Elgersburg School 2014
Hybrid Systems

Worksheet 1

Problem -1. Let $D = \{x \in \mathbb{R}^2 \,|\, x_1 = x_2 > 0\}$, $C = \mathbb{R}^n \setminus D$, $f(x) = (-1, 0)$ for every $x \in C$, $g(x) = (x_1, x_2/2)$ for every $x \in D$. Write explicit formulas for the solution to the hybrid system (C, f, D, g) from the initial point $(3, 1)$ and then from the initial point $(3, 0)$. This should include formulas for the hybrid time domain of each solution. Then, replace C by $\overline{C}$, the closure of C, and D by $\overline{D}$, the closure of D and come up with explicit formulas for different solutions from the mentioned initial points.

Problem 0. Let $C = \{x \in \mathbb{R} \,|\, |x| \geq 1\}$, $f(x) = x^2$ for every $x \in C$, $D = [0, 1]$, $g(x) = x - 1$ for every $x \in D$. Write explicit formulas for two different maximal solutions to the hybrid system (C, f, D, g) from the initial point 1 and then for the maximal solution from the initial point -2.

Problem 1. The on/off behavior of a heater designed to keep the temperature in the room between two values, say 19 and 21 Celsius can be modeled by a hybrid system with a "discrete variable" taking on the values of either 0 and 1, representing, respectively, the heater being off and the heater being on. Suppose that when the heater is off, the temperature decays to the rooms natural temperature of 15 Celsius, while when the heater is on, the temperature increases, following the Newton's law, to 30 degrees.

Write the (C, F, D, G) model representing the behavior of the room's temperature and the state of the heater.

Problem 2. The timing of flashes of a firefly is determined by the fireflys internal clock. In between flashes, the internal clock gradually increases. When it reaches a threshold, a flash occurs and the clock is instantly reset to 0. In a group of fireflies, the flash of one firefly affects the internal clock of all other fireflies. That is, when a firefly witnesses a flash from another firefly, its internal clock instantly decreases to a value closer to 0.

To model the internal clocks of n fireflies, let x_i, $i = 1, 2, \ldots, n$, represent the i-th firefly internal clock, normalize units so that the internal clock of each firefly takes values in the interval $[0, 1]$, i.e., every threshold is 1, and suppose that a flash of one firefly reduces the internal clock of all other fireflies to half of their value before the flash.

Write the (C, F, D, G) model for the behavior of the n internal clocks.

The description above was created for academic purposes only, to produce an easier to write down model. A description which may be closer to reality and which leads to a model for which one can prove synchronization of the flashes among all fireflies (using Lyapunov functions) is this: when a firefly flashes, it resets its clock to 0*, and when a firelfly witnesses a flash from another firefly, its timer is instantaneously increased by a factor* $(1 + \gamma)$*, with small* $\gamma > 0$*... unless that increase would push the timer beyond* 1*, in which case the timer is reset to* 0*.*

Problem 3. Given m matrices $A_i \in \mathbb{R}^{n \times n}$, a switching (or switched) system is represented by

$$\dot{x} = A_\sigma x.$$

A solution to the switching system consists of a piecewise continuous switching signal $\sigma : [0, \infty) \to$

$\{1, 2, \ldots, m\}$ and a piecewise smooth function $x : [0, \infty) \to \mathbb{R}^n$ such that

$$\dot{x}(t) = A_{\sigma(t)} x(t)$$

for almost every $t \in [0, \infty)$. Usually, one considers a switching system and a whole class of admissible switching signals. Using the (C, F, D, G) formalism, write down a hybrid system model of a switching system where:

(a) Intervals on which σ is continuous are at least T long.

(b) Intervals on which σ is continuous are at least T long and at most T' long.

(c) Intervals on which σ is continuous are at least T long and σ has at most K discontinuities.

(d) Intervals on which σ is continuous are at least T long, as in (a), with the additional feature that the variable representing σ in the solution to the hybrid system is bounded for $t \in [0, \infty)$... unless, of course, your answer to (a) had this feature.

Problem 4. Recall the Bouncing Ball model: $(\dot{x}_1, \dot{x}_2) = (x_2, -1)$ when $x_1 \geq 0$, $(x_1^+, x_2^+) = (x_1, -0.5x_2)$ when $x_1 = 0$, $x_2 < 0$. Here x_1 is the height above the floor, x_2 is the velocity, the gravity constant is taken to be -1 and the coefficient of restitution is taken to be 0.5, for simplicity. Show that the solution to this model from an initial point $(0, v_0)$ with $v_0 > 0$ is Zeno.

Problem 5. Let $g : \mathbb{R} \to \mathbb{R}$ be an odd function given, for $x > 0$, by

$$g(x) = \begin{cases} \dfrac{p-1}{q} & \text{for } x = \dfrac{p}{q}, \text{ q is prime, } \gcd(p, q) = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\gcd(p, q)$ stands for the greatest common divisor of p and q. Show that every solution to $x^+ = g(x)$ reaches 0 in a finite number of jumps and for each initial point $x \in \mathbb{R}$ determine the number of jump this takes.