Elgersburg Lectures – March 2010

EXERCISES

Exercise I.1: Linear models

The mathematical model ($\mathbb{R}^{\mathtt{n}}, \mathscr{B}$) is said to be

$$[\![\text{ linear }]\!] :\Leftrightarrow [\![\mathscr{B} \text{ is a linear subspace of } \mathbb{R}^{\mathtt{n}}]\!].$$

1. **Prove that a linear behavior admits a representation**

$$Rw = 0, \qquad R \in \mathbb{R}^{\bullet \times \mathtt{n}}.$$

Call this representation a kernel representation of $\mathscr{B}$, and a minimal one if, among all kernel representations of $\mathscr{B}$, `rowdimension`(R) is as small as possible.

2. **Prove that $Rw = 0$ is minimal if and only if R has full row rank.**

3. **How are the R's corresponding to minimal kernel representations of $\mathscr{B}$ related?**

4. **Define what you mean by an image representation?**
   Prove its existence.

Consider the DES with $\mathscr{U} = \{0,1\}^{32}$ and

$$\mathscr{B} = \left\{ a_1 a_2 \cdots a_{31} a_{32} \mid a_{\mathtt{k}} \in \{0,1\} \text{ and } a_{32} \overset{\mathtt{modulo}\ 2}{=} \sum_{\mathtt{k}=1}^{31} a_{\mathtt{k}} \right\},$$

the set of 32-bit strings with a parity check as last bit.

5. **In what sense is this a linear model?**

6. **Give a kernel representation of this behavior.**

7. **Give an image representation of this behavior.**

8. **Call $e = \sum_{\mathtt{k}=1}^{32} a_{\mathtt{k}}$ the syndrome associated with this 32-bit string, and explain how e can be used for error detection.**
   How many errors can this code detect?

Exercise I.2: Symmetry

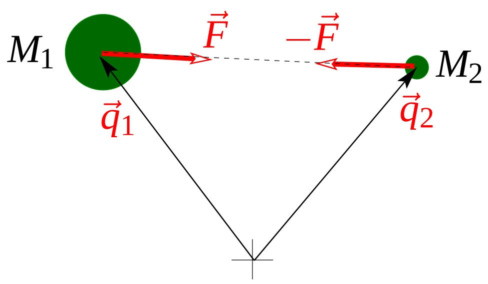
A transformation group on a set A is a set of maps that form a subgroup of the group of bijections on A. In other words, there is a group $\mathscr{G}$ and a map T from $\mathscr{G}$ to the bijections on A, such that for all $g, g_1, g_2 \in \mathscr{G}$, there holds

1. $T_1 = \mathtt{id}_A$ ($\mathtt{id}_A$ **denotes the identity map on** A),
2. $T_{g^{-1}} = T_g^{-1}$,
3. $T_{g_1 g_2} = T_{g_2} \circ T_{g_1}$ (**$\circ$ denotes map composition).**

Let $T_{\mathscr{G}}$ be a transformation group on $\mathscr{U}$. The mathematical model $(\mathscr{U}, \mathscr{B})$ is said to be symmetric with respect to $T_{\mathscr{G}}$ if

$$T_g(\mathscr{B}) = \mathscr{B} \qquad \text{for all } g \in \mathscr{G}.$$

1. **Identify an obvious symmetry for the 32-bit strings with a parity check discussed in Exercise I.1.**
2. **Formalize time-invariance as a symmetry.**
3. **Identify a few symmetries for the gravitational attraction of two bodies,**
   $$\mathscr{U} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3;\ \mathscr{B} = \left\{ (\vec{q}_1, \vec{q}_2, \vec{F}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \mid \vec{F} = M_1 M_2 \frac{(\vec{q}_2 - \vec{q}_1)}{|\vec{q}_1 - \vec{q}_2|^3} \right\}.$$

   Extend the definition of $\mathscr{B}$ by including M_1, M_2, so that exchanging the masses also becomes a symmetry.
4. **Explain in what sense Maxwell's equations are symmetric with respect to space translation and rotation.**

Exercise I.3: Memoryless systems

The dynamical system $(\mathbb{T}, \mathbb{W}, \mathscr{B})$ is said to be memoryless $:\Leftrightarrow$

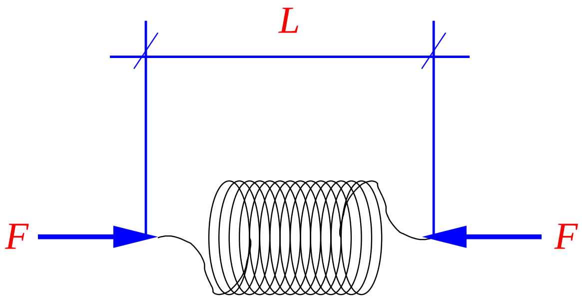
$$[\![w_1, w_2 \in \mathscr{B} \text{ and } t' \in \mathbb{T}]\!] \Rightarrow [\![w_1 \wedge_{t'} w_2 \in \mathscr{B}]\!],$$

where $w_1 \wedge_{t'} w_2$, the concatenation of w_1 and w_2 at t', is defined as

$$(w_1 \wedge_{t'} w_2)(t) = \begin{cases} w_1(t) & \text{for } t < t', \\ w_2(t) & \text{for } t \geq t'. \end{cases}$$

1. **Which of the following physical devices discussed in Lecture I define memoryless systems?**
   - **The gas law.**
   - **A resistor, an inductor, a capacitor.**
   - **The gravitational attraction of two bodies, Kepler's laws, Newton's second law, a spring, the mass-spring system.**

Consider a simple spring. ***Is it really a memoryless system?***



Consider a real variable, E, the energy stored in the spring, related to L by

$$E(L) = \int_{L^*}^{L} v(\sigma)\, d\sigma, \text{ with } F = v(L) \text{ the spring characteristic}$$

and L^* the equilibrium length (corresponding to $F = 0$).

2. **Consider F, L, E as functions of time. Prove that $\frac{d}{dt}E = F\frac{d}{dt}L$.**
3. **Prove that the spring viewed in terms of the variables (F, L) define a memoryless system, but that in terms of the variables (F, L, E) it does not define a memoryless system. Energy and power considerations can hence bring in dynamics, even in a memoryless system.**

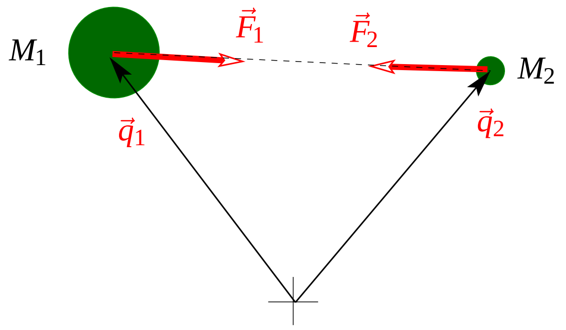
Exercise I.4: Two-body problem

The motion of a pointmass in a force field is governed by

$$M\frac{d^2}{dt^2}\vec{q} = \vec{F}(\vec{q}),$$

with M **the mass of the body,** $\vec{q}$ **the position, and** $\vec{F}:\mathbb{R}^3 \to \mathbb{R}^3$ **the force field. Studying the resulting motions of the pointmass is called the 'one-body problem'.**

Consider the motion of two bodies, under mutual attraction.



