Null-Homotopic Graphs and Triangle-Completions of Spanning Trees

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Abstract

We prove that a connected graph whose cycle space is generated by its triangles and which is either planar or of maximum degree at most four has the property that it arises from an arbitrary spanning tree by iteratively adding edges which complete triangles.

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We consider finite, simple, and undirected graphs G with vertex set $V(G)$ and edge set $E(G)$. A graph H arises from a graph G by a *triangle-completion* if $V(H) = V(G)$ and there are three vertices x, y, and z in G such that $xy, yz \in E(G)$, $xz \notin E(G)$, and $E(H) = E(G) \cup \{xz\}$. A set F of edges of a graph G is *triangle-complete*, if $xz \in F$ for all three vertices x, y, and z in G with $xy, yz \in F$ and $xz \in E(G)$. The *triangle-completion* $\Delta_G(F)$ of a set F of edges of G is the intersection of all triangle-complete sets of edges of G containing F. It is easy to see that $\Delta_G(F)$ is the edge set of a graph which arises by applying a sequence of triangle-completions to $(V(G), F)$. A graph G is said to be *tree-generated* if G has a spanning tree T such that $E(G) = \Delta_G(E(T))$. A graph G is said to be *strongly tree-generated* if G is connected and $E(G) = \Delta_G(E(T))$ for every spanning tree T of G. A graph G is *null-homotopic* [2] if its triangles generate its cycle space, i.e. if every cycle of G is the mod 2 sum of triangles of G. For notational simplicity, we denote the class of all connected null-homotopic graphs by $\mathcal{NH}$, the class of all tree-generated graphs by $\mathcal{TG}$, and the class of all strongly tree-generated graphs by $\mathcal{STG}$, respectively. Our first simple observation relates these classes.

Observation 1 *$\mathcal{STG} \subseteq \mathcal{TG} \subseteq \mathcal{NH}$.*

Proof: $\mathcal{STG} \subseteq \mathcal{TG}$ holds by definition. We prove by induction on the cyclomatic number $\mu(G) = |E(G)| - |V(G)| + 1$ that every graph $G \in \mathcal{TG}$ belongs to $\mathcal{NH}$. If $\mu(G) = 0$, then G is a tree and trivially belongs to $\mathcal{NH}$. Now let $\mu(G) > 0$. Since $G \in \mathcal{TG}$, there is an edge $xy \in E(G)$ such that $G' = G - xy \in \mathcal{TG}$ and G arises from G' by a triangle-completion, i.e. there is a vertex $z \in V(G)$ such that $C_0 : xyzx$ is a triangle in G. By induction, $G' \in \mathcal{NH}$. Let C be a cycle of G. If C is also a cycle of G', then C is the mod 2 sum of triangles of G' and hence of G. If $xy \in E(C)$, then the mod 2 sum of C and C_0, say C', is a union of cycles of G'. Hence C' is the mod 2 sum of triangles of G' and hence of G. Now mod 2 adding C_0 to this sum results in C, i.e. $G \in \mathcal{NH}$. $\square$

The starting point for the present note was the question posed by Jamison [3] whether all connected null-homotopic graphs are strongly tree-generated, i.e. whether $\mathcal{NH} \subseteq \mathcal{STG}$. In view of Observation

1, a positive answer to this question would imply $\mathcal{STG} = \mathcal{TG} = \mathcal{NH}$. We have already noted that every graph in $\mathcal{TG}$ which is not a tree has an edge whose removal results in a graph in $\mathcal{TG}$. Therefore, the graph constructed by Champetier in [1] provides a negative answer to Jamison's question. Since it belongs to $\mathcal{NH}$, is not a tree, and the deletion of any of its edges results in a graph which does not belong to $\mathcal{NH}$, Champetier's graph belongs to $\mathcal{NH} \setminus \mathcal{TG}$. This implies that the inclusion $\mathcal{TG} \subseteq \mathcal{NH}$ is strict. At this moment we do not know whether the inclusion $\mathcal{STG} \subseteq \mathcal{TG}$ is strict or not.

Jamison [3] pointed out that connected plane graphs in which all but at most one of the faces are triangles as well as connected chordal graphs are examples of graphs in $\mathcal{STG}$. We complement these remarks by the following.

Theorem 2 *Let $G \in \mathcal{NH}$.*

(i) If G is planar, then $G \in \mathcal{STG}$.

(ii) If G has maximum degree at most 4, then $G \in \mathcal{STG}$.

Before we proceed to the proof of Theorem 2, we establish two preparatory results. Since we also rely on Jamison's remarks, we provide a short proof for the sake of completeness.

Lemma 3

(i) If G is connected and $e \in E(G)$ is a bridge of G, then both components of $G - e$ belong to $\mathcal{STG}$ if and only if G belongs to $\mathcal{STG}$.

