Strict Betweennesses induced by Posets as well as by Graphs

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Abstract

For a finite poset $P = (V, \leq)$, let $\mathcal{B}_s(P)$ consist of all triples $(x, y, z) \in V^3$ such that either $x < y < z$ or $z < y < x$. Similarly, for every finite, simple, and undirected graph $G = (V, E)$, let $\mathcal{B}_s(G)$ consist of all triples $(x, y, z) \in V^3$ such that y is an internal vertex on an induced path in G between x and z. The ternary relations $\mathcal{B}_s(P)$ and $\mathcal{B}_s(G)$ are well-known examples of so-called strict betweennesses. We characterize the pairs (P, G) of posets P and graphs G on the same ground set V which induce the same strict betweenness relation $\mathcal{B}_s(P) = \mathcal{B}_s(G)$.

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1 Introduction

The axiomatic study and formalization of what *betweenness* should mean as a mathematical term goes back to Huntington and Kline [8] in 1917. Two prominent examples of such betweennesses are those induced by metrics studied by Menger [11] in 1928 and those induced by posets studied by Birkhoff [2] in 1948. While Altwegg [1] provided a complete axiomatic description of the latter kind of betweennesses which was generalized by Sholander [13] and recently by Düntsch and Urquhart [6], a similar result is unknown for the former kind (see Chvátal [3] for a detailed discussion).

In the present paper we consider so-called *strict betweennesses* on a finite ground set V defined as a ternary relation $\mathcal{B}_s \subseteq V^3$ on V such that $(x, y, z) \in \mathcal{B}_s$ implies that x, y, and z are pairwise distinct and that $(z, y, x) \in \mathcal{B}_s$. Two natural examples of strict betweennesses discussed by Chvátal in [4] are derived from posets and graphs.

For a finite poset $P = (V, \leq)$, Lihová [10] defines the *strict order betweenness* as

$$\mathcal{B}_s(P) = \left\{(x, y, z) \in V^3 \mid x < y < z \text{ or } z < y < x\right\}.$$

Using Altwegg's result [1], she gives a complete axiomatic description of strict order betweennesses in [10].

For a finite, simple, and undirected graph $G = (V, E)$, the *strict induced path betweenness* is defined as

$$\mathcal{B}_s(G) = \left\{(x, y, z) \in V^3 \mid y \text{ is an internal vertex on an induced path in } G \text{ between } x \text{ and } z\right\}.$$

Convexity notions based on induced paths were studied by Jamison-Waldner [9] and Duchet [5].

In the present note we consider the situation when these two examples of strict betweennesses coincide. More specifically, we characterize the pairs (P, G) of posets P and graphs G on the same ground set V which induce the same strict betweenness relation $\mathcal{B}_s(P) = \mathcal{B}_s(G)$. After introducing some terminology and preliminary results in Section 2 we prove our main result in Section 3.

2 Some Terminology and Preliminaries

In the sequel all posets, graphs, and digraphs will be finite. Furthermore, all graphs and digraphs will be simple.

Let $P = (V, \leq)$ be a poset. Let u and v be in V. If $u \leq v$ and $u \neq v$, then we write $u < v$. If either $u \leq v$ or $v \leq u$, then u and v are called *comparable*. The *Hasse diagram* $\mathcal{H}(P)$ of P is the digraph with vertex set V where (u, w) is an arc of $\mathcal{H}(P)$ if and only if $u < w$ and there is no element $v \in V$ with $u < v < w$. The vertex set of a component of the underlying undirected graph of the Hasse diagram $\mathcal{H}(P)$ is called *a weak component of* P. A poset is called *weakly connected* if it has exactly one weak component. A poset $P' = (V, \leq')$ is said to arise *by an inversion of a weak component of* P if there is some weak component U of P and $\leq' = (\leq \setminus (U \times U)) \cup \{u \leq v \mid u, v \in U \wedge v \leq u\}$. Note that $\mathcal{B}_s(P) = \mathcal{B}_s(P')$ in this case. If $P = (V, \leq)$ is a poset, $G = (V, E)$ is a graph, $D = (V, A)$ is a digraph, and U is a subset of V, then the subposet $P[U]$ of P induced by U is $\left(U, \leq \cap U^3\right)$, the subgraph $G[U]$ of G induced by U is $\left(U, E \cap \binom{U}{2}\right)$ where $\binom{U}{2}$ denotes the set of all 2-element subsets of U, and the subdigraph $D[U]$ of D induced by U is $(U, A \cap (U \times U))$.

