Elgersburg Summer School, March 2014
Mathematical Biology
Exercise Sheet 3– Lur'e systems

(1) For the trichotomy of stability results given in lectures: recall the key assumptions that the non–linearity $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $f(0) = 0$, $y \mapsto f(y)/y =: g(y)$ is strictly decreasing and introduce the terms

$$g_0 := \lim_{y \searrow 0} \frac{g(y)}{y}, \quad g_\infty := \lim_{y \to \infty} \frac{g(y)}{y} \quad \text{and} \quad p_e^* := \frac{1}{c^T(I-A)^{-1}b}.$$

(i) Prove that if $g_0 < p_e^*$ then zero is globally asymptotically stable.

(ii) Prove that if $g_\infty > p_e^*$ then every non–zero initial condition diverges to infinity.

(2) Suppose that $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $f(0) = 0$, $f \in C^2(\mathbb{R}_+)$, is strictly increasing, concave (that is, $-f$ is convex) and g defined by $g(y) := \frac{f(y)}{y}$ is strictly decreasing. Suppose further that there exists $\rho > 0$ such that

$$\rho y^* = f(y^*), \tag{1}$$

for some $y^* > 0$. Prove that

(i) such a positive y^* satisfying (1) is unique, and,

(ii) the sector condition around y^*

$$|f(y) - f(y^*)| \leq \rho |y - y^*|, \tag{2}$$

holds for all $y \geq 0$.

(3) Consider the Ricker function

$$[0, \infty) \ni y \mapsto f(y) := \beta y \mathrm{e}^{-\alpha y},$$

for constants $\alpha, \beta > 0$.

(i) Sketch the Ricker function.

(ii) Prove that for each $\alpha, \beta > 0$ there exists $\rho^* > 0$ such that for every $\rho \in (\rho^*, \beta)$

(a) there exists a unique positive intersection y^* as in (1), which you should find, and,

(b) the sector condition around $g(y^*)$ holds.

(iii) What is the critical value of $\rho^* > 0$ so that the sector condition (2) around the resulting y^* *fails* for the Ricker function?

(4) A matrix PPM for Anchovy (*Engraulis mordax*) is given by [1] with

$$A = \begin{bmatrix} 0.0224 & 0 & 0 & 0 & 0 & 0 & 0.0928 & 0.0098 & 0.0004 \\ 0.7710 & 0.1360 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5580 & 0.3360 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4150 & 0.6380 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2930 & 0.5760 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4130 & 0.8980 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0972 & 0.9770 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0202 & 0.9920 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0059 & 0.9930 \end{bmatrix},$$

which we have available as a `Matlab .mat` file.

(i) Demonstrate that A is primitive, calculate the dominant eigenvalue λ of A and thus verify that A models an asymptotically growing population.

(ii) It is argued that the linear matrix PPM is not accurately predicting anchovy population abundancies. Suppose that the PPM specified by A is replaced by a model that includes density–dependence. Specifically, take

$$
\begin{aligned}
A_0 &= \begin{bmatrix} 0.0224 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.7710 & 0.1360 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5580 & 0.3360 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4150 & 0.6380 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2930 & 0.5760 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.4130 & 0.8980 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0972 & 0.9770 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.0202 & 0.9920 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0059 & 0.9930 \end{bmatrix}, \\
b &= e_1 , \\
c^T &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.0928 & 0.0098 & 0.0004 \end{bmatrix} ,
\end{aligned}
\tag{3a}
$$

so that $A = A_0 + bc^T$ and introduce the Beverton–Holt density dependence for recruitment

$$
f : \mathbb{R}_+ \to \mathbb{R}_+ , \quad f(y) = \frac{\alpha y}{\beta + y} , \quad \alpha, \beta > 0 . \tag{3b}
$$

Consider now the Lur'e system

$$
x(t+1) = A_0 x(t) + b f(c^T x(t)) , \quad x(0) = x^0 , \quad t \in \mathbb{N}_0 . \tag{3c}
$$

(a) Compute $p_e^* := \frac{1}{c^T(I-A_0)^{-1}b}$ for (3).