(ii) **(Jamison [3])** *If G is a connected plane graphs in which all but at most one of the faces are triangles, then $G \in \mathcal{STG}$.*

(iii) **(Jamison [3])** *If G is a connected chordal graphs, then $G \in \mathcal{STG}$.*

Proof: (i) This follows immediately from the definition of $\mathcal{STG}$ and the fact that every spanning tree of G contains the bridge e.

(ii) Let T be a spanning tree of G. We prove $\Delta_G(E(T)) = E(G)$ by induction over $|E(G)|$. If $|E(G)| \leq 2$, then $G = T$ and hence $\Delta_G(E(T)) = E(G)$. Now let $|E(G)| \geq 3$. By (i) and induction, we may assume that G has no bridges. Let $xz \in E(G) \setminus E(T)$ be an edge in the boundary of the unique face which is not a triangle or, if no such face exists, let $xz \in E(G) \setminus E(T)$ be chosen arbitrarily. Since G has no bridges, xz belongs to a triangle $xyzx$ of G. Clearly, $G - xz$ is again a connected plane graph in which all but at most one of the faces are triangles. By induction, $\Delta_{G-xz}(E(T)) = E(G) \setminus \{xz\}$ and hence $\Delta_G(E(T)) = E(G)$. This completes the proof of (ii).

(iii) Let T be a spanning tree of G. For contradiction, we assume that $\Delta_G(E(T)) \neq E(G)$. Let $C : u_1u_2 \ldots u_lu_1$ be a shortest cycle such that $u_1u_2 \in E(G) \setminus \Delta_G(E(T))$ and $E(C) \setminus \{u_1u_2\} \subseteq \Delta_G(E(T))$. (Since T is a spanning tree of G such a cycle clearly exists.) Since $u_1u_2 \in E(G) \setminus \Delta_G(E(T))$, C cannot be a triangle, i.e. $l \geq 4$. Since G is chordal, C has a chord u_iu_j with $j - i \geq 2$. $C' : u_iu_{i+1} \ldots u_{j-1}u_ju_i$ and $C'' : u_ju_{j+1} \ldots u_{i-1}u_iu_j$ are two cycles in G which are shorter than C. We may assume that $u_1u_2 \notin E(C')$. Now the choice of C implies that $u_iu_j \in \Delta_G(E(T))$ and the existence of the cycle C'' contradicts the choice of C. $\Box$

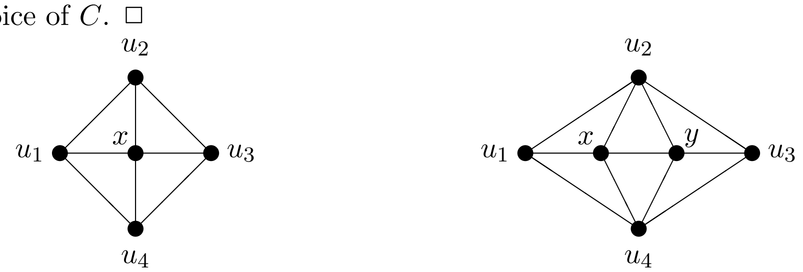

Figure 1 $C : u_1u_2u_3u_4u_1$.

Lemma 4 *Let $G \in \mathcal{NH}$ have maximum degree at most 4. If $C : u_1u_2 \dots u_lu_1$ is a chordless cycle in G with $l \geq 4$, then $l = 4$ and G contains one of the two graphs in Figure 1 as an induced subgraph.*

Proof: Let $\mathcal{C}$ be a collection of triangles whose mod 2 sum is C.

First, we assume that there are no two vertices $x, y \in V(G) \setminus V(C)$ such that the two triangles $xuvx$ and $xyvx$ belong to $\mathcal{C}$ for some edge $uv \in E(C)$. Since C is chordless of length at least 4, there is some $x \in V(G) \setminus V(C)$ such that xu_1u_2x belongs to $\mathcal{C}$. Since $xu_1, xu_2 \notin E(C)$, our assumption implies that xu_1u_lx and xu_2u_3x are triangles in $\mathcal{C}$. Since $xu_3 \notin E(C)$, our assumption implies that xu_3u_4x is a triangle in $\mathcal{C}$. Since G has maximum degree 4 and $u_1, u_2, u_3, u_4, u_l \in N_G(x)$, we obtain that $l = 4$ and that $G[\{x, u_1, u_2, u_3, u_l\}]$ is the left graph in Figure 1.

Therefore, we may assume now that $x, y \in V(G) \setminus V(C)$ are such that xu_1u_2x and xyu_2x are two triangles in $\mathcal{C}$. Since $yu_2 \notin E(C)$ and $u_1, u_3, x, y \in N_G(u_2)$, either yu_1u_2y or yu_2u_3y is a triangle in $\mathcal{C}$. If yu_1u_2y belongs to $\mathcal{C}$, then u_1u_2 does not belong to an odd number of triangles in $\mathcal{C}$ which is a contradiction. Hence yu_2u_3y is a triangle in $\mathcal{C}$. Since $l \geq 4$, neither xu_1u_3x nor yu_3u_1y belong to $\mathcal{C}$.

