Finite Sholander Trees, Trees, and their Betweennesses

Dieter Rautenbach and **Philipp Matthias Schäfer**

Institut für Mathematik, TU Ilmenau, Postfach 100565, D-98684 Ilmenau, Germany,
emails: dieter.rautenbach@tu-ilmenau.de, philipp.schaefer@gmail.com

Abstract

We provide a proof of a claim made by Sholander (Trees, lattices, order, and betweenness, *Proc. Amer. Math. Soc.* **3**, 369-381 (1952)) concerning the representability of collections of so-called segments by trees. Furthermore, we strengthen Burigana's axiomatic characterization of so-called betweennesses induced by trees (Tree representations of betweenness relations defined by intersection and inclusion, *Mathematics and Social Sciences* **185**, 5-36 (2009)) and provide a short proof.

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1 Introduction

Trees are one of the most simple yet important classes of graphs with countless applications ranging from data structures and VLSI design over mathematical psychology to gardening. Here we pick up two previous and closely related papers on trees. One [11] relatively old by Sholander in 1952 and one [3] quite recent by Burigana in 2009. Both papers propose axiomatic characterizations of so-called betweennesses which are very natural ternary relations associated with trees.

Let V be a finite set. Let $\mathcal{B} \subseteq V^3$ be a ternary relation on V. $\mathcal{B}$ is *strict* if $(x, y, z) \in \mathcal{B}$ implies that x, y, and z are pairwise distinct. Let $T = (V, E)$ be a tree, i.e. a finite, simple, undirected, connected graph without cycles [8]. The *tree betweenness of* T is

$$\mathcal{B}(T) = \left\{(x, y, z) \in V^3 \mid \ y \text{ belongs to the path in } T \text{ between } x \text{ and } z\right\} \tag{1}$$

and the *strict tree betweenness of* T is

$$\mathcal{B}_s(T) = \left\{(x, y, z) \in V^3 \mid \ y \text{ is an internal vertex of the path in } T \text{ between } x \text{ and } z\right\}. \tag{2}$$

The reasons for reconsidering Sholander's [11] and Burigana's [3] work are as follows. Sholander actually does not really consider trees in the graph-theoretical sense. Instead he studies collections of so-called *segments* which are subsets of V indexed by ordered pairs of elements of V. He considers such a collection to be a *tree* if it satisfies certain axioms and claims without proof that "*Trees in our sense which are finite are trees in König's sense.*" (cf. [11], p. 370). While collections of segments derived from trees are easily seen to be trees in Sholander's sense — which proves one direction of this claim, we present as our first result the non-trivial proof of the other direction in Section 2. Only in conjunction with such a result, Sholander's axiomatic characterization of the betweennesses associated with his trees yields an axiomatic characterization of tree betweennesses as defined above.

Burigana characterizes the strict tree betweennesses in terms of five axioms (cf. Theorem 1 in [3]). In Section 3, we strengthen his result and provide a proof which is considerably shorter than his proof in [3]. Furthermore, we extend his result to non-strict tree betweennesses.

Before we proceed to our results, we want to mention some related research in order to clarify the context. The axiomatic study of betweenness as a mathematical concept goes back to Huntington and Kline [7] in 1917. Tree betweennesses are a special case of those induced by metrics which were first studied by Menger [9] in 1928. Also partially ordered sets naturally induce betweennesses which were first studied by Birkhoff [2] in 1948. While Altwegg [1] provided a complete axiomatic description of the latter kind of betweennesses which was generalized by Sholander in [11] and recently by Düntsch and Urquhart [6], a similar result is unknown for the former kind (see Chvátal [4] for a detailed discussion). Therefore axiomatic descriptions of betweennesses induced by special metrics such as graph metrics are of interest. Next to Sholander and Burigana another characterization of tree betweennesses was obtained by Defays [5]. A general result in this context is the characterization of the interval function of a graph by Mulder and Nebeský (cf. [10] and the many references given there).

2 Finite Sholander Trees are Trees

For a set V, Sholander considers a collection $\mathcal{S}$ of so-called *segments* which are subsets of V indexed by ordered pairs of elements of V. The segment indexed by the ordered pair $(u,v) \in V^2$ will be denoted by $[u,v]$. Sholander defines a "*tree*" as a collection $\mathcal{S}$ of segments which has the following three properties (S_1), (S_2), and (S_3).

