DAEs - Control and Numerics

Exercise Sheet 1 - Solution theory

Exercise 1 (Regularity)
Check whether the matrix pairs

$$\left(\begin{bmatrix}1&1&0\\0&-1&1\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&-1\\0&-1&1\end{bmatrix}\right),\quad\left(\begin{bmatrix}2&-1&1\\3&-2&2\\0&0&0\end{bmatrix},\begin{bmatrix}1&0&0\\0&1&-1\\1&-1&1\end{bmatrix}\right)$$

are regular or singular and determine their Kronecker canonical forms by elementary row and column transformations.

Exercise 2 (Wong-sequences)
Consider

$$A := \begin{bmatrix}3&0&1&0\\0&2&2&-1\\1&2&3&0\\0&-1&0&2\end{bmatrix},\qquad E := \begin{bmatrix}1&-1&-3&0\\0&2&0&-1\\-3&-1&1&2\\-2&-2&0&2\end{bmatrix}.$$

Calculate the so-called Wong sequences, given by

$$\mathcal{V}_0 := \mathbb{R}^n,\qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i),$$
$$\mathcal{W}_0 := \{0\},\qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i),$$

where $M^{-1}(\mathcal{S}) := \{\ x \in \mathbb{R}^n \ \mid Mx \in \mathcal{S}\ \}$ is the pre-image of the some set $\mathcal{S} \subseteq \mathbb{R}^n$ under the matrix $M \in \mathbb{R}^{n\times n}$ and $M\mathcal{S} := \{\ Mx \in \mathbb{R}^n \ \mid x \in \mathcal{S}\ \}$ is the image of $\mathcal{S}$ under M.
Choose full rank matrices V, W with $\operatorname{im} V = \bigcap_{i\in\mathbb{N}} \mathcal{V}_i$ and $\operatorname{im} W = \bigcup_{i\in\mathbb{N}} \mathcal{W}_i$ and calculate PEQ and PAQ where $P := [V, W]$ and $Q := [EV, AW]^{-1}$. What do you observe?

Exercise 3 (Drazin inverse)
Consider a regular matrix pair (E, A) and any $\lambda \in \mathbb{R}$ such that $\det(\lambda E - A) \neq 0$. Prove that $\hat{E} := (\lambda E - A)^{-1}E$ and $\hat{A} := (\lambda E - A)^{-1}A$ commute, i.e. $\hat{E}\hat{A} := \hat{A}\hat{E}$.
Furthermore, calculate the Drazin inverse $\hat{E}^D$ of $\hat{E}$ and $\hat{A}^D$ of $\hat{A}$ with (E, A) as in Exercise 2 and $\lambda = -1$.

Exercise 4 (A bad distribution)
Denote with $\mathcal{C}_0^\infty$ the space of smooth functions $\varphi : \mathbb{R} \to \mathbb{R}$ with bounded support. Consider the following linear mapping

$$\mathcal{C}_0^\infty \ni \varphi \mapsto \sum_{i\in\mathbb{N}} \frac{(-1)^i}{i+1}\varphi\left(\tfrac{(-1)^i}{i+1}\right).$$

Show that this mapping is well defined for all $\varphi \in \mathcal{C}_0^\infty$, i.e. the infinite sum converges in $\mathbb{R}$.
Consider now the "restriction" of this mapping to the open interval $(0, \infty)$:

$$\mathcal{C}_0^\infty \ni \varphi \mapsto \sum_{i\in\mathbb{N}} \frac{1}{2i+1}\varphi\left(\tfrac{1}{2i+1}\right).$$

Show that this mapping is not well defined if $\varphi(0) \neq 0$.