Clearly, some relations of a poset as well as some edges of a graph may be irrelevant for the induced betweennesses. Therefore, it suffices to consider suitably reduced posets and graphs. A poset P is *reduced* if every arc of its Hasse diagram $\mathcal{H}(P)$ is contained in a directed path of order 3. Similarly, a graph G is *reduced* if no component of G of order at least two is complete. We summarize some simple observations concerning reduced posets and graphs.

Proposition 1 *(i) For every poset $P = (V, \leq)$, there is a reduced poset $P' = (V, \leq')$ with $\leq' \subseteq \leq$ and $\mathcal{B}_s(P) = \mathcal{B}_s(P')$. Furthermore, a reduced poset is uniquely determined by its strict order betweenness up to inversions of weak components.*

(ii) For every graph $G = (V, E)$, there is a reduced graph $G' = (V, E')$ with $E' \subseteq E$ and $\mathcal{B}_s(G) = \mathcal{B}_s(G')$. Furthermore, a reduced graph is uniquely determined by its strict order betweenness.

Proof: (i) Let the digraph H' arise from the Hasse diagram $\mathcal{H}(P)$ of P by deleting all arcs which do not belong to directed paths of order 3. The poset P' whose Hasse diagram is H' has the desired properties.

Let $P = (V, \leq)$ be a reduced poset. Let G denote the underlying undirected graph of the Hasse diagram $\mathcal{H}(P) = (V, A)$. By definition, uv is an edge of G if and only if there is no element $x \in V$ with $(u, x, v) \in \mathcal{B}_s(P)$ and there is some element $y \in V$ with either $(u, v, y) \in \mathcal{B}_s(P)$ or $(y, u, v) \in \mathcal{B}_s(P)$. Therefore, $\mathcal{B}_s(P)$ uniquely determines G. Let uv, vw be two distinct incident edges of G. Since

$$(((u, v), (v, w) \in A) \vee ((v, u), (w, v) \in A)) \Leftrightarrow (u, v, w) \in \mathcal{B}_s(P),$$

P is uniquely determined by $\mathcal{B}_s(P)$ up to inversions of weak components.

(ii) The graph which arises from G by deleting all edges which belong to complete components has the desired properties.

In order to prove the uniqueness, let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two graphs with $\mathcal{B}_s(G_1) = \mathcal{B}_s(G_2)$. For contradiction, we assume that $uv \in E_1 \setminus E_2$.

If uv belongs to an induced path uvw in G_1, then $(u, v, w) \in \mathcal{B}_s(G_1)$. Hence G_2 contains an induced path P between u and w such that v is an internal vertex of P. Since $uv \notin E_2$, there is a vertex x on P between u and v and $(u, x, v) \in \mathcal{B}_s(G_2) \setminus \mathcal{B}_s(G_1)$ which is a contradiction. Hence, we may assume that uv does not belong to an induced path of order 3. This implies that $N_{G_1}[u] = N_{G_1}[v]$.

If u and v have two non-adjacent common neighbours, say x and y, then $(x, u, y), (x, v, y) \in \mathcal{B}_s(G_1)$. This implies that G_2 contains two — not necessarily distinct — induced paths between x and y which contain u and v as internal vertices, respectively. Hence G_2 contains a path between u and v. Since $uv \notin E_2$, there is a vertex $x \in V$ with $(u, x, v) \in \mathcal{B}_s(G_2) \setminus \mathcal{B}_s(G_1)$ which is a contradiction. Hence all common neighbours of u and v are adjacent.

Since G_1 is reduced, some vertex in $N_{G_1}[u]$, say x, has a neighbour, say y, which does not belong to $N_{G_1}[u]$. Since uxy and vxy are induced paths in G_1, we have $(u, x, y), (v, x, y) \in \mathcal{B}_s(G_1)$. This implies that G_2 contains an induced path between u and y and an induced path between v and y. Hence G_2 contains a path between u and v. Since $uv \notin E_2$, there is a vertex $z \in V$ with $(u, z, v) \in \mathcal{B}_s(G_2) \setminus \mathcal{B}_s(G_1)$ which is a contradiction. This completes the proof. □

Note that the proof of Proposition 1 (i) immediately yields an efficient algorithm to reconstruct a poset — up to inversions of weak components — from its strict order betweenness. Since the strict order betweenness of a poset can be constructed in polynomial time, this also yields an efficient and constructive algorithm to check whether a given betweenness is a strict order betweenness.