(S_1) $\forall u,v,w \in V : \exists x \in V : [v,x] = [u,v] \cap [v,w]$
(S_2) $\forall u,v,w \in V : [u,v] \subseteq [u,w] \Rightarrow [u,v] \cap [v,w] = \{v\}$
(S_3) $\forall u,v,w \in V : [u,v] \cap [v,w] = \{v\} \Rightarrow [u,v] \cup [v,w] = [u,w]$

Let $T = (V,E)$ be a tree. For $u,v \in V$ let $[u,v]_T$ denote the set of vertices on the path in T between u and v and let $\mathcal{S}(T) = \{[u,v]_T \mid u,v \in V\}$. The next result shows that Sholander's notion of a tree is equivalent to the graph-theoretical notion of a tree.

Theorem 1 *Let V be a finite set and let $\mathcal{S} = \{[u,v] \mid u,v \in V\} \subseteq 2^V$. There is a tree $T = (V,E)$ with $[u,v] = [u,v]_T$ for all $u,v \in V$ if and only if $\mathcal{S}$ satisfies (S_1), (S_2), and (S_3).*

Proof: Since for every tree T, $\mathcal{S}(T)$ obviously satisfies (S_1), (S_2), and (S_3), the "only if"-part of the statement is clear and we proceed to the "if"-part. Therefore, let $\mathcal{S}$ satisfy (S_1), (S_2), and (S_3).

Claim 1 *For all $u,v \in V$, $u,v \in [u,v]$ and $\{u\} = [u,u]$.*

Proof of Claim 1: Since $[u,u] \subseteq [u,u]$, (S_2) implies $[u,u] \cap [u,u] = \{u\}$ and hence $[u,u] = \{u\}$. Similarly, since $[u,v] \subseteq [u,v]$, (S_2) implies $[u,v] \cap [v,v] = \{v\}$ and hence $v \in [u,v]$. By (S_1), there is some $x \in V$ with $[u,x] = [u,u] \cap [u,v]$. Now, by the previous observations,

$$\{x\} \subseteq [u,x] = [u,u] \cap [u,v] \subseteq [u,u] = \{u\}.$$

Hence $u = x \in [u,x] \subseteq [u,v]$, i.e. $u \in [u,v]$. $\square$

Claim 2 *For all $u,v \in V$, $[u,v] = [v,u]$.*

Proof of Claim 2: Clearly, we may assume that $u \neq v$. By symmetry, it suffices to prove $[v,u] \subseteq [u,v]$. By (S_1), there is some $x \in V$ with $[v,x] = [u,v] \cap [v,u]$. By Claim 1, we have $u,v,x \in [v,x]$. Since $|[v,x]| \geq 2$, Claim 1 implies $v \neq x$. Since $[v,x] \subseteq [v,u]$, (S_2) implies $[v,x] \cap [x,u] = \{x\}$. Since, by Claim 1, $x,u \in [v,x] \cap [x,u]$, this implies $x = u$. Now $[v,u] = [v,x] = [u,v] \cap [v,u]$ which implies $[v,u] \subseteq [u,v]$. $\square$

Claim 3 *If $x \in [u,v]$ for $u,v,x \in V$ with $u \neq x \neq v$, then $u \neq v$, $[u,v] = [u,x] \cup [x,v]$ and $[u,x] \cap [x,v] = \{x\}$.*

Proof of Claim 3: Since $|[u,v]| \geq 2$, Claim 1 implies $u \neq v$ and hence $|[u,v]| \geq 3$. We prove $[u,v] = [u,x] \cup [x,v]$ and $[u,x] \cap [x,v] = \{x\}$ by induction on $|[u,v]|$.

First, let $|[u,v]| = 3$, i.e. $[u,v] = \{u,v,x\}$. By Claim 2 and (S_1), there is some $y \in V$ with $[u,y] = [v,u] \cap [u,x] = [u,v] \cap [u,x]$. By Claim 1, $y \in [u,y] \subseteq [u,v]$, i.e. $y \in \{u,x,v\}$. By Claim 1, $u,x \in [u,y] = [u,v] \cap [u,x]$. Hence $|[u,y]| \geq 2$ and, by Claim 1, $y \neq u$. If $y = v$, then (S_2) applied to $[u,v] = [u,y] = [u,v] \cap [u,x] \subseteq [u,x]$ implies $[u,v] \cap [v,x] = \{x\}$. Since, by Claim 1, $v,x \in [u,v] \cap [v,x]$, this implies the contradiction $x = v$. Hence $y = x$. Since $[u,x] = [u,y] = [u,v] \cap [u,x]$ and hence $[u,x] \subseteq [u,v]$, (S_2) implies $[u,x] \cap [x,v] = \{x\}$ and (S_3) implies $[u,x] \cup [x,v] = [u,v]$, i.e. the claim holds in this case.

