Constructing a Dominating Set for bipartite graphs in several Rounds

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Abstract

Using the probabilistic method, new upper bounds on the domination number of a bipartite graph in terms of the cardinalities and the minimum degrees of the two colour classes are established.

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We consider finite, undirected and simple graphs without isolated vertices. The domination number $\gamma = \gamma(G)$ of a bipartite graph $G = (A, B, E)$ is the minimum cardinality of a set $D \subseteq V$ of vertices such that every vertex in $V \setminus D$ has a neighbour in D. This parameter is one of the most well-studied in graph theory, and the two volume monograph [5, 6] provides an impressive account of the research related to this concept.

Based on the results of [3] we adopt the approach to create a dominating set for an arbitrary graph in several rounds to bipartite graphs. The idea is to choose vertices for a dominating set D at random. In every round we only want to choose those which are not dominated by the previous ones. Extensive calculations cause a modified idea.

We choose k sets $X_1, \dots, X_k$ independently at random and add from ervery set X_i onlythose vertices to D that are not dominated by the first $(i-1)$ sets. Contrary to the results in [3] for arbitrary graphs, we use different probabilities p_i and q_i in A and B for the vertices to belong to a set X_i. With this we are able to prove the following theorem.

Theorem 1 *Let $G = (A, B, E)$ be a bipartite graph of maximum degrees Δ_1 and Δ_2 of the two colour classes A and B and girth at least six.*
For some $k \in \mathbb{N}$ let $p_1, \dots, p_k$ and $q_1, \dots, q_k \in [0, 1]$. If $p_{<1} = q_{<1} = 0$, $p_{<i} = 1 - \prod_{j=1}^{i-1}(1-p_j)$ and $q_{<i} = 1 - \prod_{j=1}^{i-1}(1-q_j)$ for $2 \le i \le k$, then

$$\gamma(G) \le \sum_{v\in A}\left(\prod_{i=1}^{k}(1-p_i)(1-q_i)^{d_G(v)} + \sum_{i=1}^{k}\Big[p_i\cdot(1-p_{<i})\cdot(1-q_{<i})^{d_G(v)} + (1-p_{<i})\cdot(1-q_{<i})^{d_G(v)}\cdot(1-p_i)\cdot\left(\left(1-q_i(1-p_{<i})^{(\Delta_B-1)}\right)^{d_G(v)} - (1-q_i)^{d_G(v)}\right)\Big]\right)$$

$$+ \sum_{v\in B} \left(\prod_{i=1}^{k} (1-q_i)(1-p_i)^{d_G(v)} + \sum_{i=1}^{k} \Big[q_i \cdot (1-q_{<i}) \cdot (1-p_{<i})^{d_G(v)} + \right.$$

$$\left. (1-q_{<i}) \cdot (1-p_{<i})^{d_G(v)} \cdot (1-q_i) \cdot \left(\left(1-p_i(1-q_{<i})^{(\Delta_A-1)}\right)^{d_G(v)} - (1-p_i)^{d_G(v)} \right) \Big] \right)$$

Proof: For $1 \le i \le k$ let X_{Ai} (X_{Bi}) be a subset of A (B) which arises by choosing every vertex of A (B) independently at random with probability p_i (q_i). Let $Y_{A1} = X_{A1}$, $Y_{B1} = X_{B1}$ and $Z_{A1} = Z_{B1} = \emptyset$. For $2 \le i \le k$ let

$$X_{A<i} = \bigcup_{j=1}^{i-1} X_{Aj}, \quad X_{B<i} = \bigcup_{j=1}^{i-1} X_{Bj},$$

$$Y_{Ai} = X_{Ai} \setminus (X_{A<i} \cup N(X_{B<i})), \quad Y_{Bi} = X_{Bi} \setminus (X_{B<i} \cup N(X_{A<i}))$$

and

$$\begin{aligned} Z_{Ai} &= N(X_{Bi}) \setminus (X_{A<i} \cup Y_{Ai} \cup N(X_{B<i} \cup Y_{Bi})) \\ Z_{Bi} &= N(X_{Ai}) \setminus (X_{B<i} \cup Y_{Bi} \cup N(X_{A<i} \cup Y_{Ai})). \end{aligned}$$