For graphs the situation is different. The proof of Proposition 1 (ii) does not immediately provide an efficient algorithm to reconstruct a graph from its strict induced path betweenness. Nevertheless, if $G = (V, E)$ is a graph, E' denotes the set of edges of G which belong to an induced path of order 3, and $E'' = E \setminus E'$, then it is easy to see that for $u, v \in V$ with $u \neq v$ we have

- $uv \in E'$ if and only if there is no $x \in V \setminus \{u, v\}$ with $(u, x, v) \in \mathcal{B}_s(G)$ and there is some $y \in V \setminus \{u, v\}$ with either $(u, v, y) \in \mathcal{B}_s(G)$ or $(y, u, v) \in \mathcal{B}_s(G)$ and
- $uv \in E''$ if and only if $uv \notin E'$, u and v belong to the same component of (V, E'), and there is no $x \in V \setminus \{u, v\}$ with $(u, x, v) \in \mathcal{B}_s(G)$.

These observations — which also allow an alternative uniqueness proof for the reduced graph in Proposition 1 — yield an efficient algorithm to reconstruct a graph from its strict induced path betweenness. Unfortunately, given a graph G and three distinct vertices x, y, and z, it is a NP-complete problem to decide whether G contains an induced path between x and z which contains y as an internal vertex [7], i.e. given a graph G, we can most likely not construct its strict induced path betweenness in polynomial time.

3 Posets P and Graphs G with $\mathcal{B}_s(P) = \mathcal{B}_s(G)$

A weak component U of a reduced poset $P = (V, \leq)$ is called *layered* if there is a partition

$$U = U_1 \cup U_2 \cup \ldots \cup U_l \tag{1}$$

of U such that

$$\mathcal{H}(P[U]) = \left(U, \bigcup_{i=1}^{l-1} U_i \times U_{i+1} \right). \tag{2}$$

Similarly, a component of a reduced graph $G = (V, E)$ with vertex set U is called *layered* if there is a partition of U as in (1) such that

$$G[U] = \left(U, \bigcup_{i=1}^{l-1} \binom{U_i \cup U_{i+1}}{2} \right). \tag{3}$$

Note that, since P or G is reduced, either $|U| = 1$ or $l \geq 3$.

The following is our main result.

Theorem 2 *If $P = (V, \leq)$ is a reduced poset and $G = (V, E)$ is a reduced graph, then $\mathcal{B}_s(P) = \mathcal{B}_s(G)$ if and only if*

(i) a subset of V is a weak component of P if and only it is the vertex set of a component of G and

(ii) for every weak component U of P there is a partition of U as in (1) such that (2) and (3) hold simultaneously.

Before we proceed to the proof of Theorem 2 we establish a series of lemmas.

Lemma 3 *If U is a weak component of a reduced layered poset $P = (V, \leq)$ and $U = U_1 \cup U_2 \cup \ldots \cup U_l$ is a partition of U such that (2) holds, then the graph $G[U]$ as in (3) is the unique reduced graph with $\mathcal{B}_s(P[U]) = \mathcal{B}_s(G[U])$.*

Proof: Since the result is trivial for $|U| = 1$, we may assume that $l \geq 3$.

Since it is straightforward to verify that the graph $G[U]$ as in (3) is reduced and satisfies $\mathcal{B}_s(P[U]) = \mathcal{B}_s(G[U])$, we proceed to the proof of the uniqueness of $G[U]$. Therefore, let $G' = (U, E')$ be a reduced graph with $\mathcal{B}_s(P[U]) = \mathcal{B}_s(G')$.