Newton's second law yields the equations of motion

$$M_1\frac{d^2}{dt^2}\vec{q}_1 = \vec{F}_1, \qquad M_2\frac{d^2}{dt^2}\vec{q}_2 = \vec{F}_2. \tag{$\diamondsuit$}$$

According to Newton's third law,

$$\vec{F}_1 + \vec{F}_2 = \vec{0}. \tag{$\spadesuit$}$$

The problem is to obtain an explicit description of the trajectories $\vec{q}_1:\mathbb{R}\to\mathbb{R}^3, \vec{q}_2:\mathbb{R}\to\mathbb{R}^3$ **that are possible. This is the so-called 'two-body problem', which, in contrast to the three-body or** `n`**-body problem, can be reduced to one-body problems.**

1. **Define the 'barycenter' of the two bodies**

$$\vec{R} = \frac{M_1\vec{q}_1 + M_2\vec{q}_2}{M_1 + M_2}. \tag{$\clubsuit$}$$

   Eliminate $\vec{q}_1, \vec{q}_2, \vec{F}_1, \vec{F}_2$ **from (**$\diamondsuit$**,** $\spadesuit$**,** $\clubsuit$**) and show that the behavior of** $\vec{R}$ **is governed by** $\frac{d^2}{dt^2}\vec{R} = 0.$

2. Consider the difference vector of the bodies

$$\vec{\Delta} = \vec{q}_1 - \vec{q}_2. \qquad (\heartsuit)$$

Assume that $\vec{F}_1$ is a function of $(\vec{q}_1, \vec{q}_2)$ only through $\vec{\Delta}$ (makes perfect sense physically). Eliminate $\vec{q}_1, \vec{q}_2, \vec{F}_2$ from ($\diamondsuit$, $\spadesuit$, $\heartsuit$) and prove that the behavior of $(\vec{\Delta}, \vec{F}_1)$ is governed by $\mu \frac{d^2}{dt^2}\vec{\Delta} = \vec{F}_1$, with $\mu = \frac{M_1 M_2}{M_1 + M_2}$; μ is called the 'reduced mass'. Prove that the motion of $\vec{\Delta}$ is that of one body with mass μ under the force field $\vec{F}_1(\vec{\Delta})$. Hence after solving 2 one-body problems, we obtain $(\vec{q}_1, \vec{q}_2)$ by

$$\vec{q}_1 = \vec{R} + \frac{M_2}{M_1 + M_2}\vec{\Delta}, \quad \vec{q}_2 = \vec{R} + \frac{M_1}{M_1 + M_2}\vec{\Delta}.$$

3. Often $\vec{F}_1$ is a central force, that is, it is of the form $\vec{F}_1 = F(||\vec{\Delta}||)\frac{\vec{\Delta}}{||\vec{\Delta}||}$.

A special case is 'Kepler's problem', with $F(||\vec{\Delta}||) = -\frac{1}{||\vec{\Delta}||^2}$ (with suitable units). This yields

$$\frac{d^2}{dt^2}\vec{\Delta} + \frac{1}{||\vec{\Delta}||^2} = 0 \qquad (\star)$$

as the equation for $\vec{\Delta}$. It can be shown that the orbits satisfying K1, K2, K3 (with suitably chosen constants) are solutions. Actually Newton *derived* ($\star$) from K1, K2, K3.

Isaac Newton (1643-1727)

Do K1, K2, K3 give all the solutions to ($\star$)? Argue from physical insight, do not attempt to answer using mathematical arguments.

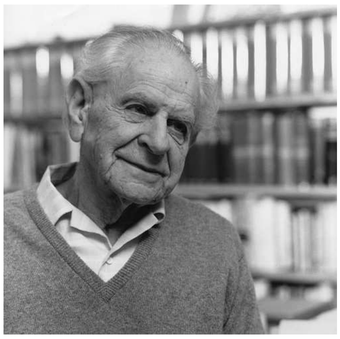
Exercise I.5: The MPUM

Assume that the set of system trajectories $\mathbb{D} = \{\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n\}$, with $\tilde{w}_k : \mathbb{T} \to \mathbb{W}$, for $k = 1, 2, \ldots, n$, is observed. The following is a version of the (deterministic) system identification problem.

Find the behavior of the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ that produced these observations.

Call $\mathscr{B}$ $[\![$*unfalsified* by $\mathbb{D}]\!]$ $:\Leftrightarrow$ $[\![\mathbb{D} \subseteq \mathscr{B}]\!]$.

Call $[\![\mathscr{B}_1$ *more powerful* than $\mathscr{B}_2]\!]$ $:\Leftrightarrow$ $[\![\mathscr{B}_1 \subset \mathscr{B}_2]\!]$.
The more a model forbids, the better it is.
According to Popper, this is against common belief.



Karl Popper (1902-1994)

Let $\mathbb{B}$ be a set of behaviors, i.e., a set of subsets of $W^{\mathbb{T}}$.
Call $\mathscr{B}^*$ $[\![$the ***most powerful unfalsified model*** (MPUM) in $\mathbb{B}$ for $\mathbb{D}]\!]$ $:\Leftrightarrow$ $[\![$

- $\mathscr{B}^* \in \mathbb{B}$,
- $\mathscr{B}^*$ is unfalsified by $\mathbb{D}$,
- $\mathscr{B}^*$ is more powerful than every other element of $\mathbb{B}$ that is unfalsified by $\mathbb{D}$.$]\!]$

1. Prove that when $\tilde{w}_k : \mathbb{Z} \to \mathbb{R}^{w}$ for $k = 1, 2, \ldots, n$, there exists an MPUM in the class of discrete-time LTIDSs.
2. With the result of Exercise II.2 part 3, you may also prove that when $\tilde{w}_k \in \mathscr{C}^{\infty}(\mathbb{R}, \mathbb{R}^{w})$ for $k = 1, 2, \ldots, n$, there exists an MPUM in $\mathscr{L}^{w}$.

Exercise I.6: PDEs

1. **A specification of the behavior in terms of an ODE or a PDE is often not very helpful, and there exist other specifications that give much more insight in the nature of $\mathscr{B}$. For example, Kepler's laws give much more insight than the associated ODE (equation ($\star$) in Exercise I.4).**

   Consider the wave equation

$$\frac{\partial^2}{\partial t^2} w = \frac{\partial^2}{\partial x^2} w.$$

   It defines a system $\left(\mathbb{R}^2, \mathbb{R}, \mathscr{B}\right)$**. Prove that**

$$\mathscr{B} = \{ w \in \mathscr{C}^\infty \left(\mathbb{R}^2, \mathbb{R}\right) \mid \exists f_-, f_+ \in \mathscr{C}^\infty (\mathbb{R}, \mathbb{R}) \text{ such that } w(t,x) = f_-(t-x) + f_+(t+x) \}.$$

   Argue that this description of $\mathscr{B}$ is more insightful than the PDE and puts in evidence the wave nature of the behavior.

2. **Write Maxwell's equations in polynomial matrix form**

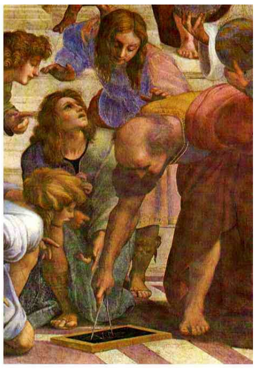
$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathrm{n}}}\right) w = 0.$$

   Specify the associated w and R.

Exercise II.1: The Smith form

Prove the Smith canonical form.

In the proof suggested below, use the following step of the *Euclidean algorithm* for polynomials.
Given $x, y \in \mathbb{R}[\xi], y \neq 0$**,** $\exists\, d, r \in \mathbb{R}[\xi]$ **such that**
$x = yd + r,$ **with** $d, r \in \mathbb{R}[\xi]$, **and**
$r = 0$ **or** $\texttt{degree}(r) < \texttt{degree}(y)$.



Euclides (325-265 BC)
painting by Rafaello

Proceed as follows.

1. **Assume** $M \neq 0$**. Prove that by pre- and postmultiplying by a permutation matrix, we may assume that the** $(1,1)$ **element of** M **is** $\neq 0$ **and has the least degree of all other nonzero elements of** M**.**

2. **Let** $M_{1,1}$ **be this** $(1,1)$ **element. Assume there is another nonzero element in the first row or the first column of** M**. Call this element** x**. Use the Euclidean algorithm to define** r **by**

$$x = M_{1,1}d + r \text{ with } r = 0 \text{ or } \texttt{degree}(r) < \texttt{degree}(M_{1,1}).$$

   Prove that there exist a unimodular matrix U **or** V **such that** UM **or** MV **has either one more zero element in the first row or column than** M**, or a** $(1,1)$ **element with degree less than the degree of** $M_{1,1}$**.**

3. **Prove that in a finite number of steps this leads to a matrix of the form**

$$\begin{bmatrix} M_{1,1} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & M' & \\ 0 & & & \end{bmatrix}.$$

4. **Obtain the Smith form by induction.**

Exercise II.2: Minimal representations of LTIDSs

1. **Let $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^1$ have kernel representations**

$$p_1(\frac{d}{dt})w = 0 \quad \textbf{and} \quad p_2(\frac{d}{dt})w = 0, \qquad p_1, p_2 \in \mathbb{R}[\xi].$$

   Obtain a minimal kernel representation for $\mathscr{B}_1 \cap \mathscr{B}_2$ and $\mathscr{B}_1 + \mathscr{B}_2$ using the greatest common divisor and the least common multiple of p_1, p_2.