Let

$$R_A = A \setminus \left(\bigcup_{j=1}^{k} X_{Aj} \cup N\left(\bigcup_{j=1}^{k} X_{Bj} \right) \right), \quad R_B = B \setminus \left(\bigcup_{j=1}^{k} X_{Bj} \cup N\left(\bigcup_{j=1}^{k} X_{Aj} \right) \right).$$

Claim 1 *For* $1 \le i \le k$ *is*

$$\begin{aligned} &N_G(X_{A1} \cup \dots \cup X_{Ai}) \cup X_{B1} \cup \dots \cup X_{Bi} \\ \subseteq &(Y_{B1} \cup Z_{B1}) \cup \dots \cup (Y_{Bi} \cup Z_{Bi}) \cup N_G((Y_{A1} \cup Z_{A1}) \cup \dots \cup (Y_{Ai} \cup Z_{Ai})) \end{aligned}$$

and

$$\begin{aligned} &N_G(X_{B1} \cup \dots \cup X_{Bi}) \cup X_{A1} \cup \dots \cup X_{Ai} \\ \subseteq &(Y_{A1} \cup Z_{A1}) \cup \dots \cup (Y_{Ai} \cup Z_{Ai}) \cup N_G((Y_{B1} \cup Z_{B1}) \cup \dots \cup (Y_{Bi} \cup Z_{Bi})). \end{aligned}$$

Proof of Claim 1: We only prove the first equation of Claim 1 by induction. The second part follows analogous. For $i = 1$ we get

$$\begin{aligned} & N[X_{A1} \cup X_{B1}] \cap A && \subseteq (Y_{A1} \cup Z_{A1}) \cup N(Y_{B1} \cup Z_{B1}) \\ \Leftrightarrow \quad & X_{A1} \cup N(X_{B1}) && \subseteq (Y_{A1} \cup Z_{A1}) \cup N(Y_{B1} \cup Z_{B1}) \end{aligned}$$

and this is easy to see, because $X_{A1} = Y_{A1} \cup Z_{A1}$ and $X_{B1} = Y_{B1} \cup Z_{B1}$. For $i \ge 2$, by induction,

$$\begin{aligned} &N(X_{A1} \cup \dots \cup X_{Ai-1}) \cup X_{B1} \cup \dots \cup X_{Bi-1} \\ &\qquad \subseteq (Y_{B1} \cup Z_{B1}) \cup \dots \cup (Y_{Bi} \cup Z_{Bi}) \cup N((Y_{A1} \cup Z_{A1}) \cup \dots \cup (Y_{Ai} \cup Z_{Ai})) \end{aligned}$$

and it suffices to show
$X_{Bi} \cup N(X_{Ai}) \subseteq (Y_{B1} \cup Z_{B1}) \cup \dots \cup (Y_{Bi} \cup Z_{Bi}) \cup N((Y_{A1} \cup Z_{A1}) \cup \dots \cup (Y_{Ai} \cup Z_{Ai}))$.

Case 1) If $x \in X_{Ai}$, then either $x \in Y_{Bi}$ or $x \in N(X_{A<i}) \cup X_{B<i}$. In both cases we are done.

Case 2) If $x \in N(X_{Ai})$, then either $x \in N(X_{A<i}) \cup X_{B<i}$ or $x \in Y_{Bi} \cup N[Y_{Ai}]$ or, by definition, $x \in Z_{Bi}$. Again in all cases we are done and the proof of the claim is complete. □

Note that, by the claim and the definition of R_A and R_B, the set

$$D = R_A \cup R_B \cup (\bigcup_{i=1}^{k}(Y_{Ai} \cup Z_{Ai})) \cup (\bigcup_{i=1}^{k}(Y_{Bi} \cup Z_{Bi}))$$

is a dominating set of G.

