Balanced Truncation for Stochastic Differential Equations with Levy Noise

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Introduction

Concept of Reachability

Concept of Observability

Balanced Truncation

Levy processes

Properties of Levy processes

- Levy processes generalize Wiener processes.
- The trajectories may have jumps but at most countably many.
- The trajectories are right-continuous and have left limits.

We will focus on square integrable Levy processes \((M(t))_{t \geq 0}\) with zero mean, which means that

\[
\mathbb{E} [M(t)] = 0 \text{ and } \mathbb{E} [M(t)^2] < \infty
\]

for all \(t \geq 0\).
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for all \(t \geq 0\).
We consider the following equation for $t \geq 0$:

$$
\begin{align*}
\frac{dX(t)}{dt} &= [AX(t) + Bu(t)] dt + \Psi X(t-)dM(t), \\
Y(t) &= CX(t),
\end{align*}
$$

where $X(0) = x_0$, $A, \Psi \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $M$ is a real-valued square integrable Levy process with zero mean and $u$ is an adapted stochastic processes with

$$
\|u\|_{L^2}^2 := \mathbb{E} \int_0^\infty \|u(t)\|^2 dt < \infty.
$$

**Definition**

see [Damm '04]

The system (1) is called asymptotically mean square stable if the homogeneous solution $X_{x_0}^h$ fulfills

$$
\mathbb{E} \|X_{x_0}^h(t)\|_2^2 \to 0
$$

for $t \to \infty$ and every initial condition $x_0 \in \mathbb{R}^n$. 
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for $t \to \infty$ and every initial condition $x_0 \in \mathbb{R}^n$. 
We introduce the stochastic processes \((\Phi(t, \tau))_{t \geq \tau}\) with values in \(\mathbb{R}^{n \times n}\), which satisfy

\[
\Phi(t, \tau) = I_n + \int_{\tau}^{t} A\Phi(s, \tau)ds + \int_{\tau}^{t} \Psi\Phi(s-, \tau)dM(s)
\]

for \(t \geq \tau \geq 0\). We write \(\Phi(t, 0) = \Phi(t)\).

**Proposition**

The solution of equation (1) is given by

\[
\Phi(t)x_0 + \int_{0}^{t} \Phi(t, s)Bu(s)ds, \quad t \geq 0.
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Concept of Reachability
Reachability Gramian

We call $P_t := \int_0^t \mathbb{E} \left[ \Phi(s)B B^T \Phi^T(s) \right] ds$ finite reachability Gramian at time $t \geq 0$.

The asymptotic stability in mean square also provides the existence of the infinite reachability Gramian

$$P := \int_0^\infty \mathbb{E} \left[ \Phi(s)B B^T \Phi^T(s) \right] ds.$$

**Proposition**

For all $t \geq 0$ it holds

$$im \, P_t = im \, P.$$

One can show that

$$0 = B B^T + P \, A^T + A \, P + \psi \, P \, \psi^T \mathbb{E} \left[ M(1)^2 \right].$$
Reachability Gramian

[Benner, Damm '11]

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$$0 = BB^T + P A^T + A P + \psi P \psi^T \mathbb{E} \left[ M(1)^2 \right].$$
By $X(T, 0, u)$ we denote the solution of the inhomogeneous system (1) at time $T$ with initial condition 0 for a given input $u$.

**Definition**

An average state $x \in \mathbb{R}^n$ is called reachable (from zero) if a time $T > 0$ and a control function $u \in L_T^2$ exist such that

$$
\mathbb{E}[X(T, 0, u)] = x.
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**Proposition**

An average state $x \in \mathbb{R}^n$ is reachable (from zero) if and only if $x \in \text{im } P$. 
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M. Redmann, *Balanced Truncation for Stochastic Differential Equations with Levy Noise*
Proposition

