Elgersburg Lectures – March 2010

Lecture I

MODELS and BEHAVIORS

Theme

FAQ: How should we think of a ‘mathematical model’,
in the sense of: as a mathematical concept?

Theme

FAQ: How should we think of a 'mathematical model',
in the sense of: as a mathematical concept?

Answer: As a subset of a universum of possible events.

This subset = the outcomes which the model allows,
= the behavior.

The aim of this lecture is to develop this mathematical formalism, with the behavior as the central concept.

Outline

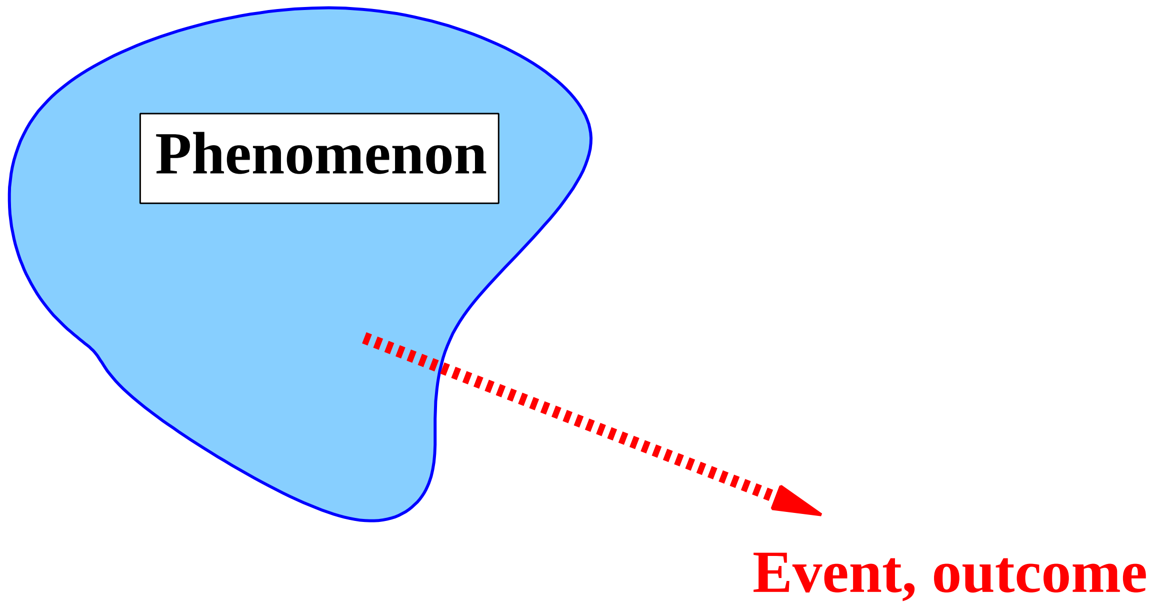
- **Mathematical models**
- **The universum and the behavior**
- **Dynamical systems**
- **Properties of dynamical systems**
- **Linear time-invariant differential systems (LTIDSs): systems described by linear constant-coefficient ODEs**
- **Other sets of independent variables**

Mathematical models

Modeling

Assume that we have a 'real' phenomenon.