Now, let $|[u,v]| > 3$. By Claim 2 and (S_1), there is some $y \in V$ with $[u,y] = [v,u] \cap [u,x] = [u,v] \cap [u,x]$. Since $[u,y] \subseteq [u,v]$, (S_2) implies $[u,y] \cap [y,v] = \{y\}$ and (S_3) implies $[u,y] \cup [y,v] = [u,v]$. If $x = y$, then the desired statement holds. Hence we assume that $x \neq y$. By Claim 2, we may assume that $x \in [u,y]$. By induction, this implies $[u,x] \cup [x,y] = [u,y]$ and $[u,x] \cap [x,y] = \{x\}$. We obtain $y \notin [u,x]$ and hence $[u,x] \subseteq [u,y] \setminus \{y\}$. Since, by Claim 1, $\{y\} \subseteq [x,y] \cap [y,v] \subseteq [u,y] \cap [y,v] = \{y\}$, we have $[x,y] \cap [y,v] = \{y\}$ and (S_3) implies $[x,y] \cup [y,v] = [x,v]$. Now

$$[u,x] \cup [x,v] = [u,x] \cup [x,y] \cup [y,v] = [u,y] \cup [y,v] = [u,v]$$

and

$$\begin{aligned} [u,x] \cap [x,v] &= [u,x] \cap ([x,y] \cup [y,v]) \\ &= ([u,x] \cap [x,y]) \cup ([u,x] \cap [y,v]) \\ &\subseteq \{x\} \cup (([u,y] \setminus \{y\}) \cap [y,v]) \\ &= \{x\} \cup (([u,y] \cap [y,v]) \setminus \{y\}) \\ &= \{x\}. \end{aligned}$$

$\square$

Claim 4 *If $a,b,c,d \in V$ are such that $a \neq b \neq c \neq d$, $[a,b] \cap [b,c] = \{b\}$, and $[b,c] \cap [c,d] = \{c\}$, then a, b, c, and d are pairwise distinct and $[a,c] \cap [c,d] = \{c\}$.*

Proof of Claim 4: By (S_3) and Claim 1,

$$a \neq c, b \neq d, [a,b] \cup [b,c] = [a,c], \text{ and } [b,c] \cup [c,d] = [b,d]. \tag{3}$$

For contradiction, we assume that $[a,c] \cap [c,d] \neq \{c\}$. By (S_1), there is some $x \in V$ with

$$[a,c] \cap [c,d] = [c,x]. \tag{4}$$

By Claim 1, $x \neq c$. If $x = a$, then Claim 2 implies $[a,c] = [x,c] = [c,x] \overset{(4)}{\subseteq} [c,d]$ and hence $b \in [a,b] \overset{(3)}{\subseteq} [a,c] \subseteq [c,d]$. By Claim 1 and the assumption, $b \notin [c,d]$ which is a contradicion. Hence $x \neq a$. If $x = d$, then $[c,d] = [c,x] \overset{(4)}{\subseteq} [a,c] = [c,a]$. (S_2) and (S_3) imply

$$\{d\} = [c,d] \cap [d,a] \text{ and } [a,c] = [c,d] \cup [d,a] = [a,d] \cup [c,d]. \tag{5}$$

Since, by (3) and the assumption, $b \in [a,c] \setminus [c,d]$, this implies $b \in [a,d]$. By Claim 3,

$$[a,d] = [a,b] \cup [b,d]. \tag{6}$$

Now, by Claim 1, $c \in [b,c] \overset{(3)}{\subseteq} [b,d] \overset{(6)}{\subseteq} [a,d]$ but, by (5), $c \notin [a,d]$ which is a contradiction. Hence $x \neq d$. By (4), $[c,x] \subseteq [a,c]$ and $[c,x] \subseteq [c,d]$. Now, by Claim 2, (S_2), and (S_3),

$$[a,x] \cap [x,c] = \{x\}, [c,x] \cap [x,d] = \{x\}, [a,x] \cup [x,c] = [a,c], \text{ and } [c,x] \cup [x,d] = [c,d]. \tag{7}$$