The expected cardinality of Y_{A1} is $p_1|A| = p_1 a$. Now let $2 \leq i \leq k$. Since the sets $X_1, \ldots, X_{i-1}$ are chosen independently, the set $X_{<i}$ arises by choosing every vertex of G independently at random with probability

$$p_{<i} = 1 - \prod_{j=1}^{i-1}(1 - p_j).$$

Hence

$$\mathbb{P}[x \in Y_{Ai}] = p_i \cdot (1 - p_{<i}) \cdot (1 - q_{<i})^{d_A(x)}$$

for every vertex $x \in A$.

Analogous we get $E(|Y_{B1}|) = q_1|B| = q_1 b$, $q_{<i} = 1 - \prod_{j=1}^{i-1}(1 - q_j)$ and

$$\mathbb{P}[x \in Y_{Bi}] = q_i \cdot (1 - q_{<i}) \cdot (1 - p_{<i})^{d_B(x)}$$

for every vertex $x \in B$.

Furthermore, a vertex $x \in A$ is in Z_{Ai} if and only if $x \notin X_{A<i} \vee x \in N_G[X_{B<i}]$ and $x \notin X_{Ai}$ and there is some non-empty set $U \subseteq N_G(x)$ with $N_G(x) \cap (N_G(X_{A<i}) \cap X_{Bi}) = U$ and $N_G(x) \cap (B \setminus X_{Bi}) = N_G(x) \setminus U$.

For some specific set U let

$$N_G(x) \setminus U = \{x_1, x_2, \ldots, x_{d_G(x)-l}\}$$

and

$$U = \{x_{d_G(x)-l+1}, x_{d_G(x)-l+2}, \ldots, x_{d_G(x)}\}.$$

By the independence of the choice of the elements of the sets X_j and by the girth condition, we obtain - in what follows we indicate the use of the independence by "(i)" and the use of the girth condition by "(g)"