2. **Generalize to**

$$p_1(\frac{d}{dt})w = 0, \ldots, p_n(\frac{d}{dt})w = 0, \qquad p_1, \ldots, p_n \in \mathbb{R}[\xi].$$

3. **Assume that $\mathscr{B}$ is defined as the solution set of an infinite number of differential equations**

$$R_\alpha(\frac{d}{dt})w = 0, \qquad \alpha \in \mathbb{A},\ R_\alpha \in \mathbb{R}[\xi]^{1 \times \mathtt{w}},$$

   with $\mathbb{A}$ any (countably or uncountably) infinite set. Prove that there exists a polynomial matrix $R \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$ (hence with a finite number of rows) such that $\mathscr{B}$ is specified by

$$R\left(\frac{d}{dt}\right)w = 0.$$

Exercise II.3: Time-reversibility

$\Sigma = (\mathbb{R}, \mathbb{W}, \mathscr{B})$ **is said to be time-reversible if** $\mathtt{w} \in \mathscr{B}$ **implies** $\mathtt{reverse}(w) \in \mathscr{B}$ **with**

$$\mathtt{reverse}(w)(t) := w(-t).$$

1. **Do Kepler's laws define a time-reversible system?**
2. **Prove that** $w + \frac{d^2}{dt^2}w = 0$ **defines a time-reversible system.**
3. **Prove that the scalar system** $p(\frac{d}{dt})w = 0$ **is time-reversible if and only if** $p \in \mathbb{R}[\xi]$ **is either an even or an odd polynomial.**
4. **Prove that the single-input/single-output system** $p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$ **is time-reversible if and only if** $p, q \in \mathbb{R}[\xi]$ **are both even or both odd.**
5. **Prove that the controllable single-input/single-output system** $p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$ **is time-reversible if and only if** $p, q \in \mathbb{R}[\xi]$ **are both even.**

Exercise II.4: Controllable subsystems

1. **Prove that**

$$[\![\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}} \text{ controllable}, \mathtt{m}(\mathscr{B}_1) = \mathtt{m}(\mathscr{B}_2), \text{ and } \mathscr{B}_1 \subseteq \mathscr{B}_2]\!] \Rightarrow [\![\mathscr{B}_1 = \mathscr{B}_2]\!].$$

   Prove that the above implication does not hold without the controllability assumption.

2. **Let** $\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^2, \mathtt{m}(\mathscr{B}_1) = \mathtt{m}(\mathscr{B}_2) = 1, \mathtt{p}(\mathscr{B}_1) = \mathtt{p}(\mathscr{B}_2) = 1$ **(hence both systems are single-input/single output systems).**

   Prove that

$$[\![\mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^{\mathtt{w}} \text{ controllable}, \mathscr{B}_1 \neq \mathscr{B}_2)]\!] \Rightarrow [\![\mathscr{B}_1 \cap \mathscr{B}_2 \text{ is autonomous}]\!].$$

   Prove that the above implication does not hold without the controllability assumption.

The image representation $w = M\left(\frac{d}{dt}\right)\ell$ **is said to be observable if**

$$[\![M(\frac{d}{dt})\ell_1 = M(\frac{d}{dt})\ell_2]\!] \Leftrightarrow [\![\ell_1 = \ell_2]\!].$$

4. **Prove that a controllable system** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ **admits an observable image representation.**

5. **Assume that the single-input/single-output system** $p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2$ **is controllable. Give an observable image representation for this system.**

Exercise II.5: Non-anticipation

1. **Consider the input/output system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}, \mathscr{B})$ defined by**

$$y = G\left(\frac{d}{dt}\right)u \quad \textbf{with } G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}.$$

   We say that y does not anticipate u if for all $(u,y) \in \mathscr{B}$ and $u' \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{m}})$ such that $u'(t) = u(t)$ for $t \leq 0$, there exists $y' \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ such that $(u', y') \in \mathscr{B}$ and $y'(t) = y(t)$ for $t \leq 0$.

   Prove that y does not anticipate u (without conditions on G !).

2. **Consider now the discrete-time input/output system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{m}} \times \mathbb{R}^{\mathtt{p}}, \mathscr{B})$ defined by**

$$y = G(\sigma)u \quad \textbf{with } G \in \mathbb{R}(\xi)^{\mathtt{p}\times\mathtt{m}}.$$

   Define non-anticipation. Prove that y does not anticipate u if and only if G is proper.

3. **Does the moving average system**

$$y(t) = \frac{1}{2N+1} \sum_{t'=-N}^{N} u(t+t')$$

   define a non-anticipating system?

4. **For what $\Delta \in \mathbb{R}$ does the differential-delay system**

$$\frac{d}{dt} y(t) = u(t+\Delta)$$

   define a nonanticiapting system?

Exercise II.6: Norm-preserving representations

The representation of this exercise uses the following 'spectral factorization'-like result.

Assume that $P \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ satisfies $P(\xi) = P(-\xi)^\top$ and $P(i\omega) > 0$ for $\omega \in \mathbb{R}$. Then there exists $F \in \mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{n}}$ such that $P(\xi) = F(-\xi)^\top F(\xi)$.

1. **Prove this factorization for the case $\mathtt{n} = 1$.**
2. **Prove that if $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is controllable, then it admits an observable 'image' representation**

$$w = N\left(\frac{d}{dt}\right)\ell$$

   with $N \in \mathbb{R}(\xi)^{\mathtt{w}(\mathscr{B})\times\mathtt{m}(\mathscr{B})}$ such that

$$N(-\xi)N(\xi) = I.$$

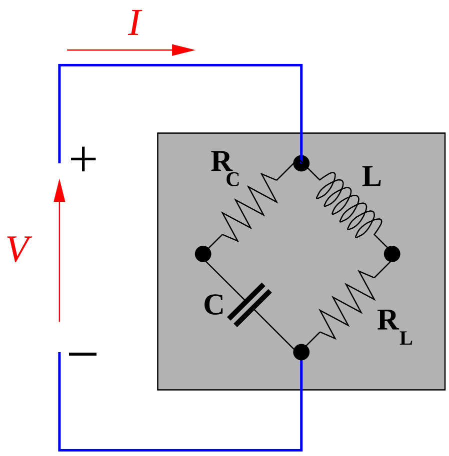
   Proceed as follows. Start with an observable image representation $w = M\left(\frac{d}{dt}\right)\ell$ with $M \in \mathbb{R}[\xi]^{\mathtt{w}(\mathscr{B})\times\mathtt{m}(\mathscr{B})}$. Then factor $M^\top(-\xi)M(\xi)$ as $F^\top(-\xi)F(\xi)$ with $F \in \mathbb{R}[\xi]^{\mathtt{m}(\mathscr{B})\times\mathtt{m}(\mathscr{B})}$. Take $N = MF^{-1}$.
3. **Prove that this representation has the property that**

$$||w||_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})} = ||\ell||_{\mathscr{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{m}})},$$

   hence the name 'norm-preserving' image representation.
4. **Prove that such a representation exists only very exceptionally with N polynomial. Norm-preserving representations have a number of applications, and require rational symbols.**

Exercise III.1: Elimination of latent variables in the RLC circuit

Consider the RLC circuit discussed in Lecture III.



The equations that describe this circuit are KCL, the constitutive equations, and the manifest variable assignment.

1. **Eliminate** I_a, I_b, I_c, I_d **and** V_1, V_2, V_3, V_4**, and arrive at**

$$C\frac{d}{dt}V = I_e + CR_C\frac{d}{dt}I_e, \qquad V = R_L I_f + L\frac{d}{dt}I_f, \qquad I = I_e + I_f.$$

   Argue the correctness of these equations from first principles.

