \in [t_0, t_1]$ and so, by continuity of u, we have $u(t) = c_0|u(t)|$ for all $t \in [t_0, t_1]$, where $c_0 := \text{sgn}(u(t_0))$. By (5.4), it now follows that

$$|g(u(t))| = c_0 g(c_0|u(t)|) \ \ \forall\, t \in [t_0, t_1]. \tag{5.5}$$

Invoking (5.1), we have $e(t)\dot{e}(t) \leq -\Lambda|e(t)|$ for all $t \geq [t_0, t_1]$, which, on integration, yields $|e(t_1)| - |e(t_0)| \leq -\Lambda|t_1 - t_0|$. The latter inequality in conjunction with the global Lipschitz property of ψ gives

$$\psi(t_0) - |e(t_0)| - (\psi(t_1) - |e(t_1)|) \leq |\psi(t_1) - \psi(t_0)| - \Lambda|t_1 - t_0| \leq 0.$$

Therefore,

$$|u(t_1)| = \frac{|e(t_1)|}{\psi(t_1) - |e(t_1)|} < \frac{|e(t_0)|}{\psi(t_0) - |e(t_0)|} = |u(t_0)|,$$

which, in conjunction with (5.5) and monotonicity of g, yields the contradiction:

$$\gamma = |g(u(t_1))| = c_0 g(c_0|u(t_1)|) \leq c_0 g(c_0|u(t_0)|) = |g(u(t_0))| = \rho < \gamma.$$

Therefore, $|g(u(t))| < \gamma$ for all $t \in [\tau, \infty)$. This completes the proof. ■

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