$$\begin{aligned}
&\mathbb{P}\left[v \in Z_{Ai} \cap (N(v) \cap N(X_{A<i}) \cap X_{Bi} = U) \wedge (N(v) \cap (B \setminus X_{Bi}) = (N(v) \setminus U))\right] \\
=&\,\mathbb{P}\left[(v \notin X_{A<i}) \wedge (v \notin N(X_{B<i})) \wedge (v \notin X_{Ai}) \wedge \left(\bigwedge_{j=1}^{d(v)-l} (v_j \notin X_{Bi})\right)\right. \\
&\qquad\qquad\left.\wedge \left(\bigwedge_{j=d^A(v)-l+1}^{d^A(v)} (v_j \in N(X_{A<i}) \cap X_{Bi})\right)\right], \\
\overset{(i)}{=}&(1-p_{<i}) \cdot (1-q_{<i})^{d(v)} \cdot (1-p_i) \cdot (1-q_i)^{(d(v)-l)} \\
&\qquad \cdot \mathbb{P}\left[\left(\bigwedge_{j=d(v)-l+1}^{d(v)} (v_j \in N(X_{A<i}) \cap X_{Bi})\right) |(v \notin X_{A<i} \wedge v \notin N(X_{B<i}))\right], \\
\overset{(i)}{=}&(1-p_{<i}) \cdot (1-q_{<i})^{(d(v))} \cdot (1-p_i) \cdot (1-q_i)^{(d(v)-l)} \\
&\cdot \prod_{j=d(v)-l+1}^{d(v)} \mathbb{P}\left[(v_j \in N(X_{A<i}) \cap X_{Bi}) \,\middle|\, \left(\bigwedge_{r=d(v)-l+1}^{j-1} (v_r \in N(X_{A<i}) \cap X_{Bi})\right)\right. \\
&\qquad\qquad\left.\wedge (v \notin X_{A<i} \wedge v \notin N(X_{B<i}))\right], \\
\overset{(i)}{=}&(1-p_{<i}) \cdot (1-q_{<i})^{d(v)} \cdot (1-p_i) \cdot (1-q_i)^{(d(v)-l)} \cdot q_i^l \\
&\cdot \prod_{j=d(v)-l+1}^{d(v)} \mathbb{P}\left[(v_j \in N(X_{A<i})) \,\middle|\, \left(\bigwedge_{r=d(v)-l+1}^{j-1} v_r \in N(X_{A<i})\right) \wedge (v \notin X_{A<i} \wedge v \notin N(X_{B<i}))\right], \\
\overset{(g)}{=}&(1-p_{<i}) \cdot (1-q_{<i})^{d(v)} \cdot (1-p_i) \cdot (1-q_i)^{(d(v)-l)} \cdot q_i^l \\
&\qquad \cdot \prod_{j=d(v)-l+1}^{d(v)} \mathbb{P}\left[(v_j \in N(X_{A<i})) \,|(v \notin X_{A<i} \wedge v \notin N(X_{B<i}))\right], \\
\overset{(g)}{=}&(1-p_{<i})(1-q_{<i})^{d(v)}(1-p_i)(1-q_i)^{(d(v)-l)} q_i^l \prod_{j=d(v)-l+1}^{d(v)} \left(1-(1-p_{<i})^{(d(v_j)-1)}\right) \\
\leq&(1-p_{<i}) \cdot (1-q_{<i})^{d(v)} \cdot (1-p_i) \cdot (1-q_i)^{(d(v)-l)} \cdot q_i^l \cdot \left(1-(1-p_{<i})^{(\Delta_B-1)}\right)^l.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\mathbb{P}[v \in Z^{Ai}] \\
\leq&(1-p_{<i}) \cdot (1-q_{<i})^{d_G(x)} \cdot (1-p_i) \cdot \sum_{l=1}^{d_G(x)} \binom{d_G(x)}{l} \cdot (1-q_i)^{(d_G(x)-l)} \cdot q_i^l \cdot \left(1-(1-p_{<i})^{(\Delta^{V_2}-1)}\right)^l \\
=&(1-p_{<i}) \cdot (1-q_{<i})^{d_G(x)} \cdot (1-p_i) \cdot \left(\left((1-q_i) + q_i\left(1-(1-p_{<i})^{(\Delta^B-1)}\right)\right)^{d_G(x)} - (1-q_i)^{d_G(x)}\right) \\
=&(1-p_{<i}) \cdot (1-q_{<i})^{d_G(x)} \cdot (1-p_i) \cdot \left(\left(1-q_i(1-p_{<i})^{(\Delta^{V_2}-1)}\right)^{d_G(x)} - (1-q_i)^{d_G(x)}\right)
\end{aligned}$$

for every vertex $x \in A$.

Finally,

$$\mathbb{P}[x \in R_A] = \prod_{i=1}^{k} (1-p_i)(1-q_i)^{d_G(x)}$$

for every vertex $x \in A$ and

$$\mathbb{P}[x \in R_B] = \prod_{i=1}^{k} (1-q_i)(1-p_i)^{d_G(x)}$$

for every vertex $x \in B$.