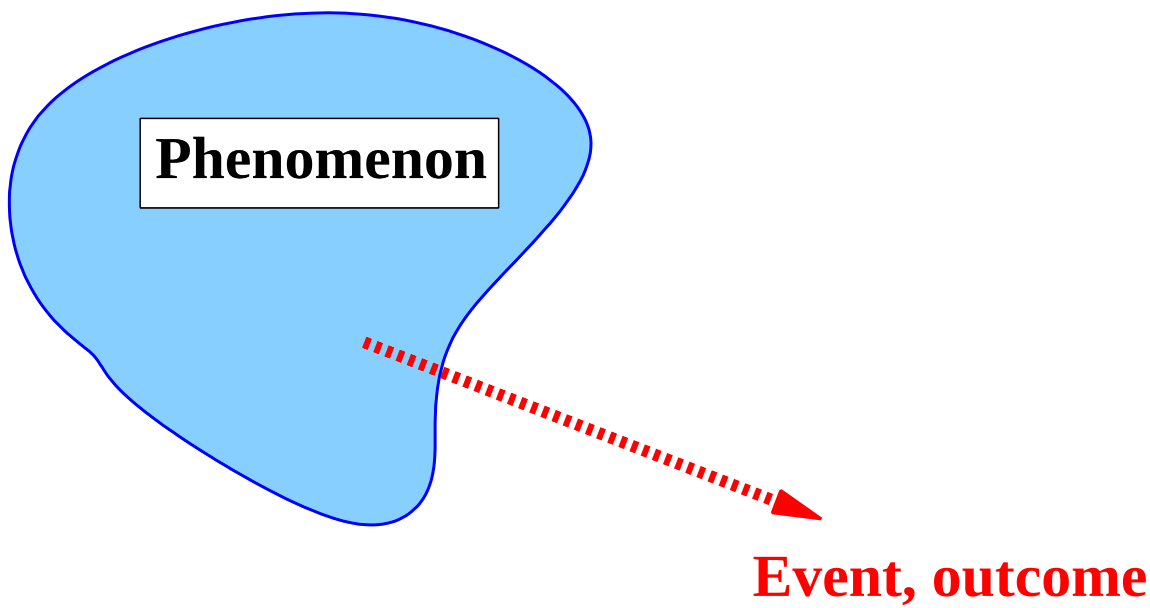
The phenomenon produces 'events' (synonym: 'outcomes').



Modeling

Assume that we have a 'real' phenomenon.

The phenomenon produces 'events' (synonym: 'outcomes').



We view a deterministic model for the phenomenon as a prescription of which events can occur,
and which events cannot occur.

The universum and the behavior

The universum and the behavior

The events are described in the language of mathematics by answering

to which set do the (unmodelled) events belong?

The universum of events that are - in principle - possible is called the ‘universum’, and is denoted by $\mathscr{U}$.

The universum and the behavior

The events are described in the language of mathematics by answering

to which set do the (unmodelled) events belong?

The universum of events that are - in principle - possible is called the ‘universum’, and is denoted by $\mathscr{U}$.

Assume that, after studying the situation, the conclusion is reached that the events are constrained, that some laws are in force. Expressing this restriction leads to a ‘model’.

The universum and the behavior

The events are described in the language of mathematics by answering

to which set do the (unmodelled) events belong?

The universum of events that are - in principle - possible is called the ‘universum’, and is denoted by $\mathscr{U}$.

Assume that, after studying the situation, the conclusion is reached that the events are constrained, that some laws are in force. Expressing this restriction leads to a ‘model’.

Modeling therefore means that certain events are declared impossible, that they cannot occur.