This implies $c \notin [a,x]$, $c \notin [x,d]$, and $[x,d] = [c,d] \setminus ([c,x] \setminus \{x\})$. Since, by Claim 1 and the assumption, $b \notin [c,d]$, we have, by (7),

$$b \notin [c,x]. \tag{8}$$

Furthermore, since, by Claim 1 and (3), $b \in [a,c]$, we have, by Claim 1, (7), and (8), $b \in [a,x]$. By Claim 3, this implies

$$[a,x] = [a,b] \cup [b,x] \text{ and } \{b\} = [a,b] \cap [b,x]. \tag{9}$$

Since, by (7), $c \notin [a,x]$ and, by (9), $[b,x] \subseteq [a,x]$, we have

$$c \notin [b,x]. \tag{10}$$

Now

$$\begin{aligned}
\{x\} &\subseteq [b,x] \cap [x,d] \\
&\overset{(9)}{\subseteq} [a,x] \cap [x,d] \\
&\overset{(7)}{\subseteq} [a,c] \cap [x,d] \\
&\overset{(7)}{=} [a,c] \cap ([c,d] \setminus ([c,x] \setminus \{x\})) \\
&= ([a,c] \cap [c,d]) \setminus ([c,x] \setminus \{x\}) \\
&\overset{(4)}{=} [c,x] \setminus ([c,x] \setminus \{x\}) \\
&= \{x\},
\end{aligned}$$

i.e. $[b,x] \cap [x,d] = \{x\}$. By (S_3), $[b,d] = [b,x] \cup [x,d]$. Since, by (10), $c \notin [b,x]$ and, by (7), $c \notin [x,d]$, we obtain $c \notin [b,d] = [b,x] \cup [x,d]$. But, by (3), $c \in [b,d] = [b,c] \cup [c,d]$ which is a contradiction. □

In view of Claim 2, we define a graph $T = (V,E)$ such that

$$\forall u \in V : \forall v \in V \setminus \{u\} : uv \in E \Leftrightarrow |[u,v]| = 2. \tag{11}$$

Let $u, v \in V$ be such that $u \neq v$. A *$\mathcal{S}$-path of order l between u and v* is a sequence $P : u_1 u_2 \ldots u_l$ such that $u = u_1$, $v = u_l$, $u_i \neq u_{i+1}$ for $1 \leq i \leq l-1$, and $[u_i, u_{i+1}] \cap [u_{i+1}, u_{i+2}] = \{u_{i+1}\}$ for $1 \leq i \leq l-2$.

Claim 5 *If $P : u_1 u_2 \ldots u_l$ is a $\mathcal{S}$-path with $l \geq 2$, then $P_i : u_1 \ldots u_{i-1} u_{i+1} \ldots u_l$ is a $\mathcal{S}$-path for $1 \leq i \leq l$ and all elements of P are distinct.*

Proof of Claim 5: Clearly, if $i \in \{1, l\}$, then P_i is a $\mathcal{S}$-path by definition. If $2 \leq i \leq l-1$, one or two applications of Claim 4 together with Claim 2 imply that P_i is a $\mathcal{S}$-path. The second part of the statement follows immediately from the definition of $\mathcal{S}$-paths and the first part of the statement. □

By Claim 5, all $\mathcal{S}$-paths have order at most $|V|$.

Claim 6 *If the $\mathcal{S}$-path $P : u_1u_2\dots u_l$ between u and v is maximal with respect to inclusion, then P is a path in T.*

Proof of Claim 6: For contradiction, we assume that $u_iu_{i+1} \not\in E$. By (11), this implies the existence of some $x \in [u_i, u_{i+1}]$ with $u_i \neq x \neq u_{i+1} \neq u_i$. By Claim 3, $[u_i, u_{i+1}] = [u_i, x] \cup [x, u_{i+1}]$ and $\{x\} = [u_i, x] \cap [x, u_{i+1}]$. If $i \geq 2$, then, by Claim 1,

$$\{u_i\} \subseteq [u_{i-1}, u_i] \cap [u_i, x] \subseteq [u_{i-1}, u_i] \cap [u_i, u_{i+1}] \subseteq \{u_i\},$$

i.e. $\{u_i\} = [u_{i-1}, u_i] \cap [u_i, x]$. Similarly, if $i \leq l-2$, then, by Claim 1,

$$\{u_{i+1}\} \subseteq [x, u_{i+1}] \cap [u_{i+1}, u_{i+2}] \subseteq [u_i, u_{i+1}] \cap [u_{i+1}, u_{i+2}] \subseteq \{u_{i+1}\},$$

i.e. $\{u_{i+1}\} = [x, u_{i+1}] \cap [u_{i+1}, u_{i+2}]$. Hence the sequence $u_1 \dots u_ixu_{i+1} \dots u_l$ is a $\mathcal{S}$-path between u and v strictly containing P which is a contradiction. $\Box$