If $xu_3 \in E(G)$, then $xu_1 \notin E(C)$ implies that u_1xyu_1 is a triangle in $\mathcal{C}$. Now u_1u_l does not belong to a triangle which is a contradiction. Hence $xu_3 \notin E(G)$ and, by symmetry, $yu_1 \notin E(G)$.

If there is some $z \in V(G) \setminus V(C)$ such that yzu_3y belong to $\mathcal{C}$, then $x \neq z$ and $zu_2 \notin E(G)$. Since $u_2, u_4, y, z \in N_G(u_3)$, $zu_2 \notin E(G)$ implies that zu_3u_4z belong to $\mathcal{C}$. Since $u_2, u_3, x, z \in N_G(y)$, $xy \notin E(C)$ implies that $xyzx$ belongs to $\mathcal{C}$. Since $xz \notin E(C)$, this implies the contradiction $u_1 = u_4$. Hence there is no such vertex z.

This implies that xu_1u_lx and yu_3u_4y are two triangles in $\mathcal{C}$. Since $xy \notin E(C)$, this implies $u_4 = u_l$, i.e. $l = 4$ and $G[\{x, y, u_1, u_2, u_3, u_4\}]$ is the right graph in Figure 1. This completes the proof. $\Box$

It is easy to see that the possible blocks of graphs in $\mathcal{NH}$ which are of maximum degree at most 3 are K_1, K_2, K_3, and $K_4 - e$.

We proceed to the proof of Theorem 2.

Proof of Theorem 2: (i) For contradiction, we assume that $G \in \mathcal{NH}$ is planar and there is a spanning tree T of G with $\Delta_G(E(T)) \neq E(G)$. Furthermore, we assume that subject to these conditions the order of G is minimal. By Lemma 3 (i) and (ii), G has no brigde and G does not have a plane embedding in which all but at most one of the faces are triangles. By the characterization of planar null-homotopic graphs in [2], there are two graphs G_1 and G_2 of order at least 4 such that $G = G_1 \cup G_2$, $G_1 \cap G_2$ is a triangle $xyzx$, G_1 is a planar null-homotopic graph, and G_2 has a plane embedding in which all but at most one of the faces are triangles. We assume that G is embedded such that in the induced embedding of G_2 all but at most one of the faces are triangles and G arises by "*glueing*" G_1 into a bounded face of G_2 which is a triangle.

Our next goal is to prove that $\Delta_G(E(T))$ contains the edge set of a spanning tree of G_1. If $T \cap G_1$ is not a spanning tree of G_1, then we may assume, by symmetry, that $xy \notin E(T)$ and that $T \cap G_2$ contains a path P between x and y. Now, Lemma 3 (ii) applied to the maximal subgraph of G_2 whose unbounded face has the boundary cycle consisting of P and xy implies that $xy \in \Delta_G(E(T))$. This implies that $\Delta_G(E(T))$ contains the edge set of a spanning tree of G_1. By the choice of G, this implies that $\Delta_G(E(T))$ contains $E(G_1)$. Hence $\Delta_G(E(T))$ contains the edge set of a spanning tree of G_2. By Lemma 3 (ii), this implies that $\Delta_G(E(T))$ contains $E(G_2)$, i.e. $\Delta_G(E(T)) = E(G)$ which is a contradiction. This completes the proof of (i).

(ii) Let $G \in \mathcal{NH}$ have maximum degree at most 4. Let T be a spanning tree of G. As in the proof of Lemma 3 (iii), we assume, for contradiction, that $\Delta_G(E(T)) \neq E(G)$. Let $C : u_1u_2 \dots u_lu_1$ be a shortest cycle such that $u_1u_2 \in E(G) \setminus \Delta_G(E(T))$ and $E(C) \setminus \{u_1u_2\} \subseteq \Delta_G(E(T))$. Since $u_1u_2 \in E(G) \setminus \Delta_G(E(T))$, C cannot be a triangle, i.e. $l \geq 4$. If G has a chord u_iu_j with $j - i \geq 2$, then we obtain the same contradiction as in the proof of Lemma 3 (iii). Hence, we may assume that C has no chord. By Lemma 4, this implies that either there is a vertex $x \in V(G)$ such that $G[V(C) \cup \{x\}]$ is the left graph in Figure 1 or there are two vertices $x, y \in V(G)$ such that $G[V(C) \cup \{x, y\}]$ is the right graph in Figure 1. Since both these graphs satisfy the assumption of Lemma 3 (ii) and $\Delta_G(E(T))$ clearly contains the edge set of a spanning tree of these graphs, Lemma 3 (ii) implies the contradiction $u_1u_2 \in \Delta_G(E(T))$. $\square$

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