2. **Next, eliminate** I_f**, and obtain**

$$C\frac{d}{dt}V = I_e + CR_C\frac{d}{dt}I_e, \qquad \frac{L}{R_L}\frac{d}{dt}I + I - \frac{1}{R_L}V = I_e + \frac{L}{R_L}\frac{d}{dt}I_e.$$

3. **Finally, distinguish two cases, eliminate** I_e**, and derive the following differential equation governing** (V, I)**.**

   - **For** $CR_C \neq \frac{L}{R_L}$**,**

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C\frac{d}{dt} + CR_C\frac{L}{R_L}\frac{d^2}{dt^2}\right) V$$
$$= \left(1 + CR_C\frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right) R_C I.$$

   - **For** $CR_C = \frac{L}{R_L}$**,** $\left(\frac{R_C}{R_L} + CR_C\frac{d}{dt}\right) V = \left(1 + CR_C\frac{d}{dt}\right) R_C I.$

Exercise III.2: Series and parallel connection

1. **Recall the following result, called the *Bézout identity.***
   It is also the topic of Exercise IV.6.

   Let $a, b \in \mathbb{R}[\xi]$. Then a and b are coprime if and only if there exist $x, y \in \mathbb{R}[\xi]$ such that

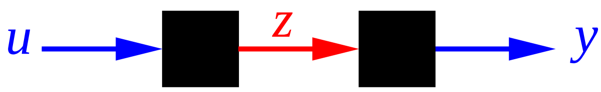
   $$ax + by = 1.$$

   Use this to find a unimodular pre-multiplication that brings $\begin{bmatrix} p \\ q \end{bmatrix}$ with $p, q \in \mathbb{R}[\xi]$ into Smith form.
   p and q need not be coprime.

Étienne Bézout (1730-1783)

2. **Consider the series connection of two SISO LTIDSs, as shown in the figure below.**

series connection



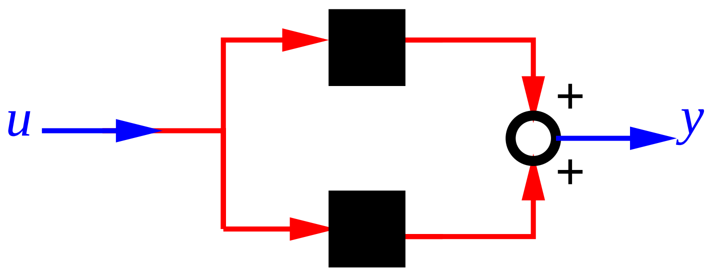
   Assume that the systems are governed by repectively

   $$p(\frac{d}{dt})z = q(\frac{d}{dt})u, \qquad d(\frac{d}{dt})y = n(\frac{d}{dt})z.$$

   Eliminate z to obtain a kernel representation for (u, y).

3. **Repeat 2. for parallel connection.**

parallel connection



Exercise III.3: The structure of $\mathscr{L}^\bullet$

Let $\mathscr{B}, \mathscr{B}_1, \mathscr{B}_2 \in \mathscr{L}^\bullet$ **and** $F \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**.**

1. **Prove that** $(\mathscr{B}_1 + \mathscr{B}_2) \in \mathscr{L}^\bullet$**.**
2. **Prove that** $\mathscr{B}_1 \cap \mathscr{B}_2 \in \mathscr{L}^\bullet$**.**
3. **Prove that** $F\left(\frac{d}{dt}\right)\mathscr{B} \in \mathscr{L}^\bullet$**.**
4. **Prove that** $F\left(\frac{d}{dt}\right)^{-1}\mathscr{B} \in \mathscr{L}^\bullet$**.**

Assume, as always with the $^\bullet$ **notation, that the relevant objects have compatible dimensions.**

Exercise III.4: Row-reduced representations

$M \in \mathbb{R}[\xi]^{n_1 \times \mathtt{n}_2}$ is said to be **row-reduced** (or row-proper) if the leading coefficient matrix of M is of full row rank. The leading coefficient matrix of M is the matrix $\tilde{M} \in \mathbb{R}^{n_1 \times \mathtt{n}_2}$ with $\mathtt{k}$-th row equal to the vector that is the coefficient of highest degree of the $\mathtt{k}$-th row of M.

1. Prove that if M is row proper, then it is of full row rank.

2. Prove that if M is of full row rank, then there exists a unimodular polynomial matrix U such that UM is row reduced.

   Proceed as follows. If $\tilde{M}$ is not of full row rank, there exists a row of $\tilde{M}$, say the $\mathtt{k}$-th row, that is a linear combination of other rows of $\tilde{M}$ that correspond to rows of M of degree lower or equal to that of the $\mathtt{k}$-th row of M. Pre-multiply $M(\xi)$ by a unimodular matrix formed by powers of ξ and the coefficients of this linear combination, so that the degree of the $\mathtt{k}$-row is decreased, while the degrees of the other rows remain the same. Continue this process until UM is row-reduced.

3. Prove that $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ admits a kernel representation $R\left(\frac{d}{dt}\right)w = 0$ with R row-reduced.

4. Assume that R is row-reduced. Let $R_{\mathtt{k}}$ be the $\mathtt{k}$-th row of R and denote its degree by $\mathtt{r}_{\mathtt{k}}$. Define $X_{\mathtt{k}}$ and X as follows

$$X_k = \begin{bmatrix} \sigma_+(R_{\mathtt{k}}) \\ \sigma_+^2(R_{\mathtt{k}}) \\ \vdots \\ \sigma_+^{\mathtt{r}_{\mathtt{k}}}(R_{\mathtt{k}}) \end{bmatrix}, \qquad X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{\mathtt{rowdimension}(R)} \end{bmatrix}.$$

   Prove that $X\left(\frac{d}{dt}\right)$ defines a minimal state map for $R\left(\frac{d}{dt}\right)w = 0$. Deduce from X a minimal input/state/output representation of $R\left(\frac{d}{dt}\right)w = 0$.

Exercise III.5: From state to input/state/output representation

Consider the state model

$$E\frac{d}{dt}x + Fx + Gw = 0.$$

The dynamical system defined by this model can be written in minimal input/state/output form

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = P\begin{bmatrix} u \\ y \end{bmatrix},$$

with P **a permutation matrix. The state** x **in the two models need not be the same.**

In this exercise a conceptual algorithm is derived for

$$(E,F,G) \mapsto (A,B,C,D,P).$$

Proceed as follows.

1. **Pre-and postmultiply** E **by a nonsingular matrix to obtain** $E \mapsto \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$**.**
   Prove that this operation yields a model of the form
   $$\frac{d}{dt}x_1 = F_{1,1}x_1 + F_{1,2}x_2 + G_1 w, \quad 0 = F_{2,1}x_1 + F_{2,2}x_2 + G_2 w.$$
2. **If** $F_{1,2} \neq 0$**, pre-and postmultiply it by a nonsingular matrix to obtain** $F_{1,2} \mapsto \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$**. Set** $x_2 = \begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix}$**, and eliminate** $x_{2,1}$**. Prove that this leads to a new model of the form** $E\frac{d}{dt}x + Fx + Gw = 0$ **with** x **of lower dimension.**
3. **Proceed in this manner until** $F_{1,2} = 0$**. Prove that this yields**
   $$\frac{d}{dt}x_1 = F_{1,1}x_1 + G_1 w, \quad 0 = F_{2,1}x_1 + F_{2,2}x_2 + G_2 w.$$
4. **If** $F_{2,2} \neq 0$**, pre-and postmultiply it by a nonsingular matrix to obtain** $F_{2,2} \mapsto \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$**. Eliminate** x_2**. Prove that this yields a model of the form**
   $$\frac{d}{dt}x = Fx + G_1 w, \quad 0 = Hx + G_2 w.$$

5. **Prove that if G_2 is not of full row rank, the second equation can be made into**

$$0 = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} x + \begin{bmatrix} \tilde{G}_2 \\ 0 \end{bmatrix} w.$$

Pre-and postmultiply H_2 by a nonsingular matrix to obtain $H_2 \mapsto \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Then cancel equations and/or components of x, so that you obtain

$$\frac{d}{dt}x = Fx + G_1 w, \quad 0 = Hx + G_2 w$$

with G_2 is of full row rank.