The control \( u \) with minimal energy to reach \( x \in \text{im } P \) at time \( T > 0 \) is given by

\[
 u(t) = B^T \Phi^T(T, t) P^\# T x, \quad t \in [0, T],
\]

with

\[
 \|u\|_{L^2_T}^2 = x^T P^\# T x.
\]

Thus, the minimal energy that is needed to steer the system to \( x \) is given by

\[
 \inf_{T > 0} x^T P^\# T x = x^T P^\# x.
\]
**Proposition**

The control $u$ with minimal energy to reach $x \in \text{im } P$ at time $T > 0$ is given by

$$u(t) = B^T \Phi^T(T, t) P_T^# x, \quad t \in [0, T],$$

with

$$\|u\|_{L_T^2}^2 = x^T P_T^# x.$$

Thus, the minimal energy that is needed to steer the system to $x$ is given by

$$\inf_{T > 0} x^T P_T^# x = x^T P^# x.$$
Concept of Observability
Due to the asymptotic stability in mean square the observability Gramian

\[ Q = \mathbb{E} \left[ \int_0^\infty \Phi^T(s) C^T C \Phi(s) ds \right] \]

exists and it is the solution of

\[ A^T Q + Q A + \Psi^T Q \Psi \mathbb{E} \left[ M(1)^2 \right] + C^T C = 0. \]
We consider an uncontrolled system (1) \((u \equiv 0)\).
Hence, \(X(t) = \Phi(t)x_0\) and
\[
\mathcal{Y}(t) = C\Phi(t)x_0.
\]

The observation energy produced by \(x_0\) is
\[
\|\mathcal{Y}\|_{L^2}^2 := \mathbb{E} \int_0^\infty \mathcal{Y}^T(t)\mathcal{Y}(t) dt = x_0^T \mathbb{E} \int_0^\infty \Phi^T(t)C^T C\Phi(t) dt x_0 \\
= x_0^T Q x_0.
\]

**Definition**

An initial condition \(x_0\) is unobservable if \(x_0 \in \ker Q\).
We consider an uncontrolled system (1) \( (u \equiv 0) \).
Hence, \( X(t) = \Phi(t)x_0 \) and
\[
Y(t) = C\Phi(t)x_0.
\]
The observation energy produced by \( x_0 \) is
\[
\|Y\|_{L^2}^2 := \mathbb{E} \int_0^\infty Y^T(t)Y(t)dt = x_0^T \mathbb{E} \int_0^\infty \Phi^T(t)C^TC\Phi(t)dt x_0 = x_0^T Qx_0.
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**Definition**

An initial condition \( x_0 \) is unobservable if \( x_0 \in \ker Q \).
Balanced Truncation
We start with a system

\[dX(t) = [AX(t) + Bu(t)] \, dt + \Psi X(t-)dM(t),\]

\[Y(t) = CX(t), \quad t \geq 0,\]

which is completely reachable and observable, which is equivalent to \( P \) and \( Q \) are positive definite.

We transfer the states with a regular matrix \( T \) and obtain a system with

\[(\tilde{A}, \tilde{B}, \tilde{\Psi}, \tilde{C}) = (TAT^{-1}, TB, T\Psi T^{-1}, CT^{-1}),\]

which has the same output as system (2).

The Gramians of the transformed system are given by

\[\tilde{P} = TPT^T \text{ and } \tilde{Q} = T^{-T} QT^{-1}.\]
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The Gramians of the transformed system are given by

\[
\tilde{P} = TPT^T \text{ and } \tilde{Q} = T^{-T} QT^{-1}.
\]
We want to choose $T$, such that the Gramians are equal, which means that $\tilde{P} = \tilde{Q} = \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$.