Claim 7 *T contains no cycle.*

Proof of Claim 7: For contradiction, we assume that $C : u_0u_1 \dots u_{l-1}u_l$ is a cycle in T, i.e. $l \geq 3$, the elements $u_0, u_1, \dots, u_{l-1}$ are distinct, $u_0 = u_l$, and $u_iu_{i+1} \in E$ for $0 \leq i \leq l-1$. By Claim 1 and (11), we obtain

$$[u_i, u_{i+1}] \cap [u_{i+1}, u_{i+2}] = \{u_i, u_{i+1}\} \cap \{u_{i+1}, u_{i+2}\} = \{u_{i+1}\}$$

for $0 \leq i \leq l-2$. Hence $u_0u_1 \dots u_{l-1}u_l$ is a $\mathcal{S}$-path and, by Claim 5, $u_0 \neq u_l$ which is a contradiction. $\Box$

Since for all $u, v \in V$ with $u \neq v$, $P : uv$ is a $\mathcal{S}$-path between u and v of order 2, Claims 6 and 7 imply that T is a tree.

Claim 8 $[u, v] = [u, v]_T$ *for all* $u, v \in V$.

Proof of Claim 8: Let $x \in [u, v]$. If either $x = u$ or $x = v$, then $x \in [u, v]_T$. Hence, we may assume, by Claim 1, that $u \neq x \neq v \neq u$. By Claim 3, $[u, v] = [u, x] \cup [x, v]$ and $\{x\} = [u, x] \cap [x, v]$, i.e. uxv is a $\mathcal{S}$-path between u and v. By Claim 6, a $\mathcal{S}$-path P between u and v containing uxv as a subsequence which is maximal with respect to inclusion, is a path in T. Hence x lies on the path in T between u and v which implies $x \in [u, v]_T$. We obtain $[u, v] \subseteq [u, v]_T$.

Conversely, let $x \in [u, v]_T$. By definition of $[u, v]_T$, x lies on the path $P : u_1u_2 \dots u_l$ in T between u and v. By (11), $P : u_1u_2 \dots u_l$ is a $\mathcal{S}$-path between u and v. By Claim 5, uxv is a $\mathcal{S}$-path between u and v which implies $\{x\} = [u, x] \cap [x, v]$. Now (S_3) implies $x \in [u, v] = [u, x] \cup [x, v]$. Hence $[u, v]_T \subseteq [u, v]$ and altogether $[u, v]_T = [u, v]$. $\Box$

This last claim completes the proof. $\Box$

Note that (11) yields an efficient way of reconstructing a tree from its collection of segments.

3 Burigana's Axioms for Strict Tree Betweennesses

Burigana considers the following five axioms for strict ternary relations $\mathcal{B} \subseteq V^3$.

(B_1) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (w, v, u) \in \mathcal{B}$
(B_2) $\forall u, v, w \in V : (u, v, w) \in \mathcal{B} \Rightarrow (v, u, w) \not\in \mathcal{B}$
(B_3) $\forall u, v, w, z \in V : (u, v, w), (v, w, z) \in \mathcal{B} \Rightarrow (u, w, z) \in \mathcal{B}$
(B_4) $\forall u, v, w, z \in V : (u, v, w), (u, w, z) \in \mathcal{B} \Rightarrow (v, w, z) \in \mathcal{B}$
(B_5) $\forall u, v, w \in V : N(u, v, w) \Rightarrow \exists c \in V : (u, c, v), (u, c, w), (v, c, w) \in \mathcal{B}$

Here $N(u,v,w)$ means $(u,v,w),(v,w,u),(w,u,v) \notin \mathcal{B}$ and $u \neq v \neq w \neq u$.

First, we note that for a strict relation $\mathcal{B}$, (B_2) is implied by (B_1) and (B_4). In fact, if we assume that $(u,v,w),(v,u,w) \in \mathcal{B}$, then (B_1) implies $(w,u,v),(w,v,u) \in \mathcal{B}$ and then (B_4) implies $(u,v,u) \in \mathcal{B}$ which is impossible for a strict relation. Therefore, the axiom (B_2) is superfluous.

Next, as we also consider non-strict relations, we reformulate axiom (B_3) appropriately as follows.

$$(B_3') \quad \forall u,v,w,z \in V : (v \neq w \text{ and } (u,v,w),(v,w,z) \in \mathcal{B}) \Rightarrow (u,w,z) \in \mathcal{B}$$

Note that for strict relations, (B_3) and (B_3') coincide.