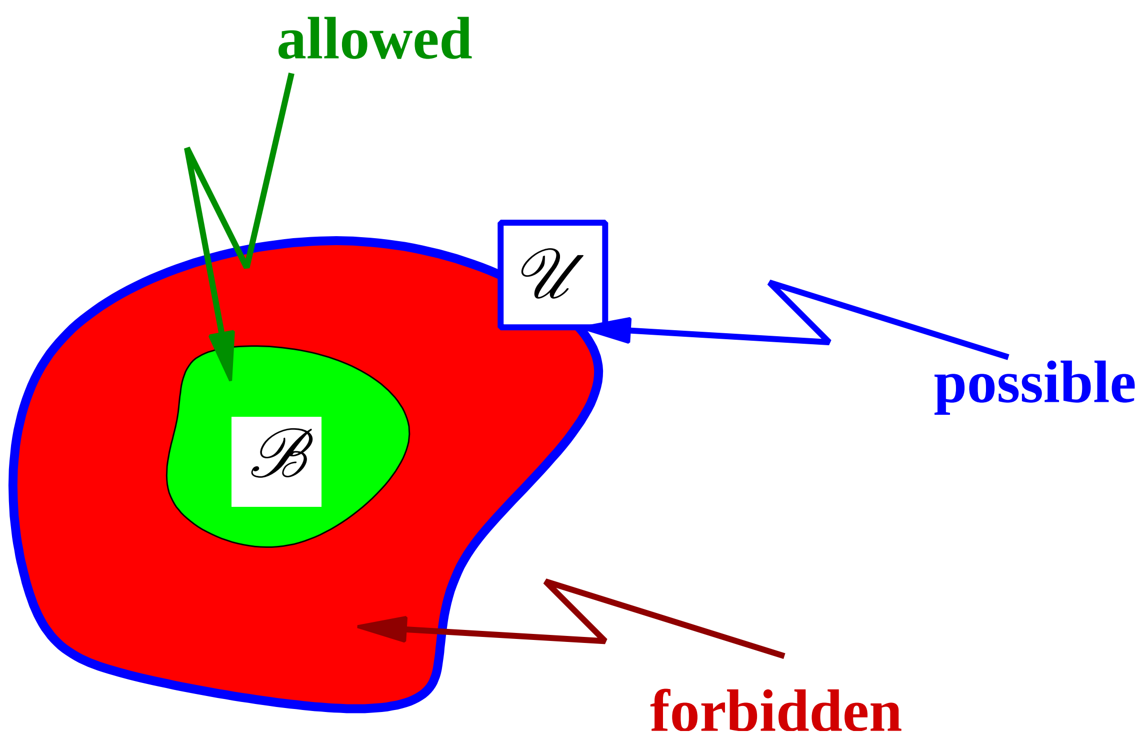
The possibilities that remain constitute the ‘behavior’ of the model, and is denoted by $\mathscr{B}$.

The behavior

A *mathematical model* $:\Leftrightarrow$ a pair $(\mathscr{U}, \mathscr{B})$ with

$\mathscr{U}$ the universum of events

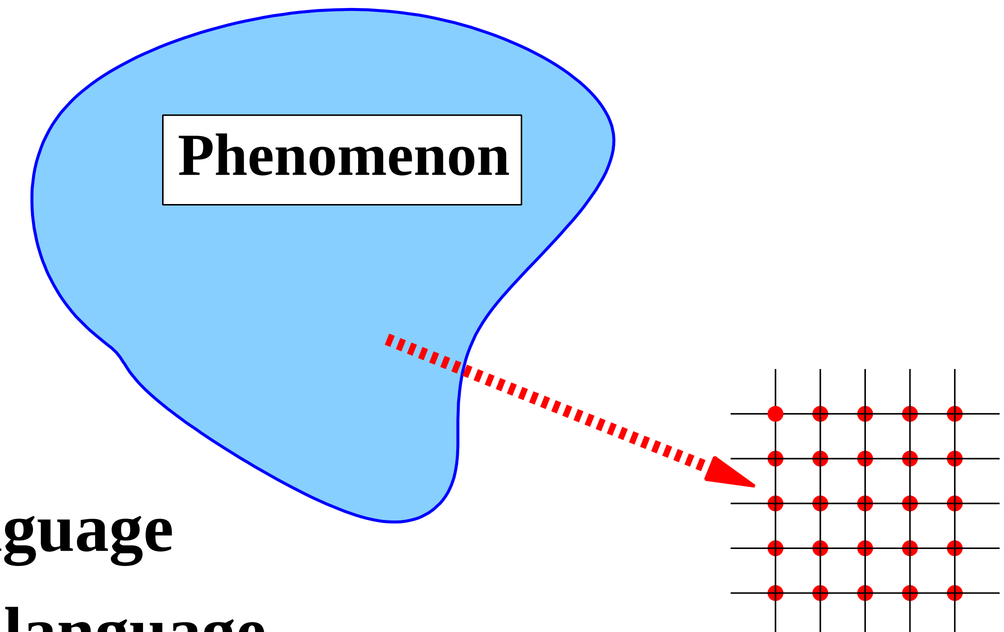
$\mathscr{B} \subseteq \mathscr{U}$ the behavior of the model



Examples

Discrete event phenomena

If $\mathscr{U}$ is a finite set, or strings of elements from a finite set, we speak about discrete event systems (DESs).