If $1 \leq i \leq l-2$ and $v_j \in U_j$ for $j \in \{i, i+1, i+2\}$, then $(v_i, v_{i+1}, v_{i+2}) \in \mathcal{B}_s(P)$. Furthermore, there is no $v \in V$ such that either $(v_i, v, v_{i+1}) \in \mathcal{B}_s(P)$ or $(v_{i+1}, v, v_{i+2}) \in \mathcal{B}_s(P)$. Hence $v_i v_{i+1} v_{i+2}$ is an induced path in G'. This implies that G' contains all edges of the form uv with $u \in U_i$ and $v \in U_{i+1}$ for some $1 \leq i \leq l-1$.

If $|U_i| \geq 2$ for some $1 \leq i \leq l-1$, $v_i, v_i' \in U_i$, and $v_{i+1} \in U_{i+1}$, then $(v_i, v_{i+1}, v_i') \not\in \mathcal{B}_s(P)$. Hence $v_i v_{i+1} v_i'$ is no induced path in G'. Since $v_i v_{i+1}$ and $v_i' v_{i+1}$ are edges of G', this implies that $v_i v_i'$ is an edge of G'. By symmetry, this implies that G' contains all edges of the form uv with $u, v \in U_i$ and $u \neq v$ for some $1 \leq i \leq l$, i.e. G' contains the graph $G[U]$ as a subgraph.

If $uv \in E'$ for some $u \in U_i$ and $v \in U_j$ with $j - i > 2$ and $u' \in U_{i+1}$, then $u < u' < v$ and hence $(u, u', v) \in \mathcal{B}_s(P)$. This implies that G' contains an induced path between u and v which has at least one internal vertex. Therefore, u and v are not adjacent in G'. By symmetry, this implies that G' coincides with $G[U]$. □

We define some specific small digraphs which will play a central role (cf. Figure 1).

$$\begin{aligned} H_1 &= (\{x_1, x_2, y_1, y_2, z\}, \{(x_1, x_2), (y_1, y_2), (x_2, z), (y_2, z)\}), \\ H_2 &= (\{x_1, x_2, y_1, y_2, z\}, \{(x_1, x_2), (y_1, y_2), (y_1, x_2), (x_2, z), (y_2, z)\}), \\ H_3 &= (\{x_1, x_2, x_3, x_4, y\}, \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (y, x_3)\}), \\ H_4(l) &= (\{x_0, x_1, \ldots, x_l, y\}, \{(x_0, x_1), (x_1, x_2), \ldots, (x_{l-1}, x_l), (x_0, y), (y, x_l)\}) \\ & \quad \text{for } l \geq 3. \end{aligned}$$

