Perturbation theory for eigenvalues of Hermitian pencils

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joint work with Shreemayee Bora, Michael Karow, and Punit Sharma
Question: Why are eigenvalues of Hermitian pencils of interest in system theory?
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**Answer**: Because Hermitian pencils are related to Hamiltonian matrices.
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**Answer:** Because Hermitian pencils are related to Hamiltonian matrices.

A matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ is called Hamiltonian if

$$\mathcal{H} = \begin{bmatrix} A & C \\ D & -A^* \end{bmatrix},$$

where $A, C, D \in \mathbb{C}^{n \times n}$ and where $C, D$ are Hermitian.
Why Hermitian pencils?

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A matrix $\mathcal{H} \in \mathbb{C}^{2n \times 2n}$ is called **Hamiltonian** if

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\mathcal{H} = \begin{bmatrix}
A & C \\
D & -A^*
\end{bmatrix},
$$

where $A, C, D \in \mathbb{C}^{n \times n}$ and where $C, D$ are Hermitian.

**Observation**: The spectrum of Hamiltonian matrices is symmetric with respect to the imaginary axis: if $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathcal{H}$, then so is $-\bar{\lambda}$ with the same multiplicities.
The Linear Quadratic Optimal Control Problem: minimize the cost functional

\[
\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \, dt
\]

subject to the dynamics

\[
\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \in [0, \infty),
\]

where \( x(t), x_0 \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, A, Q \in \mathbb{R}^{n \times n}, S \in \mathbb{R}^{n \times m}, R \in \mathbb{R}^{m \times m}, \)

\[
\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0, \quad R > 0.
\]

The solution can be obtained by solving the eigenvalue problem for the Hamiltonian matrix

\[
\mathcal{H} := \begin{bmatrix} A - BR^{-1}S^T & -BR^{-1}B^T \\ SR^{-1}S^T - Q & -A^T + SR^{-1}B^T \end{bmatrix}.
\]
**Application 1: Optimal Control**

**More general problem**: minimize the cost functional

\[
\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \, dt
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E\dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \in [0, \infty),
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\[
\begin{bmatrix} Q & S \\ ST & R \end{bmatrix} \succeq 0, \quad R \succeq 0.
\]

The solution can be obtained by solving the generalized eigenvalue problem for the matrix pencil of the form:
Application 1: Optimal Control

More general problem: minimize the cost functional

$$\int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \, dt$$

subject to the dynamics

$$E \dot{x} = Ax + Bu, \quad x(0) = x_0, \quad t \in [0, \infty),$$

where $x(t), x_0 \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, E, A, Q \in \mathbb{R}^{n \times n}, S \in \mathbb{R}^{n \times m}, R \in \mathbb{R}^{m \times m}, \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succeq 0, \quad R \succeq 0.$

$$\lambda \mathcal{E} - \mathcal{A} := \lambda \begin{bmatrix} 0 & -E^* & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} Q & A^* & S \\ A & 0 & B \\ S^* & B^* & R \end{bmatrix}.$$
Observations: on the generalized eigenvalue problem:

\[ \lambda E - A := \lambda \begin{bmatrix} 0 & -E^* & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} Q & A^* & S \\ A & 0 & B \\ S^* & B^* & R \end{bmatrix}. \]

- \( L(\lambda) := \lambda E - A \) is an **even matrix pencil**, that is, \( L(\lambda)^* = L(-\lambda) \).
- The eigenvalues of \( L(\lambda) \) are symmetric wrt the imaginary axis.
- \( L(i\lambda) = i\lambda E - A \) is a **Hermitian matrix pencil**.
- The eigenvalues of \( L(i\lambda) \) are symmetric wrt the real axis.
- If \( R \) is invertible, then the generalized eigenvalue problem can be reduced to the form

\[ \lambda \begin{bmatrix} 0 & -E^* \\ E & 0 \end{bmatrix} - \begin{bmatrix} Q - SR^{-1}S^* & A^* - SR^{-1}B^* \\ A - BR^{-1}S^* & -BR^{-1}B^* \end{bmatrix}. \]
**Application 2: Algebraic Riccati Equations**

**Algebraic Riccati Equation (ARE):** find a solution $X \in \mathbb{C}^{n \times n}$ such that

$$D + XA + A^*X - XCY = 0,$$

where $A, C, D \in \mathbb{C}^{n \times n}$ and $C, D$ are Hermitian.