Finally, we consider two more axioms.

$$(B_0) \quad \forall u,v \in V : (u,u,v) \in \mathcal{B} \text{ and } ((u,v,u) \in \mathcal{B} \Rightarrow u = v)$$
$$(B_5') \quad \forall u,v,w \in V : N(u,v,w) \Rightarrow \exists u' \in V : u \neq u' \text{ and } (u,u',v),(u,u',w) \in \mathcal{B}$$

Note that (B_5') is weaker than (B_5).

Lemma 2 *Let V be a finite set. If $\mathcal{B}_s \subseteq V^3$ is strict and satisfies (B_1), (B_3), (B_4), and (B_5'), then $\mathcal{B}_s$ satisfies (B_5).*

Proof: Let $u,v,w \in V$ be such that $N(u,v,w)$ holds. For contradiction, we assume that there is no $c \in V$ with $(u,c,v),(u,c,w),(v,c,w) \in \mathcal{B}_s$.

Claim 1 *If $u' \in V$ is such that $(u,u',v),(u,u',w) \in \mathcal{B}_s$, then there is some $u'' \in V$ such that $(u,u',u''),(u,u'',v),(u,u'',w) \in \mathcal{B}_s$.*

Proof of Claim 1: By our assumption, $(v,u',w) \notin \mathcal{B}_s$. If $(u',v,w) \in \mathcal{B}_s$, then $(u,u',v) \in \mathcal{B}_s$ and (B_3) imply $(u,v,w) \in \mathcal{B}_s$ which is a contradiction. Hence $(u',v,w) \notin \mathcal{B}_s$. Similarly, if $(u',w,v) \in \mathcal{B}_s$, then $(u,u',w) \in \mathcal{B}_s$ and (B_3) imply $(u,w,v) \in \mathcal{B}_s$ which is a contradiction. Hence $(u',w,v) \notin \mathcal{B}_s$ and we obtain $N(u',v,w)$. By (B_5'), there is some $u'' \in V$ with $(u',u'',v),(u',u'',w) \in \mathcal{B}_s$. Since $(u,u',v),(u',u'',v) \in \mathcal{B}_s$, (B_1) and (B_4) imply $(u,u',u'') \in \mathcal{B}_s$. Since $(u,u',u''),(u',u'',v),(u',u'',w) \in \mathcal{B}_s$, two applications of (B_3) imply $(u,u'',v),(u,u'',w) \in \mathcal{B}_s$. $\square$

Claim 2 *If $u,u_1,u_2,\ldots,u_l \in V$ are such that $(u,u_i,u_{i+1}) \in \mathcal{B}$ for $1 \leq i \leq l-1$, then $u_1,u_2,\ldots,u_l$ are pairwise distinct.*

Proof of Claim 2: For contradiction, we may assume $u_1 = u_l$. Since $\mathcal{B}_s$ is strict, this implies $l \geq 3$. If $(u,u_1,u_i) \in \mathcal{B}_s$ for some $2 \leq i \leq l-1$, then $(u,u_i,u_{i+1}) \in \mathcal{B}_s$ and (B_4) imply $(u_1,u_i,u_{i+1}) \in \mathcal{B}_s$. Now $(u,u_1,u_{i+1}) \in \mathcal{B}_s$, (B_1), and (B_3) imply $(u,u_1,u_{i+1}) \in \mathcal{B}_s$. By an inductive argument, this implies $(u,u_1,u_l) \in \mathcal{B}_s$. Since $\mathcal{B}_s$ is strict, this is a contradiction. $\square$

Applying (B_5') once yields the existence of some $u_1 \in V$ with $(u,u_1,v),(u,u_1,w) \in \mathcal{B}_s$. Now, by iteratively applying Claim 1, we obtain a sequence $u_1,u_2,u_3,\ldots$ of elements of V such that $(u,u_i,u_{i+1}),(u,u_{i+1},v),(u,u_{i+1},w) \in \mathcal{B}_s$ holds for $i \geq 1$. By Claim 2, all elements of this sequence are distinct, which contradicts the finiteness of V. $\square$

Theorem 3 *Let V be a finite set. Let $\mathcal{B},\mathcal{B}_s \subseteq V^3$ be such that $\mathcal{B}_s$ is strict.*

(i) There is a tree $T = (V,E)$ such that $\mathcal{B}_s(T) = \mathcal{B}_s$ if and only if $\mathcal{B}_s$ satisfies (B_1), (B_3), (B_4), and (B_5').