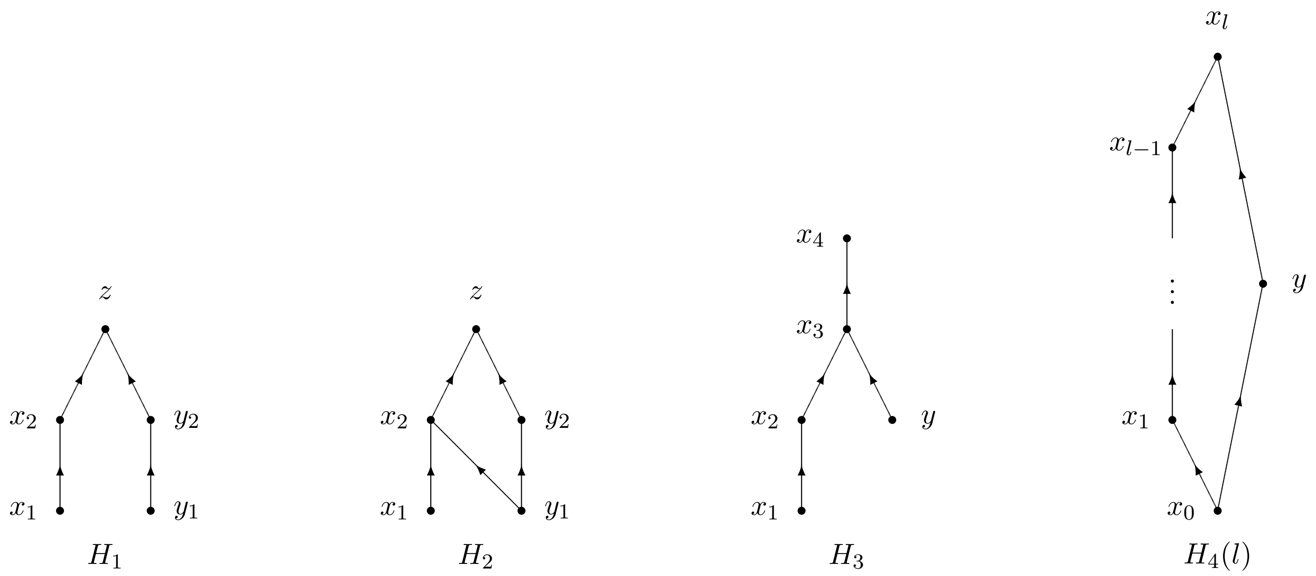

Figure 1 The digraphs H_1, H_2, H_3, and $H_4(l)$.

For a digraph $H = (V, A)$, let H^{-1} denote the digraph with the same vertex set V and arc set $A^{-1} = \{(v, u) \mid (u, v) \in A\}$.

Lemma 4 *If P is a reduced poset whose Hasse diagram $\mathcal{H}(P)$ belongs to*

$$\mathcal{H} = \{H_i, H_i^{-1} \mid 1 \leq i \leq 3\} \cup \{H_4(l) \mid l \geq 3\},$$

then there exists no graph G such that $\mathcal{B}_s(P) = \mathcal{B}_s(G)$.

Proof: We will only give details for H_1 and H_2. The remaining cases can be proved similarly and are left to the reader. Therefore, let P be such that $\mathcal{H}(P)$ is either H_1 or H_2. For contradiction, we assume the existence of a graph G with $\mathcal{B}_s(P) = \mathcal{B}_s(G)$.

Since $(x_1, x_2, z) \in \mathcal{B}_s(P)$ and there is no element x_2' different from x_2 such that $(x_1, x_2', z) \in \mathcal{B}_s(P)$, x_1x_2z is an induced path in G. Similarly, y_1y_2z is an induced path in G. Since $(x_2, z, y_2) \notin \mathcal{B}_s(P)$, x_2y_2 is an edge of G. Since $(x_1, x_2, y_2) \notin \mathcal{B}_s(P)$, x_1y_2 is an edge of G. Now $(x_1, y_2, z) \in \mathcal{B}_s(G) \setminus \mathcal{B}_s(P)$ which is a contradiction. □

Lemma 5 *Let $P = (V, \leq)$ be a reduced weakly connected poset. If P is not layered, then its Hasse diagram $\mathcal{H}(P) = (V, A)$ contains an induced subdigraph $H' = (V', A')$ such that*

(i) H' is isomorphic to one of the digraphs in $\mathcal{H}$ and

(ii) H' is the Hasse diagram of the subposet P' of P induced by V', i.e. $\mathcal{H}(P)[V'] = \mathcal{H}(P[V'])$.

Proof: We call an induced subdigraph H' of the Hasse diagram $\mathcal{H}(P)$ which satisfies (ii) *faithful*. For contradiction, we assume that P is a reduced weakly connected poset which is not layered and does not contain an induced subdigraph H' as specified in the statement, i.e. it does not contain a faithful induced subdigraph from $\mathcal{H}$.

For $x \in V$, let height(x) denote the maximum order of a chain in P ending in x. Note that height(x) coincides with the maximum order of a directed path in $\mathcal{H}(P)$ ending in x. Furthermore, note that height$(y) \geq$ height$(x) + 1$ for every arc (x, y) of $\mathcal{H}(P)$.

We consider two different cases.

Case 1 height$(y) >$ height$(x) + 1$ *for some arc (x, y) of $\mathcal{H}(P)$.*

Since height$(y) >$ height$(x) + 1$, a chain of maximum order ending in y also contains two elements u and v distinct from x such that (v, u) and (u, y) are arcs of $\mathcal{H}(P)$. Since $\mathcal{H}(P)$ is the Hasse diagram of P, x and u are incomparable and $x \not\leq v$. Since height$(y) >$ height$(x) + 1$, $v \not\leq x$, i.e. x and v are incomparable.

Since P is reduced, there is an element w such that either (w, x) or (y, w) is an arc of $\mathcal{H}(P)$.