The solution can be obtained by solving the eigenvalue problem for the **Hamiltonian matrix**

$$\mathcal{H} := \begin{bmatrix} A & C \\ D & -A^* \end{bmatrix}.$$

If the columns of

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{C}^{2n \times n}$$

form a basis of an invariant subspace $\mathcal{H}$ associated with the eigenvalues in the left half plane and if $X_1$ is invertible, then $X = X_2X_1^{-1}$ is a solution of the ARE such that $A + GX \subseteq \mathbb{C}^-$. 

Application 2: gyroscopic systems

**Gyroscopic systems**: have the form

\[ I \ddot{x} + G \dot{x} + K x = 0, \]

where \( G, K \in \mathbb{C}^n \) and where \( G = -G^* \) and \( K = K^* \).

Introducing the new variables \( y_1 = \dot{x} - \frac{1}{2} G x \) and \( y_2 = x \) this system can be reduced to the first order system

\[ \dot{y} + \mathcal{H} y = 0 \]

with the **Hamiltonian matrix**

\[ \mathcal{H} = \begin{bmatrix} \frac{1}{2} G & K + \frac{1}{4} G^2 \\ I_n & \frac{1}{2} G \end{bmatrix}. \]

Consequently, the gyroscopic system is stable if all eigenvalues of \( \mathcal{H} \) are on the imaginary axis and semisimple.
**Experiment**: Consider two Hamiltonian systems:

\[
\dot{x} = \mathcal{H}_1 x = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix} x, \quad \dot{y} = \mathcal{H}_2 y = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
\end{bmatrix} y.
\]

**Observation**: Both matrices have the semisimple eigenvalues \(+i\) and \(-i\) with algebraic multiplicity 2 each. Consequently both systems are **stable**.

**Question**: How do the matrices (Systems) behave under perturbations, i.e., what happens to the eigenvalues of

\[\mathcal{H}_1 + \Delta \mathcal{H}_1, \quad \mathcal{H}_2 + \Delta \mathcal{H}_2?\]
Result 1: Eigenvalue distribution of 1000 perturbations, when $\Delta \mathcal{H}_i$ was a random matrix of norm $\frac{1}{4}$.

The perturbed systems are not stable.
Result 2: Eigenvalue distribution of 1000 perturbations, when $\Delta \mathcal{H}_i$ was a random Hamiltonian matrix of norm $\frac{1}{4}$.

The second perturbed system remains stable, the first one typically not.
Why?

For the understanding, we need ...
Indefinite Linear Algebra

- name **Indefinite Linear Algebra** invented by Gohberg, Lancaster, Rodman in 2005;

- \( \pm H^* = H \in \mathbb{F}^{n \times n} \) invertible defines an **inner product** on \( \mathbb{F}^n \):

  \[
  [x, y]_H := y^* H x \quad \text{for all } x, y \in \mathbb{F}^n;
  \]

  Here, \( * \) either denotes the transpose \( T \) or the conjugate transpose \( * \);

<table>
<thead>
<tr>
<th>( H = H^* )</th>
<th>Hermitian sesquilinear form</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = -H^* )</td>
<td>skew-Hermitian sesquilinear form</td>
</tr>
<tr>
<td>( H = H^T )</td>
<td>symmetric bilinear form</td>
</tr>
<tr>
<td>( H = -H^T )</td>
<td>skew-symmetric bilinear form</td>
</tr>
</tbody>
</table>

- the inner product **may be indefinite** (needs not be positive definite).
The adjoint: For $X \in \mathbb{F}^{n \times n}$ let $X^*$ be the matrix satisfying

$$[v, Xw]_H = [X^*v, w]_H \quad \text{for all } v, w \in \mathbb{F}^n.$$

We have $X^* = H^{-1}X^T H$ resp. $X^* = H^{-1}X^* H$.

