Elgersburg School 2014
Hybrid Systems

Worksheet 2

Problem 1. Consider the hybrid system given by

$$C = (-\infty, 1) \quad f(x) = x(1-x) \;\; \forall x \in C$$
$$D = [1, \infty) \qquad g(x) = 0 \;\; \forall x \in D$$

Do all (maximal) solutions converge to 0? Is 0 uniformly asymptotically stable?

Problem 2. For the thermostat model, where $x = (T, q) \in \mathbb{R}^2$ represents the temperature and the off or on state of the thermostat, and the data (with $k > 0$ being some constant) is

$$C = [19, \infty) \times \{0\} \cup (-\infty, 21] \times \{1\} \quad f(T, q) = \begin{cases} (k(15 - T), 0) & \text{if } q = 0 \\ (k(30 - T), 0) & \text{if } q = 1 \end{cases}$$
$$D = (-\infty, 19) \times \{0\} \cup (21, \infty) \times \{1\} \quad g(T, q) = (T, 1 - q)$$

show that the compact set $[19, 21] \times \{0, 1\}$ is uniformly asymptotically stable.

Problem 3. Consider two 2×2 matrices given by

$$A_1 = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 4 \\ -1 & 0 \end{pmatrix}.$$

(a) Verify that solutions to $\dot{x} = A_1 x$ rotate counterclockwise along the ellipses $x_1^2 + \frac{x_2^2}{2^2} = \text{const}$ and solutions to $\dot{x} = A_2 x$ rotate clockwise along the ellipses $\frac{x_1^2}{2^2} + x_2^2 = \text{const}$.

(b) Find a switching signal, that is, a function $\sigma : [0, \infty) \to \{1, 2\}$ so that the solution to the time-dependent differential equation $\dot{x} = A_{\sigma(t)} x$ from the initial condition $(2, 1)$ diverges to ∞.

Problem 4. Recall the Bouncing Ball model: $(\dot{x}_1, \dot{x}_2) = (x_2, -1)$ when $x_1 \geq 0$, $(x_1^+, x_2^+) = (x_1, -0.5x_2)$ when $x_1 = 0$, $x_2 < 0$. Here x_1 is the height above the floor, x_2 is the velocity, the gravity constant is taken to be -1 and the coefficient of restitution is taken to be 0.5, for simplicity.

(a) Find a $\mathcal{KL}$ bound on the solutions to the Bouncing Ball model.

(b) Is $V_1(x) = \frac{1}{2}x_2^2 + x_1$ (which represents the total energy of the ball) a Lyapunov function verifying the uniform asymptotic stability of the origin?

(c*) Is $V_2(x) = (1 + \frac{3}{5\pi}\arctan(x_2))V_1(x)$ a Lyapunov function for that purpose? Alternatively, try $V_4(x) = \sqrt{V_1(x)} + \varepsilon x_2$ with some small $\varepsilon > 0$.

Problem 5. Suppose that the function $V : \mathbb{R}^n \to [0, \infty)$ is a Lyapunov function for each of the differential equations $\dot{x} = f_q(x)$, $q = 1, 2, \ldots, m$, where $f_q : \mathbb{R}^n \to \mathbb{R}^n$ are some continuous functions. *(So, V is positive definite, radially unbounded, continuously differentiable, and for each $q = 1, 2, \ldots, m$, $\nabla V(x) \cdot f_q(x) \leq -V(x)$ for every $x \in \mathbb{R}^n$.)* Show that the origin is uniformly asymptotically stable for the differential inclusion

$$\dot{x} \in F(x),$$

where, for every $x \in \mathbb{R}^n$, the set $F(x)$ is the convex hull of $f_1(x), f_2(c), \ldots, f_m(x)$.