6. **Post-multiply G_2 by a permutation matrix to obtain $G_2 \mapsto \begin{bmatrix} G_{2,1} & G_{2,2} \end{bmatrix}$ with $G_{2,2}$ nonsingular. Call (the permuted) $w = \begin{bmatrix} u \\ y \end{bmatrix}$. Prove that the second equation can be brought to the form**

$$y = Cx + Du.$$

Substitute this equation in $\frac{d}{dt}x = Fx + G_1 w$ and victoriously arrive at

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = P \begin{bmatrix} u \\ y \end{bmatrix}.$$

Observe that (A, C) is not necessarily observable. You can reduce x in this model further, using the Kalman decomposition, to make (A, C) observable.

Exercise III.6: McMillan degree

The McMillan degree of a system $\mathscr{B} \in \mathscr{L}^\bullet$ is the dimension of the minimal state dimension of $\mathscr{B}$.
Define the map

$$\mathtt{n} : \mathscr{L}^\bullet \to \mathbb{N} \text{ by } \mathtt{n}(\mathscr{B}) := \text{ the McMillan degree of } \mathscr{B}.$$

Brockway McMillan

$\mathtt{n}(\mathscr{B})$ defines, with $\mathtt{w}(\mathscr{B}), \mathtt{m}(\mathscr{B})$ and $\mathtt{p}(\mathscr{B})$, an important 'integer invariant' of $\mathscr{B}$.

Let $R\left(\frac{d}{dt}\right) w = 0$ be a minimal kernel representation of $\mathscr{B}$.

1. **Prove that $\mathtt{n}(\mathscr{B})$ equals the maximal degree of all $\mathtt{p}(\mathscr{B}) \times \mathtt{p}(\mathscr{B})$ minors of R.**

   Hint: Prove this, using Exercise III.4, for R row-reduced, and subsequently deduce the general case.

2. **Prove that $\mathtt{n}(\mathscr{B}) = \mathtt{degree}(\mathtt{deteminant}(P))$ with P defined by the input/output representation**

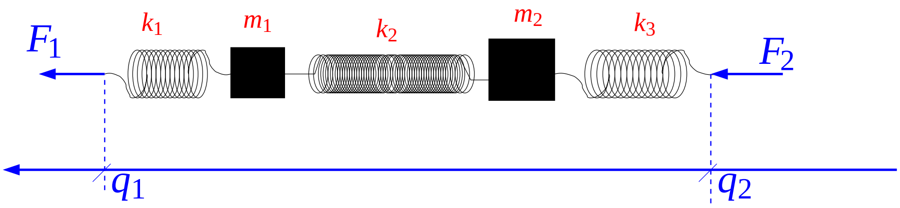
$$P\left(\frac{d}{dt}\right) y = Q\left(\frac{d}{dt}\right) u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

   of $\mathscr{B}$ with the transfer function $P^{-1}Q$ proper.

3. **Prove that if $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$ is autonomous, then**

$$\mathtt{n}(\mathscr{B}) = \mathtt{dimension}(\mathscr{B}).$$

Exercise IV.1: Modeling a mass-spring-damper system

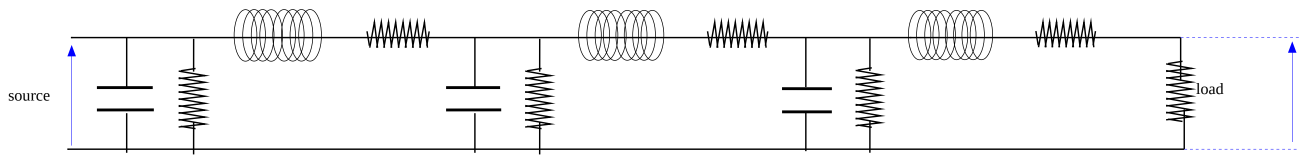


Consider the mass-spring device shown above. Assume that it operates horizontally from equilibrium in its linear mode. The problem is to model the relation between the forces F_1, F_2, and the horizontal positions q_1, q_2.

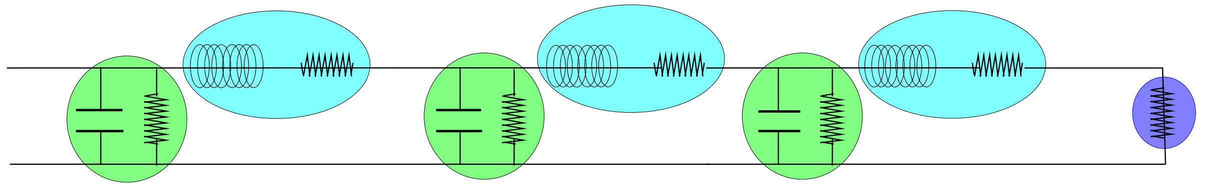
1. **View this system as consisting of 5 subsystems. Define the graph with leaves that determines the interconnection architecture.**
2. **Choose latent and manifest variables, and write the module equations.**
3. **Write the interconnection laws.**
4. **Eliminate the latent variables and obtain behavioral equations involving only the manifest variables.**

Exercise IV.2: Modeling a transmission line

Consider the transmission line modeling problem discussed during Lecture IV.



1. **View the transmission line as an interconnection of 7 subsystems as shown below.**



   Determine the graph with leaves that defines the interconnection architecture.

2. **There are 3 kinds of subsystems: blue, green, and cyan. Model each of these subsystems.**

3. **Specify the interconnection laws.**

4. **Specify the manifest variables.**

5. **Obtain the full set of equations leading, after elimination of the latent variables, to the desired differential equation that describes the behavior of (w_1, w_2)**

$$r_1\left(\frac{d}{dt}\right)w_1 = r_2\left(\frac{d}{dt}\right)w_2.$$

Exercise IV.3: Voltages and potentials

We view an electrical circuit as a device that interacts with its environment through wires, called terminals. In Lecture IV, we have posed that the interaction variables consist of (i) a current and (ii) a potential for each terminal. Currents are measurable by ammeters, but voltmeters only measure voltages across points, that is, potential differences. So, there is something to be said for taking as the variables that describe an electrical circuit (i) the currents and (ii)' the voltages. For clarity we denote potentials by E and voltage by V.

Assume that there are N terminals. Let the interaction variables be the currents $I_{\mathtt{k}}, \mathtt{k} \in \{1,2,\ldots,N\}$ and the voltages $V_{\mathtt{k},\ell}, \mathtt{k},\ell \in \{1,2,\ldots,N\}$. $I_{\mathtt{k}}$ denotes the current that flows into the circuit along the $\mathtt{k}$-th terminal, while $V_{\mathtt{k},\ell}$ denotes the voltage across terminals $\mathtt{k}$ and ℓ.