Matrices with symmetries in indefinite inner products:

<table>
<thead>
<tr>
<th>$A$ $H$-selfadjoint</th>
<th>adjoint</th>
<th>$y^T H x$</th>
<th>$y^* H x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^* = A$</td>
<td>$A^T H = HA$</td>
<td>$A^* H = HA$</td>
<td></td>
</tr>
<tr>
<td>$S$ $H$-skew-adjoint</td>
<td>$S^* = -S$</td>
<td>$S^T H = -HS$</td>
<td>$S^* H = -HS$</td>
</tr>
<tr>
<td>$U$ $H$-unitary</td>
<td>$U^* = U^{-1}$</td>
<td>$U^T H U = H$</td>
<td>$U^* H U = H$</td>
</tr>
</tbody>
</table>
• A matrix $\mathcal{H} \in \mathbb{F}^{2n \times 2n}$ is Hamiltonian if and only if
\[ \mathcal{H}^T J = -J \mathcal{H}, \quad \text{where } J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \]

• Hamiltonian matrices are skew-adjoint with respect to the skew-symmetric bilinear form induced by $J$.

<table>
<thead>
<tr>
<th>J-selfadjoint</th>
<th>$N^T J = JN$</th>
<th>skew-Hamiltonian</th>
</tr>
</thead>
<tbody>
<tr>
<td>J-skew-adjoint</td>
<td>$H^T J = -JH$</td>
<td>Hamiltonian</td>
</tr>
<tr>
<td>J-unitary</td>
<td>$S^T JS = J$</td>
<td>symplectic</td>
</tr>
</tbody>
</table>

• **Symplectic matrices** occur in **discrete optimal control problems**.
“Hermitian matrix pencil \( \hat{=} \) \( H \)-selfadjoint matrix”:

**Observation 1:** \( A \in \mathbb{C}^{n \times n} \) is \( H \)-selfadjoint

\[ \iff A^* H = H A \]

\[ \iff (HA)^* = HA \]

**Observation 2:** If \( H \) is invertible:

\[ Ax = \lambda x \]

\[ \iff HAx = \lambda Hx \]

**Conversely:** If \( \lambda H - G \) is a Hermitian pencil (d.h. \( H = H^* \) and \( G = G^* \)) and if \( H \) is invertible, then \( A := H^{-1} G \) is \( H \)-selfadjoint:

\[ A^* H = GH^{-1}H = G = HH^{-1}G = HA. \]
Why bother?

“Forget structure and use general methods”
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No! We prefer structure-preserving methods, because ...
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- **perturbation theory is different if structure is preserved**
  - because some eigenvalues occur in pairs
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  - because some eigenvalues occur in pairs
  - because there is sign characteristic
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- structure-preserving algorithms produce physically meaningful results
- perturbation theory is different if structure is preserved
  - because some eigenvalues occur in pairs
  - because there is sign characteristic

What is sign characteristic?
Definite Linear Algebra: Any Hermitian matrix is unitarily diagonalizable and all its eigenvalues are real.

Indefinite Linear Algebra: Selfadjoint matrices with respect to indefinite inner products may have complex eigenvalues and need not be diagonalizable.

Example:

\[ H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \]

\( A_1 \) and \( A_2 \) are \( H \)-selfadjoint, i.e., \( A_i^* H = HA_i \).
Transformations that preserve structure:

- for bilinear forms: \((H, A) \mapsto (P^T HP, P^{-1} AP), \quad P \text{ invertible};\)
- for sesquilinear forms: \((H, A) \mapsto (P^* HP, P^{-1} AP), \quad P \text{ invertible};\)

\[ A \] is \(\begin{cases} 
H\text{-selfadjoint} \\
H\text{-skew-adjoint} \\
H\text{-unitary} 
\end{cases}\) \(\iff\) \(P^{-1} AP\) is \(\begin{cases} 
P^* HP\text{-selfadjoint} \\
P^* HP\text{-skew-adjoint} \\
P^* HP\text{-unitary} 
\end{cases}\)

