Elgersburg Summer School, March 2014
Mathematical Biology
Exercise Sheet 4– Solutions

(1) (i) This question is covered in the lecture, only we take $\gamma = 1$. See slides 7 and 11–13, where the conditions–to–be–derived are listed as *(I)–(IV)*.

(ii) Apply part (i) to the model data given. For the final part, we require $k^2 > 0$ such that

$$h(k^2) := d\left[k^4 - \frac{1}{d}(df_u + g_v)k^2 + \frac{1}{d}(f_u g_v - f_v g_u)\right] < 0\,, \qquad (*)$$

(see slides 13 and 14). Inserting the values given and completing the square in $(*)$ gives

$$\left(k - \frac{1}{2}\right)^2 < -4\left(-\frac{1}{2} + \frac{55}{100}\right) + 4\left(-\frac{1}{4} + \frac{1}{2}\right)^2 = \frac{1}{20}\,,$$

as required. The condition on k^2 suggests that the spatial domain must be large, but not too large.

(2) (i) The cross–termed partial derivatives f_v and g_u must be negative.

(ii) We assume that there is a stable steady state, so that

$$f_u + g_v < 0, \qquad f_u g_v - f_v g_u > 0\,. \qquad (\dagger)$$

By assumption, $f_v g_u > 0$ and thus

$$f_u g_v > f_v g_u > 0\,. \qquad (\ddagger)$$

If f_u and g_v are both positive, then the first condition in $(\dagger)$ cannot hold, so at least one must be negative, which when combined with $(\ddagger)$, we conclude that both f_u and g_v are negative. However, it now follows that

$$df_u + g_v < 0\,,$$

for all $d > 0$ and thus DDI is not possible.

(3) (i) Solving the equations

$$a - X^* - \frac{4X^*Y^*}{1 + (X^*)^2} = 0 \quad \text{and} \quad b\left(X^* - \frac{X^*Y^*}{1 + (X^*)^2}\right) = 0\,,$$

for X^* and Y^* gives $X^* = \frac{a}{5}$ and $Y^* = 1 + \frac{a^2}{25}$ as required.

(ii) The linearisation calculation is straightforward and is not given.

(iii) Necessary and sufficient conditions for

$$\frac{1}{1+\beta^2}\begin{pmatrix} -5 + 3\beta^2 & -4\beta \\ 2\gamma\beta^2 b & -\gamma\beta b \end{pmatrix},$$

(where $\beta = \frac{a}{5}$) to be stable are that the trace is negative and determinant is positive. We compute that the determinant is given by

$$\frac{1}{1+\beta^2}(5\gamma\beta b - 3\gamma\beta^3 b + 8\gamma\beta^3 b) = \frac{5}{1+\beta^2}(\gamma\beta b + \gamma\beta^3 b) > 0\,,$$

always. Computing the trace gives, therefore, the required characterisation.

(iv) These conditions follow as in those in question (1), inserting the particular model data.

(v) Not given here. Ask to see.

(4) By assumption there exists $P = P^* > 0$ such that

$$A^*P + PA < 0, \quad B^*P + PB = -Q\,, \tag{$\circ$}$$

for some $Q = Q^* > 0$ (the inequality involving A is not important here). Pre– and post–multiplying the equality in ($\circ$) by B^{-*} and B^* respectively gives

$$A^*P + PA < 0, \quad B^{-*}P + PB^{-1} = -\underbrace{B^{-*}QB^{-1}}_{>0}\,,$$

proving that P is a CLF for A and B^{-1}.

(5) If $P = P^* > 0$ is a Lyapunov function for A, meaning that

$$A^*P + PA < -\beta^2 I\,,$$

for some $\beta > 0$, then we compute that

$$\begin{aligned} B^*P + PB &= A^*P + PA + (B - A^*)P + P(B - A) \\ &< -\beta^2 I + \|P\| \cdot \|B - A\|\,. \end{aligned}$$

If $\|A - B\|$ is small, then P is a Lyapunov function for B as well, and thus $A + \alpha B$ is stable for any $\alpha > 0$. Now we take the contrapositive.

(6) Suppose that $P = P^* > 0$ solves the Riccati equation

$$A^*P + PA + \rho^2 E^*E + PDD^*P = 0\,,$$

for some $\rho > 0$. Let $A_\Delta := A + D\Delta E$ and consider

$$\begin{aligned} A_\Delta^* P + PA_\Delta &= A^*P + PA + E^*\Delta^*D^*P + PD\Delta E \\ &= -\rho^2 E^*E - PDD^*P + E^*\Delta^*D^*P + PD\Delta E\,. \end{aligned}$$

Therefore, for $x \in \mathbb{C}^n$

$$\begin{aligned} \langle (A_\Delta^* P + PA_\Delta)x, x\rangle &= \langle (-\rho^2 E^*E - PDD^*P + E^*\Delta^*D^*P + PD\Delta E)x, x\rangle \\ &= -\rho^2\|Ex\|^2 + \|\Delta Ex\|^2 - \|\|(D^*P + \Delta E)x\|^2 \\ &\leq -(\rho^2 - \|\Delta\|^2)\|Ex\|^2\,, \end{aligned} \tag{§}$$

which is negative semi–definite when $\|\Delta\| < \rho$. Therefore,

$$A_\Delta^* P + PA_\Delta \leq 0\,,$$

and a more involved argument using (§) demonstrates that in fact $\sigma(A_\Delta) \subseteq \mathbb{C}_0^-$. Ask for more details!

(7) The linearisation computations are elementary. For the numerical solutions see lecture slides 24 and 25.

(8) As above, the linearisation computations are straightforward. For the numerical solutions see lecture slides 29 and 32.