1. **Assume that the $V_{\mathtt{k},\ell}$'s satisfy the following law (which may be thought of as Kirchhoff's voltage law).**

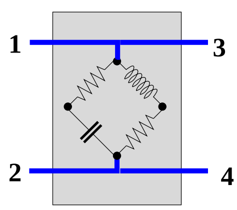
   $$\forall\, \mathtt{k}_1,\mathtt{k}_2,\ldots,\mathtt{k}_n \in \{1,2,\ldots,N\}, \text{ there holds } V_{\mathtt{k}_1,\mathtt{k}_2}+V_{\mathtt{k}_2,\mathtt{k}_3}+\cdots+V_{\mathtt{k}_n,\mathtt{k}_1}=0.$$

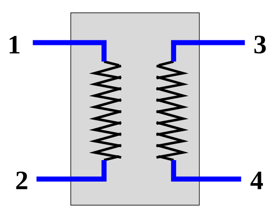
   Prove that the $V_{\mathtt{k},\ell}$'s satisfy this law if and only if there exist potentials $E_{\mathtt{k}}, \mathtt{k} \in \{1,2,\ldots,N\}$ such that

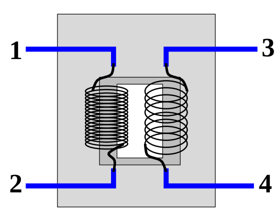
   $$V_{\mathtt{k},\ell} = E_{\mathtt{k}} - E_\ell, \quad \forall\, k,\ell \in \{1,2,\ldots,N\}.$$

2. **Prove that the $E_{\mathtt{k}}$'s are not uniquely determined by the $V_{\mathtt{k},\ell}$'s.**

3. **Set up the behavioral equations in terms of the voltages. Determine for each of the circuits shown below the extent to which the potentials are uniquely defined.**







Exercise IV.4: Regularity

In this exercise, we deal with LTIDSs. The notation is the one used in Lecture IV. In the lecture, we considered a special case of regularity, aimed at full control and leading to a controlled system that is autonomous. However, there is a more general notion of regularity, requiring

$$\mathtt{p}(\mathscr{P} \cap \mathscr{C}) = \mathtt{p}(\mathscr{P}) + \mathtt{p}(\mathscr{C}).$$

1. **Prove that the regularity notion used in the lecture is a special case of this more general notion.**
2. **Assume that $\mathscr{P}$ is controllable. Prove that for any $\mathscr{P}'$ that is a subsystem of $\mathscr{P}, \mathscr{P}' \subseteq \mathscr{P}$, there exists a regular controller $\mathscr{C}$ such that $\mathscr{P} \cap \mathscr{C} = \mathscr{P}'$.**
3. **Prove that if $\mathscr{P}$ is not controllable, the zero system $\mathscr{P}' = \{0\}$ can not be implemented in a regular way, i.e., there exists a regular controller $\mathscr{C}$ such that $\mathscr{P} \cap \mathscr{C} = \{0\}$.**

Exercise IV.5: Characteristic polynomial assignment

Consider the plant $\mathscr{P} \in \mathscr{L}^{\mathtt{w}}$. As all LTIDSs, it can be decomposed into

$$\mathscr{P} = \mathscr{P}_{\text{controllable}} \oplus \mathscr{P}_{\text{autonomous}}.$$

Let $\mathscr{C} \in \mathscr{L}^{\mathtt{w}}$ be a regular controller and consider the autonomous controlled system $\mathscr{P} \cap \mathscr{C} \in \mathscr{L}^{\mathtt{w}}$.

1. **Prove that for any monic $\pi \in \mathbb{R}[\xi]$ there exists such a $\mathscr{C}$ such that**

   $$\chi_{\mathscr{P} \cap \mathscr{C}} = \pi$$

   if and only if $\chi_{\mathscr{P}_{\text{autonomous}}}$ is a factor of π. Deduce the pole placement theorem from here.

2. **Repeat the same question with the characteristic polynomial replaced by the minimal polynomial.**

Exercise IV.6: Proper controllers

In this exercise we study the Bézout equation

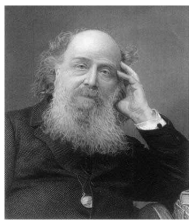
$$ax + by = f.$$

Recall that we already encountered Monsieur Bézout in Exercise III.2. Consider $a, b \in \mathbb{R}[\xi]$ fixed. In this exercise, we study the solvability of this equation for a given $f \in \mathbb{R}[\xi]$, with $x, y \in \mathbb{R}[\xi]$ the unknows.

1. **Assume $\mathtt{degree}(a) = \mathtt{n}$, $\mathtt{degree}(b) = \mathtt{m}$. Consider the map $(x, y) \mapsto ax + by$ acting on $x, y \in \mathbb{R}[\xi], \text{degree}(x) < \mathtt{m}, \text{degree}(y) < \mathtt{n}$. Prove we can then view $(x, y) \mapsto ax + by$ as a linear map that maps $\mathbb{R}^{\mathtt{n}+\mathtt{m}}$ into itself. Prove that the coprimeness of a, b implies that this map is injective. Conclude that it is also surjective and bijective.**

2. **Let $a(\xi) = a_0 + a_1\xi + \cdots + a_{\mathtt{n}}\xi^{\mathtt{n}}, \quad b(\xi) = b_0 + b_1\xi + \cdots + b_{\mathtt{m}}\xi^{\mathtt{m}}$.
Now form the matrix**

$$\begin{bmatrix} a_0 & a_1 & \cdots & a_{\mathtt{n}} & 0 & \cdots & & \\ 0 & a_0 & a_1 & \cdots & a_{\mathtt{n}} & 0 & \cdots & \\ 0 & 0 & a_0 & a_1 & \cdots & a_{\mathtt{n}} & 0 & \cdots \\ & & & \vdots & & & & \\ & & \cdots & 0 & a_0 & a_1 & \cdots & a_{\mathtt{n}} \\ b_0 & b_1 & \cdots & b_{\mathtt{m}} & 0 & \cdots & & \\ 0 & b_0 & b_1 & \cdots & b_{\mathtt{m}} & 0 & \cdots & \\ 0 & 0 & b_0 & b_1 & \cdots & b_{\mathtt{m}} & 0 & \cdots \\ & & & \vdots & & & & \\ & & \cdots & 0 & b_0 & b_1 & \cdots & b_{\mathtt{m}} \end{bmatrix} \begin{matrix} \left.\right\} \mathtt{m} \textbf{ rows} \\ \\ \left.\right\} \mathtt{n} \textbf{ rows} \end{matrix} .$$



James Sylvester (1814-1897)

**This matrix is called the *Sylvester matrix*. Its determinant is called the *resultant* of the polynomials a and b. Prove that the resultant of a and b is non-zero if and only if a and b are coprime.
Hint: write Bézout in matrix notation.**

3. **Prove that this implies that $a, b \in \mathbb{R}[\xi]$ are coprime if and only if there exist $x, y \in \mathbb{R}[\xi]$ such that $ax + by = 1$ (the so-called Bézout identity).**

4. **Prove that if $\mathtt{degree}(a) = \mathtt{n}$, and $\mathtt{degree}(b) < \mathtt{n}$, there exist, for all $f \in \mathbb{R}[\xi]$ with $\mathtt{degree}(f) = 2\mathtt{n}-1$, $x, y \in \mathbb{R}[\xi]$ with $\mathtt{degree}(x) = \mathtt{n}-1$, and $\mathtt{degree}(y) \leq \mathtt{n}-1$, such that $ax + by = f$.**

In the classical Linear Systems Theory courses, the pole placement problem is usually studied in the following setting. The plant is given by

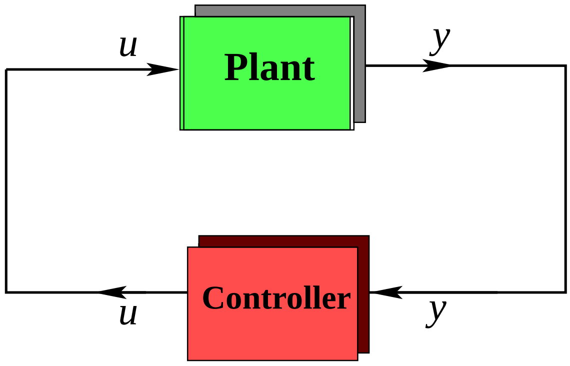
$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx,$$

and the controller by

$$\frac{d}{dt}z = Fz + Gy, \quad u = Hz + Jy.$$

A major difference from our polynomial matrix oriented approach is that here the transfer function of the plant is assumed to be strictly proper, while the controller is restricted to be proper. This guarantees, for example, that the McMillan degree of the controlled system is the sum of the McMillan degree of the plant and of the McMillan the controller. In this exercise, we discuss the properness issue in a polynomial context for single-input/single-output systems.

5. **Consider the single-input/single-output controllable plant with strictly proper transfer function $g = \frac{q}{p}$. Let $\mathtt{degree}(p) = \mathtt{n}$. Prove that for any monic $f \in \mathbb{R}[\xi], \mathtt{degree}(f) \geq 2\mathtt{n})$, there exist a controller with proper transfer function such that the controlled system has characteristic polynomial f. Interpret this result in terms of the feedback system shown below.**



Exercise V.1: Preservation of properties under interconnection

The notation is the one used in Lectures IV and V. $\mathscr{B}$ is the original behavior, $\mathscr{B}'$ the behavior obtained after interconnection of terminals.