Here \(P^* = P^T\) or \(P^* = P^*\), respectively.
Theorem (Gohberg, Lancaster, Rodman, 1983, Thompson, 1976)
Let $A \in \mathbb{C}^{n \times n}$ be $H$-selfadjoint. Then there exists $P \in \mathbb{C}^{n \times n}$ invertible such that

$$P^{-1}AP = A_1 \oplus \cdots \oplus A_k, \quad P^*HP = H_1 \oplus \cdots \oplus H_k,$$

where either

1) $A_i = J_{n_i}(\lambda)$, and $H_i = \varepsilon R_{n_i}$, where $\lambda \in \mathbb{R}$ and $\varepsilon = \pm 1$; or

2) $A_i = \begin{bmatrix} J_{n_i}(\mu) & 0 \\ 0 & J_{n_i}(\overline{\mu}) \end{bmatrix}$, $H_i = \begin{bmatrix} 0 & R_{n_i} \\ R_{n_i} & 0 \end{bmatrix}$, where $\mu \not\in \mathbb{R}$.

Here $J_m(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & \lambda \end{bmatrix} \in \mathbb{C}^{m \times m}$ and $R_m = \begin{bmatrix} 0 & 1 \\ \vdots & \ddots \\ 1 \end{bmatrix} \in \mathbb{C}^{m \times m}$. 
Canonical forms

There are similar results for $H$-skewadjoint and $H$-unitary matrices and for real or complex bilinear forms.

**Spectral symmetries:**

<table>
<thead>
<tr>
<th></th>
<th>$y^T H x$</th>
<th>$y^* H x$</th>
<th>$y^T H x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>field</td>
<td>$F = \mathbb{C}$</td>
<td>$F = \mathbb{C}$</td>
<td>$F = \mathbb{R}$</td>
</tr>
<tr>
<td>$H$-selfadjoints</td>
<td>$\lambda$</td>
<td>$\lambda, \bar{\lambda}$</td>
<td>$\lambda, \bar{\lambda}$</td>
</tr>
<tr>
<td>$H$-skew-adjoints</td>
<td>$\lambda, -\lambda$</td>
<td>$\lambda, -\bar{\lambda}$</td>
<td>$\lambda, -\lambda, \bar{\lambda}, -\bar{\lambda}$</td>
</tr>
<tr>
<td>$H$-unitaries</td>
<td>$\lambda, \lambda^{-1}$</td>
<td>$\lambda, \bar{\lambda}^{-1}$</td>
<td>$\lambda, \lambda^{-1}, \bar{\lambda}, \bar{\lambda}^{-1}$</td>
</tr>
</tbody>
</table>
**Sign characteristic**: There are additional invariants for real eigenvalues of $H$-selfadjoint matrices: signs $\varepsilon = \pm 1$.

**Example**:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad H_\varepsilon = \begin{bmatrix} \varepsilon & 0 \\ 0 & -1 \end{bmatrix}, \quad \varepsilon = \pm 1;$$

- There is no transformation $P^{-1}AP = A$, $P^*H_+P = H_-1$, because of Sylvester's Law of Inertia;

- each Jordan block associated with a real eigenvalue of $A$ has a corresponding sign $\varepsilon \in \{+1, -1\}$;

- the collection of all the signs is called the **sign characteristic** of $A$;
The sign characteristic

**Interpretation of the sign characteristic** for simple eigenvalues:

- let \((\lambda, v)\) be an eigenpair of the selfadjoint matrix \(A\), where \(\lambda \in \mathbb{R}\):
  - let \(\varepsilon\) be the sign corresponding to \(\lambda\);
  - the inner product \([v, v]_H\) is positive if \(\varepsilon = +1\);
  - the inner product \([v, v]_H\) is negative if \(\varepsilon = -1\).

**Analogously:**

- purely imaginary eigenvalues of \(H\)-skew-adjoint matrices have signs;
- unimodular eigenvalues of \(H\)-unitary matrices have signs.
What happens under structured perturbations?