(b) For what ranges of the parameter quotient $\frac{\alpha}{\beta}$ does (3) demonstrate zero GAS?

(c) For what ranges of the parameter quotient $\frac{\alpha}{\beta}$ does (3) have a GAS non–zero equilibrium? In these regions give formulae for the equilibria.

(d) Does (3) ever exhibit population blowup? If not, why not?

(e) Simulate (3) numerically for several different initial conditions x^0 and choices of $\frac{\alpha}{\beta}$ and plot your results.

(iii) Suppose that the parameter quotient $\frac{\alpha}{\beta}$ is such that (3) has a GAS non–zero equilibrium. We wish to implement a harvesting strategy for anchovy, which we model as

$$
x(t+1) = A_0 x(t) + b f(c^T x(t)) - du(t) , \quad x(0) = x^0 , \quad t \in \mathbb{N}_0 , \tag{4}
$$

where $d \in \mathbb{R}^n_+$ and $u(t) \geq 0$ is the harvesting effort at time–step t. We desire a persistent population in the presence of harvesting.

(a) What is the effect of a constant (or open–loop) harvesting strategy, $u(t) = u^*$ for all $t \in \mathbb{N}_0$? Describe your results analytically or numerically.

(b) Demonstrate that for $d = e_j$, $j \in \{1, 2, \ldots, 9\}$ the state feedback harvesting strategy

$$
u(t) = h^T x(t) , \quad t \in \mathbb{N}_0 ,
$$

where $h \in \mathbb{R}^n_+$ is to–be–determined can give rise to a new non–zero GAS which you should compute. Are there choices of d and h that lead to population extinction?

(iv) In many biological situations, the state x of (3c) is not known precisely, for example because the initial population distribution x^0 in (3c) is unknown. In fact, population managers may only have access to an observed portion of the state, say

$$
y(t) = Mx(t) , \quad t \in \mathbb{N}_0 ,
$$

where $M \in \mathbb{R}_+^{p\times n}$ is known. As such, it is of interest to use a dynamic Luenberger observer [2, 3] with state z to asymptotically estimate the state, where z is given by

$$z(t+1) = A_0 z(t) + bf(c^T z(t)) + HM(x(t) - z(t)), \quad z(0) = 0, \quad t \in \mathbb{N}_0. \tag{5}$$

In (5), $H \in \mathbb{R}_+^{n\times p}$ is chosen such that

$$A_1 := A_0 - HM \in \mathbb{R}_+^{n\times n}. \tag{6}$$

Prove that if f in (3c) is globally Lipschitz with Lipschitz constant $\ell > 0$ and furthermore

$$c^T(I - A_1)^{-1} b\ell < 1,$$

then

(a) the observer $z(t)$ is componentwise nonnegative for all $t \in \mathbb{N}_0$,

(b) the error $e(t) := x(t) - z(t) \to 0$ as $t \to \infty$, and,

(c) if f is increasing then the error $e(t)$ is componentwise nonnegative for all $t \in \mathbb{N}_0$.

Design an observer for the specific model (3) for different choices of M and H and simulate your results numerically.

References

[1] N. C. Lo, P. E. Smith, and J. L. Butler, "Population growth of northern anchovy and pacific sardine using stage-specific matrix models," *Marine ecology progress series. Oldendorf*, vol. 127, no. 1, pp. 15–26, 1995.

[2] D. Luenberger, "Observing the state of a linear system," *Military Electronics, IEEE Transactions on*, vol. 8, no. 2, pp. 74–80, 1964.

[3] D. Luenberger, "An introduction to observers," *Automatic Control, IEEE Transactions on*, vol. 16, no. 6, pp. 596–602, 1971.