If (y, w) is an arc of $\mathcal{H}(P)$, then $\mathcal{H}(P)$ contains H_3^{-1} as a faithful induced subdigraph, which is a contradiction. Hence (w, x) is an arc of $\mathcal{H}(P)$. Since $\mathcal{H}(P)$ does not contain H_1^{-1} or H_2^{-1} as a faithful induced subdigraph, v and w are comparable. Furthermore, since height$(y) >$ height$(x) + 1$, $w \leq v$. Let $w_0 w_1 \ldots w_r$ be a directed path in $\mathcal{H}(P)$ such that $w = w_0$ and $v = w_r$. Let the index i with $0 \leq i \leq r$ be maximum such that w_i is comparable with x. Clearly, i is well-defined and $i \leq r - 1$. Since height$(y) >$ height$(x) + 1$, $w_i \leq x$ and $\mathcal{H}(P)[\{x, y, u, w_i, w_{i+1}, \ldots, w_r\}]$ is isomorphic to $H_4(r - i + 2)$ with $r - i + 2 \geq 3$. This contradiction completes the proof in this case.

Case 2 height$(y) =$ height$(x) + 1$ *for every arc (x, y) of $\mathcal{H}(P)$.*

Since P is not layered, there are two elements x and y such that height$(y) =$ height$(x) + 1$ and (x, y) is no arc of $\mathcal{H}(P)$. We assume that x and y are chosen such that the distance between x and y in the underlying undirected graph G of $\mathcal{H}(P)$ is as small as possible. Let $W : x_1 x_2 \ldots x_l$ be a shortest path in G with $x = x_1$ and $y = x_l$. Note that $l \geq 4$.

If height$(x_2) =$ height$(x_1) - 1$ and height$(x_{l-1}) =$ height$(x_l) + 1$, then W contains a vertex x_i with $3 \leq i \leq l - 3$ such that height$(x_i) =$ height(x_1) and (x_i, y) is no arc of $\mathcal{H}(P)$. This contradicts the choice of x and y.

If height$(x_2) =$ height$(x_1) + 1$ and height$(x_{l-1}) =$ height$(x_l) + 1$, then the choice of x and y implies that $l = 4$ and (x_2, x_3) is an arc of $\mathcal{H}(P)$. Since P is reduced, there is an element z such that either (z, y) or (x_3, z) is an arc of $\mathcal{H}(P)$. In the first case $\mathcal{H}(P)$ contains either H_1 or H_2 as a faithful induced subdigraph and in the second case $\mathcal{H}(P)$ contains H_3 as a faithful induced subdigraph which is a contradiction. If height$(x_2) =$ height$(x_1) - 1$ and height$(x_{l-1}) =$ height$(x_l) - 1$, we can argue symmetrically.

Finally, if height$(x_2) =$ height$(x_1) + 1$ and height$(x_{l-1}) =$ height$(x_l) - 1$, then the choice of x and y implies that $l = 4$ and (x_3, x_2) is an arc of $\mathcal{H}(P)$. Since P is reduced, there are two not necessarily distinct elements z and z' such that either (x_2, z) and (y, z') are arcs of $\mathcal{H}(P)$ or (z, x) and (z', x_3) are arcs of $\mathcal{H}(P)$. In these cases $\mathcal{H}(P)$ contains one of the digraphs H_1, H_1^{-1}, H_2, and H_2^{-1} as an induced subdigraph. This final contradiction completes the proof. $\square$

Lemma 6 *Let $P = (V, \leq)$ be a reduced weakly connected poset. Let $H' = (V', A')$ be an induced subdigraph of its Hasse diagram $\mathcal{H}(P) = (V, A)$ such that H' is the Hasse diagram of the subposet P' of P induced by V', i.e. $\mathcal{H}(P)[V'] = \mathcal{H}(P[V'])$.*

If $G = (V, E)$ is a graph such that $\mathcal{B}_s(P) = \mathcal{B}_s(G)$, then the subgraph G' of G induced by V' satisfies $\mathcal{B}_s(P') = \mathcal{B}_s(G')$.

Proof: We prove the two inclusions $\mathcal{B}_s(P') \subseteq \mathcal{B}_s(G')$ and $\mathcal{B}_s(G') \subseteq \mathcal{B}_s(P')$.