(ii) There is a tree $T = (V,E)$ such that $\mathcal{B}(T) = \mathcal{B}$ if and only if $\mathcal{B}$ satisfies (B_0), (B_1), (B_3'), (B_4), and (B_5').

Proof: (i) Since every strict tree betweenness obviously satisfies (B_1), (B_3), (B_4), and (B_5'), the "only if"-part of the statement is clear and we proceed to the "if"-part. Therefore, let $\mathcal{B}_s$ satisfy (B_1), (B_3), (B_4), and (B_5'). By Lemma 2, $\mathcal{B}_s$ satisfies (B_5). Clearly, we may assume that $|V| \geq 3$. Let $T = (V, E)$ be a graph such that for all $u, v \in V$ with $u \neq v$ we have

$$uv \in E \Leftrightarrow \forall x \in V : (u, x, v) \notin \mathcal{B}_s. \tag{12}$$

Let $u, v \in V$ be such that $u \neq v$ and $uv \notin E$. A *$\mathcal{B}_s$-path of order l between u and v* is a sequence $P : u_1u_2 \dots u_l$ such that $u = u_1$, $v = u_l$, $(u_i, u_{i+1}, u_{i+2}) \in \mathcal{B}_s$ for $1 \leq i \leq l-2$, and $l \geq 3$. By (12), there is at least one $\mathcal{B}_s$-path of order 3 between u and v.

Claim 1 *All elements of a $\mathcal{B}_s$-path $P : u_1u_2 \dots u_l$ are distinct.*

Proof of Claim 1: We assume, for contradiction, that $u_i = u_j$ for some $1 \leq i < j \leq l$. Since $\mathcal{B}_s$ is strict, $j - i \geq 3$. Since $(u_i, u_{i+1}, u_{i+2}), (u_{i+1}, u_{i+2}, u_{i+3}) \in \mathcal{B}_s$, an application of (B_3) implies $(u_i, u_{i+2}, u_{i+3}) \in \mathcal{B}_s$. Now, if $j - i \geq 4$, then $(u_i, u_{i+2}, u_{i+3}), (u_{i+2}, u_{i+3}, u_{i+4}) \in \mathcal{B}_s$ and another application of (B_3) implies $(u_i, u_{i+3}, u_{i+4}) \in \mathcal{B}_s$. Iteratively applying (B_3) in this way eventually yields $(u_i, u_{j-1}, u_j) = (u_i, u_{j-1}, u_i) \in \mathcal{B}_s$ which is a contradiction, because $\mathcal{B}_s$ is strict. $\Box$

By Claim 1, all $\mathcal{B}_s$-paths have order at most $|V|$.

Claim 2 *If the $\mathcal{B}_s$-path $P : u_1u_2 \dots u_l$ is maximal with respect to inclusion, then P is a path in T.*

Proof of Claim 2: For contradiction, we assume that $u_iu_{i+1} \notin E$ some $1 \leq i \leq l-1$. By (12), this implies that there is some $x \in V$ such that $(u_i, x, u_{i+1}) \in \mathcal{B}_s$. If $i \leq l-2$, then $(u_i, u_{i+1}, u_{i+2}) \in \mathcal{B}_s$ and (B_4) imply that $(x, u_{i+1}, u_{i+2}) \in \mathcal{B}_s$. Similarly, if $i \geq 2$, then $(u_{i-1}, u_i, u_{i+1}) \in \mathcal{B}_s$, (B_1), and (B_4) imply that $(u_{i-1}, u_i, x) \in \mathcal{B}_s$. Now the sequence $u_1 \dots u_ixu_{i+1} \dots u_l$ is a $\mathcal{B}_s$-path between u and v strictly containing P which is a contradiction. $\Box$

Claim 3 *T contains no cycle.*

Proof of Claim 3: For contradiction, we assume that $C : u_0u_1 \dots u_{l-1}u_0$ is a shortest cycle in T. If $l = 3$, then (12) implies $N(u_0, u_1, u_2)$ and (B_5) implies a contradiction to $u_0u_1 \in E$. Hence $l \geq 4$. By (12) and (B_5), we obtain $(u_i, u_{i+1}, u_{i+2}) \in \mathcal{B}_s$ for $0 \leq i \leq l-1$ where the indices are to be understood modulo l. Iteratively applying (B_3) yields $(u_0, u_{l-1}, u_0) \in \mathcal{B}_s$ which is a contradiction, because $\mathcal{B}_s$ is strict. $\Box$

By Claims 2 and 3, T is a tree.