**Example:** symplectic matrices $S \in \mathbb{R}^{2n \times 2n};$

- consider a slightly perturbed matrix $\tilde{S}$ that is still symplectic;

- the behavior of the unimodular eigenvalues under perturbation depends on the sign characteristic;

- if two unimodular eigenvalues meet, the behavior is different if the corresponding signs are opposite or equal.
What happens under structured perturbations?

- let $S$ have two close **unimodular eigenvalues** with **opposite signs**;
- if $S$ is perturbed and the two eigenvalues meet, they generically form a Jordan block; then they may split off as a pair of nonunimodular reciprocal eigenvalues;
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What happens under structured perturbations?

- let $S$ have two close unimodular eigenvalues with equal signs;
- if $S$ is perturbed and the two eigenvalues meet, they cannot form a Jordan block, and they must remain on the unit circle;
What happens under structured perturbations?

- let $S$ have two close _unimodular eigenvalues_ with equal signs;
- if $S$ is perturbed and the two eigenvalues meet, they _cannot_ form a Jordan block, and they _must_ remain on the unit circle;
What happens under structured perturbations?

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What happens under structured perturbations?

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What happens under structured perturbations?

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Interesting observation revisited

**Experiment**: Consider two Hamiltonian systems:

\[
\dot{x} = \mathcal{H}_1 x = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix} x,
\quad
\dot{y} = \mathcal{H}_2 y = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix} y.
\]

**Observation 1**: Both matrices have the semisimple eigenvalues $+i$ and $-i$ with algebraic multiplicity 2 each. Consequently both systems are **stable**.

**Observation 2**: One can check:

- the sign characteristic of the eigenvalue $i$ of $\mathcal{H}_1$ consists of mixed signs $+1$ and $-1$ (same story for $-i$);

- the sign characteristic of the eigenvalue $i$ of $\mathcal{H}_2$ consists of identical signs $+1$ and $+1$ (similar story for $-i$: signs $-1$ and $-1$).
Interesting observation revisited

**Result 2**: Eigenvalue distribution of 1000 perturbations, when $\Delta \mathcal{H}_i$ was a random Hamiltonian matrix of norm $\frac{1}{4}$.

Left: mixed sign characteristic; Right: “definite” sign characteristic
## Generalization to matrix pencils

<table>
<thead>
<tr>
<th>matrix case:</th>
<th>$H$-selfadjoint</th>
<th>$H$-skew-adjoint</th>
<th>$H$-unitary</th>
</tr>
</thead>
<tbody>
<tr>
<td>pencil case:</td>
<td>Hermitian</td>
<td>*-even</td>
<td>*-palindromic</td>
</tr>
<tr>
<td>pencil:</td>
<td>$\lambda G - H$</td>
<td>$\lambda K - H$</td>
<td>$\lambda A - A^*$</td>
</tr>
<tr>
<td>structure:</td>
<td>$G = G^<em>$, $H = H^</em>$</td>
<td>$K = -K^<em>$, $H = H^</em>$</td>
<td>$A \in \mathbb{C}^{n \times n}$</td>
</tr>
<tr>
<td>spectral symmetry:</td>
<td>$\lambda, \bar{\lambda}$</td>
<td>$\lambda, -\bar{\lambda}$</td>
<td>$\lambda, \bar{\lambda}^{-1}$</td>
</tr>
<tr>
<td>“critical curve”:</td>
<td>real line</td>
<td>imaginary line</td>
<td>unit circle</td>
</tr>
</tbody>
</table>

**Sign characteristic:**
eigenvalues on the “critical curve” will have signs $\varepsilon = \pm 1$.

Generalization to matrix polynomials possible.
Finally: The Task
• Consider a linear time-invariant control system

\[ \dot{x} = Ax + Bu, \quad x(0) = 0, \]
\[ y = Cx + Du, \]

• Assume \( \sigma(A) \subseteq \mathbb{C}^- \) and \( D \) is invertible.

• The system is called passive if the Hamiltonian matrix

\[ H = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} := \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B^T \\ -C^T R^{-1}C & -(A - BR^{-1}C)^T \end{bmatrix} \]

has no pure imaginary eigenvalues, where \( R = D + D^T \).

• Often, an approximation \( \tilde{H} \) of \( H \) is computed and often in this approximation process the passivity is lost.