Let $(x, y, z) \in \mathcal{B}_s(P')$. By definition, H' contains a directed path $v_0 v_1 \ldots v_l$ such that $\{x, z\} = \{v_0, v_l\}$ and $y = v_i$ for some $1 \leq i \leq l-1$. Since, for $0 \leq i \leq l-2$, $v_i v_{i+1} v_{i+2}$ is a directed path in H' and hence also in $\mathcal{H}(P)$, we have $(v_i, v_{i+1}, v_{i+2}) \in \mathcal{B}_s(P)$. This implies that G contains an induced path W_i between v_i and v_{i+2} with v_{i+1} as an internal vertex. Since (v_i, v_{i+1}) and (v_{i+1}, v_{i+2}) are arcs of the Hasse diagram $\mathcal{H}(P)$, W_i has length exactly 2, i.e. $W_i = v_i v_{i+1} v_{i+2}$. For contradiction, we assume that $v_1 v_2 \ldots v_l$ is not an induced path in $G' = G[V']$. Let $v_i v_j$ be an edge of G for some $0 \leq i, j \leq l$ with $j - i \geq 2$ such that $j - i$ is as small as possible. By the above observation, $j - i \geq 3$ which implies that $v_j v_i v_{i+1}$ is an induced path in G. Since $(v_j, v_i, v_{i+1}) \in \mathcal{B}_s(G) = \mathcal{B}_s(P)$ and $v_i < v_{i+1}$, this implies the contradiction $v_j < v_i$. Hence $v_1 v_2 \ldots v_l$ is an induced path in G' and thus $(x, y, z) \in \mathcal{B}_s(G')$.

For the converse, let $(x, y, z) \in \mathcal{B}_s(G')$. By definition, $G' = G[V']$ and hence also G contains an induced path between x and z containing y as an internal vertex. Since $\mathcal{B}_s(P) = \mathcal{B}_s(G)$, we obtain $(x, y, z) \in \mathcal{B}_s(P)$ and hence also $(x, y, z) \in \mathcal{B}_s(P')$. $\square$

After these preparations we are now in a position to prove our main result.

Proof of Theorem 2: The "if"-part of the statement follows easily from Lemma 3. We proceed to the proof of the "only if"-part of the statement. Therefore, let $P = (V, \leq)$ be a reduced poset and let $G = (V, E)$ be a reduced graph such that $\mathcal{B}_s(P) = \mathcal{B}_s(G)$.

Since P is reduced, if (u, v) is an arc or the Hasse diagram $\mathcal{H}(P)$ of P, then u and v both belong to some relation in $\mathcal{B}_s(P)$. This implies that u and v belong to the same component of G.

Conversely, let uv be an edge of G. If the edge uv belongs to an induced path of order 3, then u and v both belong to some relation in $\mathcal{B}_s(G)$ and u and v also belong to the same weak component of P. Hence, we may assume $N_G[u] = N_G[v]$. If u and v have two non-adjacent common neighbours, say x and y, then $(x, u, y), (x, v, y) \in \mathcal{B}_s(G)$ and u and v also belong to the same weak component of P. Hence, we may assume that all common neighbours of u and v are adjacent. Since G is reduced, some vertex in $N_G[u]$, say x, has a neighbour, say y, which does not belong to $N_G[u]$. We obtain $(u, x, y), (v, x, y) \in \mathcal{B}_s(G)$ and u and v also belong to the same weak component of P.

These two observations imply (i).

Let U be a weak component of P. Clearly, $\mathcal{B}_s(P) = \mathcal{B}_s(G)$ implies $\mathcal{B}_s(P[U]) = \mathcal{B}_s(G[U])$.

If $P[U]$ is not layered, then Lemma 5 implies that its Hasse diagram $\mathcal{H}(P[U])$ contains an induced subdigraph $H' = (V', A')$ such that H' is isomorphic to one of the digraphs in $\mathcal{H}$ and H' is the Hasse diagram of the subposet P' of $P[U]$ induced by V'. Since the Hasse diagram of P is the disjoint union of the Hasse diagrams of the posets induced by its weak components, $\mathcal{H}(P[U]) = \mathcal{H}(P)[U]$. Therefore, H' is an induced subdigraph of $\mathcal{H}(P)$ and H' is the Hasse diagram of the subposet P' of P induced by V'. Now Lemma 4 and Lemma 6 imply a contradiction. Hence $P[U]$ is layered.

Finally, Lemma 3 implies (ii) which completes the proof. $\square$

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