**Theorem**

The balancing transformation $T$ and its inverse are given by

$$T = \Sigma^{\frac{1}{2}} K^T U^{-1},$$

$$T^{-1} = U K \Sigma^{-\frac{1}{2}},$$

where $P = UU^T$ and $U^T QU = K \Sigma^2 K^T$. 
We consider the following partitions:

\[
T = \begin{bmatrix}
W^T \\
T_2^T
\end{bmatrix}, \quad T^{-1} = \begin{bmatrix}
V & T_1
\end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{pmatrix}
\tilde{X} \\
X_1
\end{pmatrix},
\]

where \( W^T \in \mathbb{R}^{r \times n} \), \( V \in \mathbb{R}^{n \times r} \) and \( \tilde{X} \in \mathbb{R}^r \).

Hence,

\[
\begin{pmatrix}
d\tilde{X}(t) \\
dX_1(t)
\end{pmatrix} = \begin{bmatrix}
W^T AV & W^T AT_1 \\
T_2^T AV & T_2^T AT_1
\end{bmatrix} \begin{pmatrix}
\tilde{X}(t) \\
X_1(t)
\end{pmatrix} dt + \begin{bmatrix}
W^T B \\
T_2^T B
\end{bmatrix} u(t) dt
\]

\[
+ \begin{bmatrix}
W^T \psi V & W^T \psi T_1 \\
T_2^T \psi V & T_2^T \psi T_1
\end{bmatrix} \begin{pmatrix}
\tilde{X}(t-) \\
X_1(t-)
\end{pmatrix} dM(t)
\]

and

\[
\mathcal{Y}(t) = \begin{bmatrix}
CV & CT_1
\end{bmatrix} \begin{pmatrix}
\tilde{X}(t) \\
X_1(t)
\end{pmatrix}.
\]
We consider the following partitions:

\[ T = \begin{bmatrix} W^T \\ T_2^T \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} V & T_1 \end{bmatrix} \quad \text{and} \quad \hat{X} = \begin{pmatrix} \tilde{X} \\ X_1 \end{pmatrix}, \]

where \( W^T \in \mathbb{R}^{r \times n}, V \in \mathbb{R}^{n \times r} \) and \( \tilde{X} \in \mathbb{R}^r \).

Hence,

\[
\begin{align*}
\begin{pmatrix} d\tilde{X}(t) \\ dX_1(t) \end{pmatrix} &= \begin{bmatrix} W^T AV & W^T A T_1 \\ T_2^T AV & T_2^T A T_1 \end{bmatrix} \begin{pmatrix} \tilde{X}(t) \\ X_1(t) \end{pmatrix} dt + \begin{bmatrix} W^T B \\ T_2^T B \end{bmatrix} u(t) dt \\
&\quad + \begin{bmatrix} W^T \psi V & W^T \psi T_1 \\ T_2^T \psi V & T_2^T \psi T_1 \end{bmatrix} \begin{pmatrix} \tilde{X}(t-) \\ X_1(t-) \end{pmatrix} dM(t)
\end{align*}
\]

and

\[
\mathcal{Y}(t) = \begin{bmatrix} CV & CT_1 \end{bmatrix} \begin{pmatrix} \tilde{X}(t) \\ X_1(t) \end{pmatrix}.
\]
Selecting the first $r$ rows and neglecting the $X_1$ terms provide the following reduced order model:

$$d\tilde{X}(t) = W^TAV\tilde{X}(t)dt + W^T Bu(t)dt + W^T\Psi V\tilde{X}(t-)dM(t)$$

$$\tilde{Y}(t) = CV\tilde{X}(t).$$

Open Questions

- Is the reduced order model stable in general?
  deterministic case: The reduced order model is stable.