• **Aim**: modify \( \tilde{H} \) by a Hamiltonian small norm and small rank perturbation to a nearby Hamiltonian matrix having no pure imaginary eigenvalues.

• **Solved by Alam, Bora, Karow, Mehrmann, Moro 2010.**
The generalized problem

**Task**: find the smallest perturbation that moves eigenvalues on the critical curve of a structured matrix pencils off the critical curve.
The generalized problem

**Task**: find the smallest perturbation that moves eigenvalues on the critical curve of a structured matrix pencils off the critical curve.

- start with Hermitian pencils, i.e., move eigenvalues from the real line into the complex plane
The generalized problem

**Task**: find the smallest perturbation that moves eigenvalues on the critical curve of a structured matrix pencils off the critical curve.

- start with Hermitian pencils, i.e., move eigenvalues from the real line into the complex plane
- first step: find the smallest structured perturbation that makes \( \lambda \in \mathbb{C} \) an eigenvalue of a given Hermitian pencil \( A_1 + zA_2 \)
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- start with Hermitian pencils, i.e., move eigenvalues from the real line into the complex plane
- first step: find the smallest structured perturbation that makes $\lambda \in \mathbb{C}$ an eigenvalue of a given Hermitian pencil $A_1 + zA_2$

**Problem:** Let $A_1, A_2 \in \mathbb{C}^{n \times n}$ be Hermitian and let $\lambda \in \mathbb{C}$. Calculate

$$\eta_S(A_1, A_2, \lambda) = \inf_{\Delta_1, \Delta_2 \in S} \left\{ \sqrt{\|\Delta_1\|^2 + \|\Delta_2\|^2} \mid \det ((A_1 - \Delta_1) + \lambda(A_2 - \Delta_2)) = 0 \right\}$$

and construct the corresponding $\Delta_1$ and $\Delta_2$ that attain the infimum.
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and construct the corresponding $\Delta_1$ and $\Delta_2$ that attain the infimum.

We call $\eta_S(A_1, A_2, \lambda)$ the **structured backward error** of the Hermitian pencil $A_1 + zA_2$ with respect to $\lambda$ and $S$. 
The generalized problem

**Case 1:** \( S = \mathbb{C}^{n \times n} \) and \( \lambda \in \mathbb{C} \): easy!

\[
\eta(A_1, A_2, \lambda) = \frac{\sigma_{\min}(A_1 + \lambda A_2)}{\sqrt{1 + \lambda^2}},
\]

because

\[
\Delta_1(x) = \frac{1}{x^* x \sqrt{1 + \lambda^2}} (A_1 + \lambda A_2) x x^*, \quad \Delta_2(x) = \frac{\lambda}{x^* x \sqrt{1 + \lambda^2}} (A_1 + \lambda A_2) x x^*
\]

is the smallest perturbation that makes the pair \((\lambda, x)\) an eigenpair of the pencil and

\[
\eta(A_1, A_2, \lambda) = \min_{x \neq 0} \sqrt{\|\Delta_1(x)\|^2 + \|\Delta_2(x)\|^2}.
\]
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$$\eta(A_1, A_2, \lambda) = \min_{x \neq 0} \sqrt{\|\Delta_1(x)\|^2 + \|\Delta_2(x)\|^2}.$$

**Case 2**: $S = \text{Herm}(n)$ and $\lambda \in \mathbb{R}$: easy!

$$\eta(A_1, A_2, \lambda) = \frac{\sigma_{\min}(A_1 + \lambda A_2)}{\sqrt{1 + \lambda^2}},$$

because for $\lambda \in \mathbb{R}$ the above perturbations are Hermitian.
The generalized problem

Case 3: $S = \text{Herm}(n)$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$: difficult!

- Adhikari, Alam 2009: complicated formulas for the structured backward error of eigenpairs $(\lambda, x)$ for Hermitian pencils, when $\lambda \notin \mathbb{R}$
- $\eta(A_1, A_2, \lambda)$ could be obtained by minimizing these formulas over all $x \in \mathbb{C}^n \setminus \{0\}$.
- This minimization problem is not feasible.