1. **Consider the interconnection of terminals of an electrical circuit. Prove that if $\mathscr{B}$ is linear and time-invariant, so is $\mathscr{B}'$. Prove that if $\mathscr{B}$ is a LTIDS, so is $\mathscr{B}'$. Prove that if $\mathscr{B}$ satisfies KVL, so does $\mathscr{B}'$. Prove that if $\mathscr{B}$ satisfies KCL, so does $\mathscr{B}'$.**
2. **Consider the interconnection of terminals of a mechanical system. Prove that if $\mathscr{B}$ satisfies IUM, so does $\mathscr{B}'$.**
3. **Consider Newton's second law. Prove IUM.**
4. **Apparently, Kepler's laws do not satisfy IUM. Discuss why this is and in what sense IUM holds for Kepler's laws.**

Exercise V.2: Tellegen's theorem

Tellegen's theorem is one of the most powerful theorems of electrical circuit theory. It applies to a graph with in the branches 2-terminal 1-ports.
For ease of reference, we recall the required notions.

Bernard D.H. Tellegen
1900-1990

A *directed graph* $\mathscr{G}$, also called a digraph, is defined as

$$\mathscr{G} = (\mathbb{V}, \mathbb{E}, f_+, f_-)$$

with $\mathbb{V}$ the set of *vertices*, $\mathbb{E}$ the set of *edges*, $f_+, f_- : \mathbb{E} \to \mathbb{V}$, the incidence maps, consist of a *source map* f_+ and *sink map* f_-. A digraph is drawn with directed edges pointing from the source to the sink. Denote by $|\mathbb{V}|$ and $|\mathbb{E}|$ the number of vertices and edges, and enumerate the sets of vertices and edges as

$$\mathbb{V} = \{v_1, v_2, \ldots, v_{|\mathbb{V}|}\}, \qquad \mathbb{E} = \{e_1, e_2, \ldots, e_{|\mathbb{E}|}\}.$$

Assume that associated with a circuit, there is a digraph. Associate with each of the edges a current (counted positive when the current flows into the direction of the edge), and a voltage across the edge. This yields the vectors of real numbers

$$I_{\mathbb{E}} = \begin{bmatrix} I_{e_1} \\ I_{e_2} \\ \vdots \\ I_{|\mathbb{E}|} \end{bmatrix}, \qquad V_{\mathbb{E}} = \begin{bmatrix} V_{e_1} \\ V_{e_2} \\ \vdots \\ V_{|\mathbb{E}|} \end{bmatrix}.$$

Kirchhoff's current law (KCL) imposes constraints on the vectors $I_{\mathbb{E}}$ by requiring that the sum of the currents in the edges incident to any vertex is zero. Kirchhoff's voltage law (KVL) imposes constraint on the vectors $V_{\mathbb{E}}$. KVL can be expressed in a number of ways. A convenient one is to require

that there exists a potential, that is, a vector of real numbers

$$E_{\mathbb{V}} = \begin{bmatrix} E_{v_1} \\ E_{v_2} \\ \vdots \\ E_{|\mathbb{V}|} \end{bmatrix},$$

such that the voltage across an edge is equal to the difference of the potentials to which the edge is incident (counted positive when the potential of the source is $\geq$ the potential of sink).

Let $\mathscr{I} \subseteq \mathbb{R}^{|\mathbb{E}|}$ and $\mathscr{V} \subseteq \mathbb{R}^{|\mathbb{E}|}$ be the set vectors $I_{\mathbb{E}}$ and $V_{\mathbb{E}}$ that satisfy KCL and KVL. Tellegen's theorem states that $\mathscr{I}$ and $\mathscr{V}$ are linear subspaces and that

$$\mathscr{I} = \mathscr{V}^{\perp}.$$

1. **Prove Tellegen's theorem.**

   Proceed as follows.

   (a) **Introduce the *incidence matrix* A of a digraph defined as $|\mathbb{V}| \times |\mathbb{E}|$ matrix with elements $\{-1, 0, +1\}$ according to**

   $$A_{\mathrm{k},\ell} = \begin{cases} +1 & \textbf{if } f_{+} = (e_{\ell})\, v_{\mathrm{k}}, \\ -1 & \textbf{if } f_{-} = (e_{\ell})\, v_{\mathrm{k}}, \\ 0 & \textbf{otherwise}. \end{cases}$$

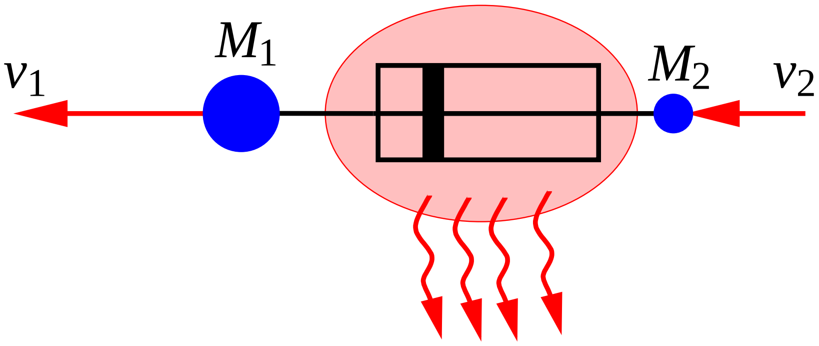
   Prove that KCL states that $AI_{\mathbb{E}} = 0$, while KVL states that $V_{\mathbb{E}} = A^{\top} E_{\mathbb{V}}$.

   (b) **Prove that the kernel of a matrix is orthogonal to the image of the transpose.**

   (c) **Deduce Tellegen's theorem.**

2. **Illustrate Tellegen's theorem by means of the circuit studied in Lecture III.**

Exercise V.3: Heat produced by a damper



Consider the system shown above. It consists of 2 masses, connected by a damper. The motion is assumed to take place horizontally. The damper is assume to be a linear damper with damping coefficient $D > 0$.

1. **Denote the positions of the pointmasses by q_1, q_2. There are no external forces that act on the masses. Obtain the differential equations that govern (q_1, q_2).**
2. **Solve this differential equation with initial conditions** $q_1(0), q_2(0), \frac{d}{dt}q_1(0), \frac{d}{dt}q_2(0)$
3. **What is the power going into the damper?**
4. **Assume that all the energy absorbed by the damper is converted into heat. Compute the energy absorbed on the interval $[0, \infty)$ as a function of the parameters $M_1, M_2, D, q_1(0), q_2(0), \frac{d}{dt}q_1(0), \frac{d}{dt}q_2(0)$. Note that this energy only depends on $M_1, M_2, \frac{d}{dt}q_1(0), \frac{d}{dt}q_2(0)$.**

Exercise V.4: Port KVL and port KCL

We only consider LTIDSs. Let $\mathscr{B} \subseteq \left(\mathbb{R}^N \times \mathbb{R}^N\right)^{\mathbb{R}}$ be the behavior of an N-terminal electrical circuit. Assume that $\mathscr{B}$ admits a voltage-driven representation

$$I = Z\left(\frac{d}{dt}\right)V$$

with $Z \in \mathbb{R}(\xi)^{N\times N}$.

Recall that the set of terminals $\{1,2,\ldots,p\}$ satisfies port-KCL $:\Leftrightarrow$

$$[\![\left(V_1,\ldots,V_p,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N\right) \in \mathscr{B}]\!] \Rightarrow [\![I_1+\cdots+I_p=0]\!]$$

and port-KVL $:\Leftrightarrow$

$$[\![\left(V_1,\ldots,V_p,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N\right) \in \mathscr{B} \text{ and } \alpha \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R})]\!] \Rightarrow [\![\left(V_1+\alpha,\ldots,V_p+\alpha,V_{p+1},\ldots,V_N,I_1,\ldots,I_p,I_{p+1},\ldots,I_N\right) \in \mathscr{B}]\!].$$

1. **Prove that if $Z(i\omega)+Z^\top(-i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, then**

   port-KCL $\Leftrightarrow$ port-KVL.