- What is the structure of the Gramians of the reduced model?
  deterministic case: $P_R = Q_R = diag(\sigma_1, \ldots, \sigma_r)$

- What is the structure of the Hankel values of the reduced model?
  deterministic case: The Hankel values are $\sigma_1, \ldots, \sigma_r$. 
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   - Deterministic case: The Hankel values are \( \sigma_1, \ldots, \sigma_r \).
Example

The following matrices provide a balanced system:

\[
A = \begin{pmatrix}
-5.12 & 2.99 & 2.05 \\
1.25 & -4.86 & 0.96 \\
1.26 & 0.07 & -9.52
\end{pmatrix},
B = \begin{pmatrix}
-3.39 & -1.19 & -0.61 \\
1.31 & -4.60 & -0.09 \\
-0.68 & 1.70 & 4.57
\end{pmatrix},
\]
\[
\Psi = \begin{pmatrix}
-2.20 & 2.09 & -0.57 \\
1.30 & -0.67 & -2.06 \\
0.30 & -0.57 & -1.14
\end{pmatrix},
C = \begin{pmatrix}
-2.77 & 1.31 & 1.62 \\
-3.94 & -2.18 & 0.74 \\
-1.31 & -2.49 & 1.74
\end{pmatrix}.
\]

The Gramians are given by

\[
P = Q = \Sigma = \begin{pmatrix}
5.97 & 0 & 0 \\
0 & 4.23 & 0 \\
0 & 0 & 1.48
\end{pmatrix}.
\]

The reduced order model has the following properties:

\[
P_R = \begin{pmatrix}
5.46 & -0.38 \\
-0.38 & 3.43
\end{pmatrix},
Q_R = \begin{pmatrix}
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The reduced order model has the following properties:

\[
P_R = \begin{pmatrix}
  5.46 & -0.38 \\
  -0.38 & 3.43 \\
\end{pmatrix},
\]

\[
Q_R = \begin{pmatrix}
  5.88 & 0.06 \\
  0.06 & 4.16 \\
\end{pmatrix},
\]

\[
HV_R = \begin{pmatrix}
  5.68 \\
  3.76 \\
\end{pmatrix}.
\]
We introduce the following partitions of a balanced realization:

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \Psi = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.
\]

We write \( E = \mathbb{E} \| \mathcal{Y}(t) - \tilde{\mathcal{Y}}(t) \|_2 \) and obtain

\[
E = \mathbb{E} \left\| C \int_0^t \Phi(t,s)Bu(s)ds - C_1 \int_0^t \tilde{\Phi}(t,s)B_1u(s)ds \right\|_2 \\
\leq \mathbb{E} \int_0^t \left\| (C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1)u(s) \right\|_2 ds \\
\leq \mathbb{E} \int_0^t \left\| C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1 \right\|_F \| u(s) \|_2 ds \\
\leq \left( \mathbb{E} \int_0^t \left\| C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1 \right\|^2_F ds \right)^{1/2} \left( \mathbb{E} \int_0^t \| u(s) \|^2_2 ds \right)^{1/2}.
\]
We introduce the following partitions of a balanced realization:

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\]

We write \( \mathcal{E} = \mathbb{E} \| \mathcal{Y}(t) - \tilde{\mathcal{Y}}(t) \|_2 \) and obtain

\[
\begin{align*}
\mathcal{E} &= \mathbb{E} \left\| C \int_0^t \Phi(t,s)Bu(s)ds - C_1 \int_0^t \tilde{\Phi}(t,s)B_1u(s)ds \right\|_2 \\
&\leq \mathbb{E} \int_0^t \left\| \left( C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1 \right)u(s) \right\|_2 ds \\
&\leq \mathbb{E} \int_0^t \left\| C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1 \right\|_F \left\| u(s) \right\|_2 ds \\
&\leq \left( \mathbb{E} \int_0^t \left\| C\Phi(t,s)B - C_1\tilde{\Phi}(t,s)B_1 \right\|_F^2 ds \right)^{1/2} \left( \mathbb{E} \int_0^t \left\| u(s) \right\|_2^2 ds \right)^{1/2}.
\end{align*}
\]
We have

$$\sup_{t \geq 0} \mathbb{E} \left\| \mathcal{Y}(t) - \tilde{\mathcal{Y}}(t) \right\|_2^2 \
\leq \left( \mathbb{E} \int_0^\infty \left\| C \Phi(t, s) B - C_1 \tilde{\Phi}(t, s) B_1 \right\|_F^2 \, ds \right)^{\frac{1}{2}} \| u \|_{L^2}.$$