Consequence: We need a different strategy!
Reformulating the problem

\[
\begin{align*}
\det \left( (A_1 - \Delta_1) + \lambda(A_2 - \Delta_2) \right) &= 0 \\
\iff (A_1 + \lambda A_2)x &= (\Delta_1 + \lambda \Delta_2)x \quad \text{for some } x \neq 0 \\
\iff x &= (A_1 + \lambda A_2)^{-1}(\Delta_1 + \lambda \Delta_2)x \\
\iff v_1 &= \Delta_1 M(v_1 + \lambda v_2) \quad \text{and} \quad v_2 = \Delta_2 M(v_1 + \lambda v_2)
\end{align*}
\]

abbreviating \( M := (A_1 + \lambda A_2)^{-1}, v_1 = \Delta_1 x, \) and \( v_2 = \Delta_2 x. \)

**Consequence:**

\[
\begin{align*}
\det \left( (A_1 - \Delta_1) + \lambda(A_2 - \Delta_2) \right) &= 0 \\
\iff \text{there exist } v_1, v_2 \text{ satisfying } v_1 + \lambda v_2 \neq 0 \text{ and} \\
v_1 &= \Delta_1 M(v_1 + \lambda v_2) \quad \text{and} \quad v_2 = \Delta_2 M(v_1 + \lambda v_2)
\end{align*}
\]
Reformulating the problem

**two structured mapping problems:** for \( x = M(v_1 + \lambda v_2) \) and \( y = v_k \) find \( H = \Delta_k \in \text{Herm}(n) \) such that \( y = Hx \) for \( k = 1, 2 \).

**Theorem** (see Mackey, Mackey, Tisseur, 2008). Let \( x, y \in \mathbb{C}^n, x \neq 0 \). Then there exists a Hermitian matrix \( H \in \text{Herm}(n) \) such that \( Hx = y \) if and only if \( \text{Im} \left( x^*y \right) = 0 \). If the latter condition is satisfied then

\[
\min \left\{ \|H\| \mid H \in \text{Herm}(n), \ Hx = y \right\} = \frac{\|y\|}{\|x\|}
\]

and the minimum is attained for

\[
H_0 = \frac{\|y\|}{\|x\|} \left[ \frac{y}{\|y\|} \ rac{x}{\|x\|} \right] \left[ \frac{x^*y}{\|x\|\|y\|} \ 1 \right]^{-1} \left[ \frac{y}{\|y\|} \ rac{x}{\|x\|} \right]^*.
\]

if \( x \) and \( y \) are linearly independent and for \( H_0 = \frac{yx^*}{x^*x} \) otherwise.
Reformulating the problem

We need \( v_1^*M(v_1 + \lambda v_2) \) and \( v_2^*M(v_1 + \lambda v_2) \) to be real.

This is equivalent to requiring that

\[
v^* H_1 v = 0 \quad \text{and} \quad v^* H_2 v = 0,
\]

where

\[
v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \quad H_1 = i \begin{bmatrix} M - M^* & \lambda M \\ \bar{\lambda}M^* & 0 \end{bmatrix}, \quad H_2 = i \begin{bmatrix} 0 & -M^* \\ M & \lambda M - \bar{\lambda}M^* \end{bmatrix}.
\]

Moreover, the minimal norm \( \| \Delta_1 \|^2 + \| \Delta_2 \|^2 \) for a pair \((\Delta_1, \Delta_2)\) satisfying \( v_1 = \Delta_1 M(v_1 + \lambda v_2) \) and \( v_2 = \Delta_2 M(v_1 + \lambda v_2) \) is given by

\[
\| \Delta_1 \|^2 + \| \Delta_2 \|^2 = \frac{v^*v}{v^*H_0v}, \quad \text{where} \quad H_0 = \begin{bmatrix} M^*M & \lambda M^*M \\ \bar{\lambda}M^*M & |\lambda|^2M^*M \end{bmatrix}.
\]
Reformulating the problem