   So, in particular, if the circuit is passive, port-KCL $\Leftrightarrow$ port-KVL.

   Proceed as follows.

   (a) Let $c \in \mathbb{R}^N$. Prove that

   $$[\![c^\top Z(i\omega)=0]\!] \Leftrightarrow [\![c^\top Z(i\omega)c=0]\!] \Leftrightarrow [\![Z(i\omega)c=0]\!].$$

   (b) It suffices to give the remainder of the proof under the assumption that V and I are related by

   $$I(t) = \int_{-\infty}^{+\infty} G(t-t')V(t')\,dt'$$

   for some $G \in \mathscr{C}^\infty\left(R,R^{N\times n}\right)$. Observe that port-KCL $\Leftrightarrow c^\top G = 0$ for a suitable $c \in \mathbb{R}^N$. Hence, by what was just proven, $Gc = 0$.

Therefore

$$\int_{-\infty}^{+\infty} G(t-t')c\alpha(t')\,dt' = 0 \;\; \forall\, \alpha \in \mathscr{C}^{\infty}(\mathbb{R},\mathbb{R})\,.$$

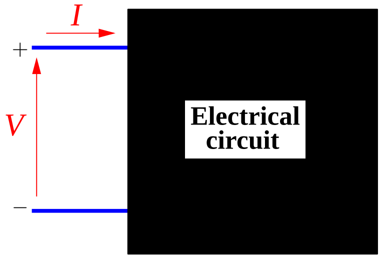
Prove that yields port-KVL. Now reverse this path to prove that port-KVL ⇒ port-KCL.

2. **Now apply these ideas to mechanical devices. Argue that the analogue of (port-)KVL (call it KDL – Kirchhoff's displacement law) is quite unlikely since it implies a much stronger property than the universal IUM. Prove that (massless) dampers, springs, and inerters satisfy both KFL and KDL, and that these are again equivalent. However, devices that are not massless satisfy neither (port-)KFL nor (port)-KDL.**

   This illustrates a basic difference between electrical and mechanical systems. This asymmetry makes electrical-mechanical analogies a hopeless, or at least a very tenuous, endeavor.

Exercise V.5: LC-synthesis

Consider a 2-terminal electrical circuit as shown below. Assume that KVL and KCL hold. Then we can take as manifest variables the voltage V across the external terminals and the current I that flows into the circuit along one terminal (and leaves the circuit along the other terminal).



Assume that the behavior is a LTIDS with behavioral equations

$$p\left(\frac{d}{dt}\right)V = q\left(\frac{d}{dt}\right)I, \qquad \textbf{with } p,q \in \mathbb{R}[\xi].$$

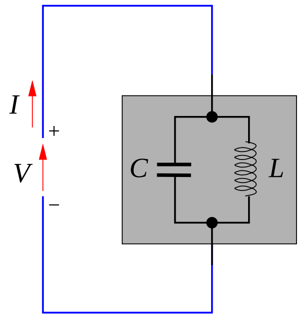
Assume controllability, in other words, assume p and q are coprime. The rational function

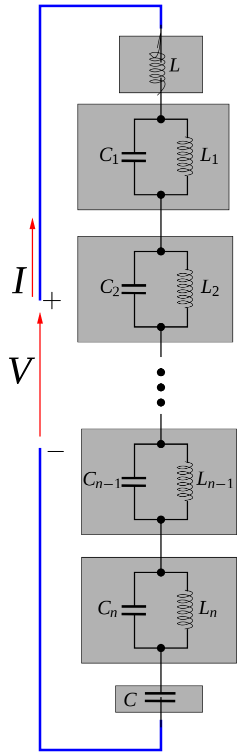
$$Z = \frac{q}{p}$$

is called the *impedance* of the circuit. Of course, we could have defined the behavioral equations as

$$V = Z\left(\frac{d}{dt}\right)I, \qquad \textbf{with } Z \in \mathbb{R}(\xi).$$

1. **Compute the impedance of the 'elementary' LC section shown below.**





2. **Compute the impedance of the 'ladder' LC circuit shown below.**

3. **Prove that every impedance of the form**

$$Z(\xi) = a_\infty \xi + a_0 \xi + \frac{a_1 \xi}{\xi^2 + \omega_1^2} + \frac{a_2 \xi}{\xi^2 + \omega_2^2} + \cdots + \frac{a_n \xi}{\xi^2 + \omega_n^2},$$

with the a_k's ≥ 0 and the ω_k's distinct and $\neq 0$, is realizable as an LC ladder.

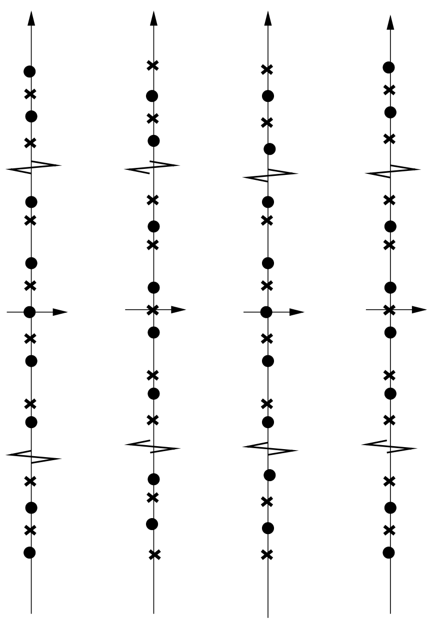
Exercise V.6: Lossless systems

We use the notation of Exercise V.5.

1. The circuit is said to be lossless if
$$\int_{-\infty}^{+\infty} V(t)I(t)\,dt = 0 \quad \text{for all } (V,I) \in \mathscr{B} \text{ with compact support}.$$
Prove that the circuit is lossless if and only if $Z(\xi) = -Z(-\xi)$.

2. If the circuit is also passive, then we know that Z is also positive real. Prove that a circuit is lossless and passive if and only if Z has the partial fraction expansion
$$Z(\xi) = a_\infty \xi + a_0 \xi + \frac{a_1 \xi}{\xi^2 + \omega_1^2} + \frac{a_2 \xi}{\xi^2 + \omega_2^2} + \cdots + \frac{a_n \xi}{\xi^2 + \omega_n^2},$$
with the a_k's ≥ 0 and the ω_k's distinct and $\neq 0$.

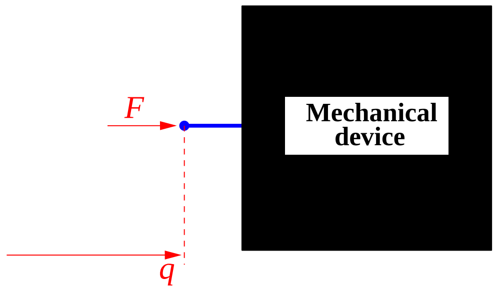
3. Prove that the pole/zero pattern of such an impedance is one of the following forms.



4. It can be shown (you need not do so) that a circuit composed of positive L's and C's is lossless and passive. Use the result of Exercise V.5

to give a necessary and sufficient condition for the realizability of an impedance as an LC circuit.

Consider a 1-terminal mechanical device as shown below. Take as manifest variables the force F acting on the external terminal and the position q of the external terminal.



Assume that the behavior is a LTIDS with behavioral equations

$$d\left(\frac{d}{dt}\right)F = n\left(\frac{d}{dt}\right)q, \quad \text{with } n,d \in \mathbb{R}[\xi].$$

Assume controllability, in other words, assume n and d are coprime. Define the rational function

$$G = \frac{q}{p}.$$

Of course, we could have defined the behavioral equations as

$$q = G\left(\frac{d}{dt}\right)F, \quad \text{with } G \in \mathbb{R}(\xi).$$

5. **The mechanical device is said to be lossless if**

$$\int_{-\infty}^{+\infty} F(t)\frac{d}{dt}q(t)\,dt = 0 \quad \text{for all } (F,q) \in \mathscr{B} \text{ with compact support}.$$

Prove that the device is lossless if and only if

$$G(\xi) = G(-\xi).$$

Relate this to time-reversibility as studied in Exercise II.3