Claim 4 $\mathcal{B}_s = \mathcal{B}_s(T)$

Proof of Claim 4: Let $(u, v, w) \in \mathcal{B}_s$. By definition, $uw \notin E$. Since uvw is a $\mathcal{B}_s$-path, Claim 2 implies that T contains a path between u and w which contains v as an internal vertex, i.e. $(u, v, w) \in \mathcal{B}_s(T)$.

Conversely, let $(u, v, w) \in \mathcal{B}_s(T)$. Let $P : u_1u_2 \dots u_l$ with $u = u_1$, $w = u_l$, and $v = u_j$ for some $2 \leq j \leq l-1$ be a path in T between u and w which contains v as an internal vertex. By (12) and (B_5), we obtain $(u_i, u_{i+1}, u_{i+2}) \in \mathcal{B}_s$ for $1 \leq i \leq l-2$. Now, applying (B_1) and iteratively applying (B_3) implies $(u, v, w) = (u_1, u_j, u_l) \in \mathcal{B}_s$ $\Box$

This last claim completes the proof of (i).

(ii) Since every tree betweenness obviously satisfies (B_0), (B_1), (B_3'), (B_4), and (B_5'), the "only if"-part of the statement is clear and we proceed to the "if"-part. Therefore, let $\mathcal{B}$ satisfy (B_0), (B_1), (B_3'), (B_4), and (B_5'). Let

$$\mathcal{B}_s = \mathcal{B} \setminus \left\{(u, v, w) \in V^3 \mid u = v \text{ or } u = w \text{ or } v = w\right\}.$$

Claim 5 *$\mathcal{B}_s$ satisfies (B_1), (B_3), (B_4), and (B'_5).*

Proof of Claim 5: $\mathcal{B}_s$ clearly satisfies (B_1).

If $(u, v, w), (v, w, z) \in \mathcal{B}_s$, then $u \neq w \neq z$ and (B'_3) for $\mathcal{B}$ implies $(u, w, z) \in \mathcal{B}$. If $u = z$, then (B_0) for $\mathcal{B}$ implies $u = w$ which is a contradiction. Hence $u \neq z$ and $(u, w, z) \in \mathcal{B}_s$, i.e. $\mathcal{B}_s$ satisfies (B_3).

If $(u, v, w), (u, w, z) \in \mathcal{B}_s$, then $v \neq w \neq z$ and (B_4) for $\mathcal{B}$ implies $(v, w, z) \in \mathcal{B}$. If $v = z$, then (B_0) for $\mathcal{B}$ implies $v = w$ which is a contradiction. Hence $v \neq z$ and $(v, w, z) \in \mathcal{B}_s$, i.e. $\mathcal{B}_s$ satisfies (B_4).

If $N(u, v, w)$, then (B'_5) for $\mathcal{B}$ implies the existence of some $u' \in V$ such that $u \neq u'$ and $(u, u', v), (u, u', w) \in \mathcal{B}$. If $u' = v$, then $(u, v, w) \in \mathcal{B}$ which is a contradiction. If $u' = w$, then $(u, w, v) \in \mathcal{B}$ which is a contradiction. Hence $u' \neq v$ and $u' \neq w$, $(u, u', v), (u, u', w) \in \mathcal{B}_s$, and $\mathcal{B}_s$ satisfies (B'_5). □

By the proof of (i), the tree T defined as in (12) satisfies $\mathcal{B}_s = \mathcal{B}_s(T)$. Let $u, v \in V$ be such that $u \neq v$. By definition of $\mathcal{B}(T)$ and (B_0), $(u, u, u), (u, u, v) \in \mathcal{B} \cap \mathcal{B}(T)$ and $(u, v, u) \notin \mathcal{B} \cup \mathcal{B}(T)$. Hence $\mathcal{B} = \mathcal{B}(T)$ and the proof of (ii) is complete. □

Note that (12) yields an efficient way of reconstructing a tree from its (strict) tree betweenness. Theorem 3 (i) immediately implies Burigana's result whose proof in [3] extends over about seven pages. (Note that for the proof of Burigana's result, Lemma 2 is not even needed.)

Corollary 4 (Burigana [3]) *Let V be a finite set. A strict relation $\mathcal{B} \subseteq V^3$ is a strict tree betweenness if and only if it satisfies (B_1), (B_2), (B_3), (B_4), and (B_5).*

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