It holds

$$\# = \left( \text{tr} \left( C \Sigma C^T \right) + \text{tr} \left( C_1 P_R C_1^T \right) - 2 \text{tr} \left( C P_M C_1^T \right) \right)^{\frac{1}{2}},$$

where

$$0 = B B_1^T + P_M A_{11}^T + A P_M + \Psi P_M \Psi_{11}^T \mathbb{E} \left[ M(1)^2 \right].$$
We have

$$\sup_{t \geq 0} \mathbb{E} \left\| \mathcal{Y}(t) - \tilde{\mathcal{Y}}(t) \right\|_2^2 \leq \left( \mathbb{E} \int_0^\infty \left\| C\Phi(t, s)B - C_1\tilde{\Phi}(t, s)B_1 \right\|_F^2 ds \right)^{1/2} \left\| u \right\|_{L^2}.$$

It holds

$$\# = \left( tr \left( C\Sigma C^T \right) + tr \left( C_1 P_R C_1^T \right) - 2 \ tr \left( CP_M C_1^T \right) \right)^{1/2},$$

where

$$0 = BB_1^T + P_M A_{11}^T + A P_M + \Psi P_M \Psi_{11}^T \mathbb{E} \left[ M(1)^2 \right].$$
We assume

$$\Psi = \begin{bmatrix} \Psi_{11} & 0 \\ 0 & \Psi_{22} \end{bmatrix}.$$  

We additionally need the partitions

$$\Sigma = \begin{bmatrix} \Sigma_1 \\ \Sigma_2 \end{bmatrix} \quad \text{and} \quad P_M = \begin{bmatrix} P_{M,1} \\ P_{M,2} \end{bmatrix}.$$  

Then, $P_R = Q_R = \Sigma_1$ and

$$\#^2 = \text{tr}((C_2^T C_2 + 2P_{M,2}A_{21}^T)\Sigma_2).$$
How to check stability of the reduced order model?

**Theorem**

Let $Y$ be an arbitrary symmetric and negative definite matrix. Then, the reduced order model is asymptotically mean square stable, if and only if the solution $X$ of

$$A_{11}^T X + X A_{11} + \psi_{11}^T X \psi_{11} \mathbb{E} \left[ M(1)^2 \right] = Y$$

is unique, symmetric and positive definite.

see [Damm '04]
We consider a company, which produces \( n \) goods. \( X_i \) represents the asset of the company corresponding to good \( i \).

We assume that the vector of all assets satisfies

\[
dX(t) = [u(t) - \text{diag}(\lambda_1, \ldots, \lambda_n)X(t)] \, dt + X(t-)dM(t)
\]

\[
X(0) = x_0,
\]

where \( \lambda_i \) is the depreciation rate of \( X_i \) and \( u \) is the vector of investments.

We observe that the system is asymptotically mean square stable if and only if \( \lambda_i > 0.5 \mathbb{E} \left[ M(1)^2 \right] \) for all \( i = 1, \ldots, n \).

The output equation we consider is given by

\[
Y(t) = (1, \ldots, 1) \, X(t).
\]
We consider a company, which produces $n$ goods. $X_i$ represents the asset of the company corresponding to good $i$.

We assume that the vector of all assets satisfies

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$$X(0) = x_0,$$

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Example

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The output equation we consider is given by

\[
Y(t) = (1, \ldots, 1) \, X(t).
\]
We assume that $n = 80$, $\mathbb{E}[M(1)^2] = 1$ and $\lambda_i \in (0.5, 1.5]$:

<table>
<thead>
<tr>
<th>Dimension of reduced order model</th>
<th>#</th>
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</thead>
<tbody>
<tr>
<td>40</td>
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</tr>
<tr>
<td>20</td>
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</tr>
<tr>
<td>10</td>
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<td>5</td>
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Thank you for your attention!

References