By linearity of expectation, we obtain

$$\begin{aligned}
\gamma(G) &\leq \mathbb{E}[|D|] \\
&= \mathbb{E}[|R^{V_1}|] + \mathbb{E}[|R^{V_2}|] + \sum_{i=1}^{k} \Big(\mathbb{E}[|Y_i^{V_1}|] + \mathbb{E}[|Y_i^{V_2}|] \Big) + \sum_{i=1}^{k} \Big(\mathbb{E}[|Z_i^{V_1}|] + \mathbb{E}[|Z_i^{V_2}|] \Big) \\
&\leq \sum_{x \in A} \Bigg(\prod_{i=1}^{k} (1-p_i)(1-q_i)^{d_G(x)} + \sum_{i=1}^{k} \Big[p_i \cdot (1-p_{<i}) \cdot (1-q_{<i})^{d_G(x)} + \\
&\quad (1-p_{<i}) \cdot (1-q_{<i})^{d_G(x)} \cdot (1-p_i) \cdot \Big(\Big(1 - q_i (1-p_{<i})^{(\Delta_B - 1)} \Big)^{d_G(x)} - (1-q_i)^{d_G(x)} \Big) \Big] \Bigg) \\
&+ \sum_{x \in B} \Bigg(\prod_{i=1}^{k} (1-q_i)(1-p_i)^{d_G(x)} + \sum_{i=1}^{k} \Big[q_i \cdot (1-q_{<i}) \cdot (1-p_{<i})^{d_G(x)} + \\
&\quad (1-q_{<i}) \cdot (1-p_{<i})^{d_G(x)} \cdot (1-q_i) \cdot \Big(\Big(1 - p_i (1-q_{<i})^{(\Delta_A - 1)} \Big)^{d_G(x)} - (1-p_i)^{d_G(x)} \Big) \Big] \Bigg)
\end{aligned}$$

and the proof is complete. □

Theorem 1 still leaves the task to find good values for the probabilites $p_1, \ldots, p_k$ and $q_1, \ldots, q_k$. In order to compare it for instance to the bound of Alon and Spencer ($\frac{\gamma(G)}{n} \leq \frac{\ln(\delta+1)+1}{\delta+1}$ short AS, vgl. [1]), we present some numerical results for r-s-regular graphs in two rounds.

Table 1 gives the numerically optimal value for the bound on $\frac{\gamma(G)}{|V|}$ in Theorem 1. For comparison we also list the result by Harant and Pruchnewski in [4] for bipartite graphs (short HP). To show the improvement compared with the result in [3] for regular graphs, that bound is also shown.

r	s	AS	HP	k rounds (regular)	k rounds (bipartite)
1	2	0,84657359	0,333333	0,650898	0,333333
1	3		0,25	0,573343	0,25
1	5		0,166666	0,464088	0,166666
1	10		0,090909	0,323649	0,090909
2	2	0,69953743	0,5	0,596325	0,426062
2	3		0,4	0,541691	0,37984
2	5		0,285714	0,457228	0,285714
2	10		0,166666	0,337451	0,166666
3	3	0,59657359	0,5	0,499870	0,367340
3	5		0,375	0,432477	0,320023
3	10		0,230769	0,333735	0,230769
5	5	0,465293245	0,417649	0,385762	0,292534
5	10		0,319350	0,311052	0.244798
10	10	0,30889957	0,285899	0,256895	0,203927

Table 1 Numerical results for Theorem 1

References

[1] N. Alon and J. Spencer, The Probabilistic Method, John Wiley and Sons, Inc., 1992.

[2] V.I. Arnautov, Estimation of the exterior stability number of a graph by means of the minimal degree of the vertices, (Russian), *Prikl. Mat. Programm.* **11** (1974), 3-8.

[3] S. Artmann, F. Göring, J. Harant, D. Rautenbach and I. Schiermeyer, Random procedures for dominating sets in graphs, submitted.

[4] J. Harant and A. Pruchnewski, A note on the domination number of a bipartite graph, *Ann. Comb.* **5** (2001), 175-178.

[5] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, Inc., New York, 1998.

[6] T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in graphs advanced topics, Marcel Dekker, Inc., New York, 1998.

[7] C. Payan, Sur le nombre d'absorption d'un graphe simple, (French), *Cah. Cent. Étud. Rech. Opér.* **17** (1975), 307-317.