Elgersburg Summer School, March 2014
Mathematical Biology
Exercise Sheet 3– Solutions

(1) Completed in the lecture.

(2) We additionally require that f is *strictly* concave, or that $y \mapsto \frac{f(y)}{y}$ is strictly decreasing. The question as written admits the case that f is linear, so that $f(x) = \alpha x$ for some $\alpha > 0$, which does not satisfy the claims.

(i) When the function g is strictly decreasing then so is the function

$$\mathbb{R}_+ \ni y \mapsto h(y) := g(y) - \rho\,.$$

A positive y^* satisfying (1) is a zero of h, which is thus unique as h is decreasing.

(ii) There are four conditions to check. The fact that $h(y) > 0$ for all $y \in [0, y^*)$ implies that

$$\begin{aligned} & f(y) \geq \rho y, \quad \forall\, y \in [0, y^*) \\ \Rightarrow \quad & f(y) - f(y^*) \geq \rho(y - y^*) = -\rho(y^* - y)\,, \quad \forall\, y \in [0, y^*). \end{aligned} \tag{†}$$

Similarly, as f is (strictly) increasing, trivially

$$f(y) \leq f(y^*) + \rho(y^* - y)\,, \quad y \in [0, y^*) \quad \Rightarrow \quad f(y) - f(y^*) \leq \rho(y^* - y)\,. \tag{††}$$

Combining (†) and (††) gives

$$\begin{aligned} & -\rho(y^* - y) \leq f(y) - f(y^*) \leq \rho(y^* - y)\,, \quad \forall\, y \in [0, y^*) \\ \iff \quad & |f(y) - f(y^*)| \leq \rho(y^* - y) = \rho|y - y^*|\,, \quad \forall\, y \in [0, y^*)\,. \end{aligned} \tag{$\circ$}$$

The sector trivially holds at $y = y^*$. For $y > y^*$ we have that $h(y) < 0$ and thus

$$f(y) \leq \rho y, \quad \forall\, y \geq y^*\,.$$

That f is increasing trivally gives that

$$f(y) \geq f(y^*) \geq f(y^*) - \rho(y - y^*)\,, \quad \forall\, y \geq y^*. \tag{§}$$

By the Mean Value Theorem, there exists $\xi \in (0, y^*)$ such that

$$f'(\xi) = \frac{f(y^*) - f(0)}{y^* - 0} = \rho\,.$$

That $-f$ is convex implies that $f'' \leq 0$ and thus f' is decreasing. Therefore,

$$f'(y^*) \leq f'(\xi) = \rho\,,$$

and thus for every $y > y^*$ there exists $\theta \in (y^*, y)$ such that

$$\frac{f(y) - f(y^*)}{y - y^*} = f'(\theta) \leq f'(y^*) = \rho\,. \tag{§§}$$

Combining (§) and (§§) gives

$$|f(y) - f(y^*)| \leq \rho|y - y^*|\,, \quad \forall\, y > y^*\,,$$

which teogther with ($\circ$) completes the proof.

(3) Covered in the lecture.

(4) (i) That A is primitive can be seen by direct calculation (compute powers of A), or by first noting that $A \geq L$, L a Leslie matrix, which is irreducible as $a_{1,9} > 0$. Furthermore, L is primitive, as it has a nonzero diagonal entry (for example a_{11}), and hence so is A. The dominant eigenvalue of A is $1.0026 > 1$ and hence the population is asymptotically growing.

(ii) (a) We have that $G(1) = 1.2312$ so that $\frac{1}{G1} = 0.8122$.

(b) The Lur'e system (3) has zero GAS when $G(1)g_0 < 1$ where recall

$$g_0 := \lim_{y \searrow 0} \frac{f(y)}{y}\,.$$

For the sample nonlinearity in (3b), we see that

$$g_0 := \lim_{y \searrow 0} \frac{f(y)}{y} = \lim_{y \searrow 0} \frac{\alpha}{\beta + y} = \frac{\alpha}{\beta}\,.$$

So zero GAS occurs when $\frac{\alpha}{\beta} < 0.8122$.

(c) The Lur'e system (3) has non–zero GAS for any $\frac{\alpha}{\beta} > 0.8122 = \frac{1}{G(1)}$. The equilibrium y^* satisfies

$$\frac{\alpha y^*}{\beta + y^*} = \frac{y^*}{G(1)} \quad \Rightarrow \quad y^* = \alpha G(1) - \beta\,.$$

(d) No, because here $g_\infty = 0$ (see lecture notes). In other words, the curve f is either always below the line with slope $\frac{1}{G(1)}$ through the origin, or there is a unique finite positive intersection.

(e) Matlab code is available.

(iii) (a) The new equilibria are determined by the solution(s) $y^* > 0$ of

$$y = G(1)f(y) - c^T(I - A_0)^{-1}du^*\,, \quad \text{or} \quad \frac{y}{G(1)} = f(y) - \frac{c^T(I - A_0)^{-1}d}{G(1)}u^*\,,$$

which is a shift (down by $c^T(I - A_0)^{-1}du^* \geq 0$) of the original situation considered. Note that there may be no positive intersection here, or many. We expect an ISS style estimate around the original equilibrium. Seemingly, numerically we may not have convergence.

(b) Inserting the state–feedback gives the model

$$x(t+1) = (A_0 - dh^T)x(t) + bf(c^T x(t))\,, \quad x(0) = x^0\,, \quad t \in \mathbb{N}_0\,. \qquad (\diamond)$$

For meaningful models we require $A_0 - dh^T \in \mathbb{R}_+^{n \times n}$ (otherwise we may get negative solutions). For $h, d \in \mathbb{R}_+^n$ it follows that

$$r(A_0 - dh^T) \leq r(A_0) < 1\,,$$

(see Exercise Sheet 2). Therefore, the trichotomy from MBL3 applies, with a new steady state gain $G_h(1) := c^T(I - (A_0 - dh^T))^{-1}b$. Note that as

$$A_0 - dh^T \leq A_0 \quad \Rightarrow \quad (I - (A_0 - dh^T))^{-1} = \sum_{j \in \mathbb{N}_0} (A_0 - dh^T)^j \leq \sum_{j \in \mathbb{N}_0} A_0^j = (I - A_0)^{-1}\,,$$

$$\Rightarrow \quad G_h(1) \leq G(1)\,,$$

and thus we may "leave" the non–zero GAS regime if h is sufficiently large and instead result in zero GAS.

(iv) (a) The observer has dynamics

$$z(t+1) = (A_0 - HM)z(t) + bf(c^T z(t)) + HMx(t)\,, \quad z(0) = 0\,, \quad t \in \mathbb{N}_0\,,$$

and as $A_0 - HM \in \mathbb{R}_+^{n\times n}$ and $HMx(t) \in \mathbb{R}_+^{n\times n}$ it follows that $z(t) \geq 0$ for every $t \in \mathbb{N}_0$.

(b) The error $e := x - z$ has dynamics given by

$$e(t+1) = (A_0 - HM)e(t) + b[f(c^T x(t)) - f(c^T z(t))]\,, \quad e(0) = 0\,, \quad t \in \mathbb{N}_0\,. \quad (\diamond)$$

A small gain argument proves that $e(t) \to 0$ as $t \to \infty$, since f is Lipschitz with constant ℓ and

$$c^T (I - A_1)^{-1} b\ell < 1\,.$$

Ask for more details!

(c) This follows from induction on t using the dynamics $(\diamond)$ and the fact that $e(0) = x_0 \geq 0$.