**Consequence:**

\[
\eta(A_1, A_2, \lambda)^2 = \inf \left\{ \frac{v^*v}{v^*H_0v} \mid v \in \mathbb{C}^{2n}, v^*H_0v \neq 0, v^*H_1v = 0, v^*H_2v = 0 \right\}
\]

\[
= \sup \left\{ \frac{v^*H_0v}{v^*v} \mid v \in \mathbb{C}^{2n} \setminus \{0\}, v^*H_1v = 0, v^*H_2v = 0 \right\}^{-1}
\]

**Idea:** Compute the maximal eigenvalue \( \lambda_{\text{max}} \) of the matrix

\[
H_0 + t_1H_1 + t_2H_2
\]

and minimize this over \( t_1, t_2 \in \mathbb{R} \).
**The main results**

**Theorem:** Bora, Karow, M., Sharma, 2012. Let $H_0, H_1, H_2 \in \mathbb{C}^{n \times n}$ be Hermitian. Assume that $H_0$ is nonzero and positive semidefinite, and that any linear combination $\alpha_1 H_1 + \alpha_2 H_2$, $(\alpha_1, \alpha_2) \in \mathbb{R}^2 \setminus \{0\}$ is indefinite. (Here, “indefinite” is used in the sense “strictly not semi-definite”.) Then the following statements hold.

1. The function $L(t_1, t_2) := \lambda_{\text{max}}(H_0 + t_1 H_1 + t_2 H_2)$ has a minimum $\lambda^\ast_{\text{max}}$.

2. If the minimum $\lambda^\ast_{\text{max}}$ of $L(t_1, t_2)$ is attained at $(t_1^\ast, t_2^\ast) \in \mathbb{R}^2$, then there exists an associated eigenvector $w \in \mathbb{C}^n \setminus \{0\}$ of $H_0 + t_1^\ast H_1 + t_2^\ast H_2$ with

   $$w^* H_1 w = 0 = w^* H_2 w.$$ 

3. We have

   $$\lambda^\ast_{\text{max}} = \sup \left\{ \frac{w^* H_0 w}{w^* w} \mid w \neq 0, w^* H_1 w = 0, w^* H_2 w = 0 \right\} .$$

   In particular, the supremum is a maximum and attained for the eigenvector $w$ from (2).
**The main results**

**Theorem:** Bora, Karow, M., Sharma, 2012. Let $A_1, A_2 \in \mathbb{C}^{n \times n}$ be Hermitian, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and Suppose $\det(A_1 + \lambda A_2) \neq 0$, and let $M = (A_1 + \lambda A_2)^{-1}$. Then

$$\eta(A_1, A_2, \lambda) = \left( \min_{t_1, t_2 \in \mathbb{R}} \lambda_{\max}(H_0 + t_1 H_1 + t_2 H_2) \right)^{-1/2},$$

where

$$H_0 = \begin{bmatrix} M^* M & \lambda M^* M \\ \bar{\lambda} M^* M & |\lambda|^2 M^* M \end{bmatrix}, \quad H_1 = i \begin{bmatrix} M - M^* & \lambda M \\ -\bar{\lambda} M^* & 0 \end{bmatrix},$$

$$H_2 = i \begin{bmatrix} 0 & -M^* \\ M & \lambda M - \bar{\lambda} M^* \end{bmatrix}.$$

**Consequence:** Distance problem solved, the corresponding perturbation can be constructed from the eigenvector $w$ from the previous result.
Experiments:
Nr. 1 unstructured, structured
Nr. 2 unstructured, structured
Nr. 3 unstructured, structured
Nr. 4 unstructured, structured
Nr. 5 structured
Conclusions

- the effects of structured perturbations of Hermitian pencils (and other structured matrices and pencils) may be significantly different compared to unstructured perturbations

- **sign characteristic** is a crucial fact in the theory of structured perturbations;

- formulas for the structured backward error for Hermitian pencils are available;

- extension to matrix polynomials possible (involves minimization over more than two parameters).
Conclusions

• the effects of structured perturbations of Hermitian pencils (and other structured matrices and pencils) may be significantly different compared to unstructured perturbations

• **sign characteristic** is a crucial fact in the theory of structured perturbations;

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Thank you for your attention!