Examples:

- **Words in a natural language**
- **Sentences in a natural language**
- **DNA sequences**
- **$\LaTeX$ code**

Discrete event phenomena

- **Words in a natural language**

$\mathscr{U} = \mathbb{A}^*$ **(:= all finite strings with letters from** $\mathbb{A}$**)**
with $\mathbb{A} = \{a, \ldots, z, A, \ldots, Z\}$.

$\mathscr{B}$ **= all words recognized by the spelling checker,**
for example, behavior $\in \mathscr{B}$**, SPQR** $\notin \mathscr{B}$**.**
$\mathscr{B}$ **is basically specified by enumeration.**

Discrete event phenomena

- **Words in a natural language**

$\mathscr{U} = \mathbb{A}^*$ **(:= all finite strings with letters from** $\mathbb{A}$**)**
with $\mathbb{A} = \{a, \ldots, z, A, \ldots, Z\}$.

$\mathscr{B}$ **= all words recognized by the spelling checker,**
for example, behavior $\in \mathscr{B}$**, SPQR** $\notin \mathscr{B}$**.**
$\mathscr{B}$ **is basically specified by enumeration.**

- **Sentences in a natural language**

$\mathscr{U} = \mathbb{A}^*$ **(:= all finite strings with letters from** $\mathbb{A}$**)**
with $\mathbb{A} = \{a, \ldots, z, A, \ldots, Z, \ , .;:\ `"' - ()!?,$ **etc.**$\}$.

$\mathscr{B}$ **= all legal sentences.**
Specifying $\mathscr{B}$ **is a complicated matter, involving grammars.**

Discrete event phenomena

- **DNA sequences**

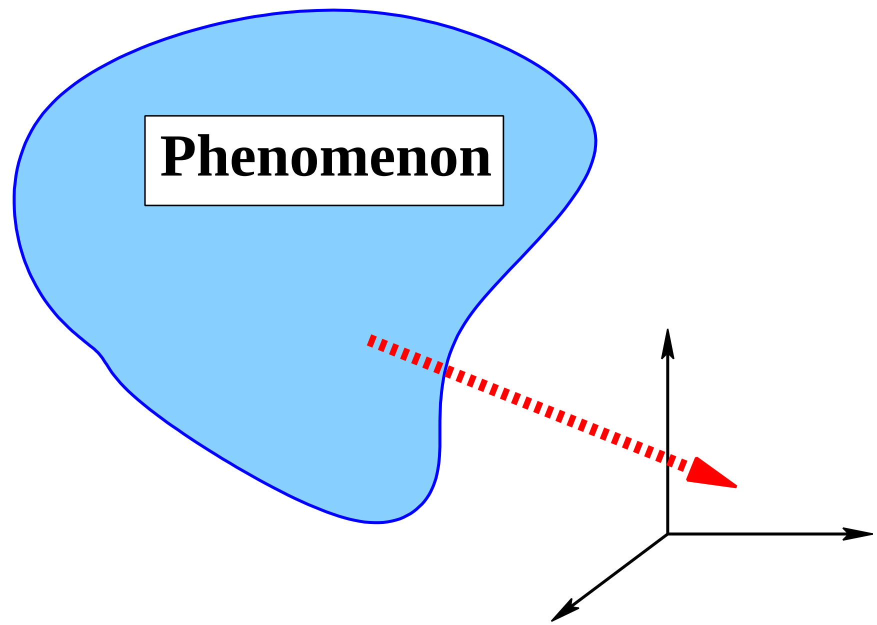
$\mathbb{A} = \{A, G, C, T\}, \;\; \mathscr{U} = \mathbb{A}^*, \;\; \mathscr{B} = ???$

- **LaTeXcode**

$\mathscr{B} =$ **all LaTeXfiles that 'compile'.**

Continuous phenomena

If $\mathscr{U}$ is a (subset of) a finite-dimensional real (or complex) vector space, we speak about continuous models.



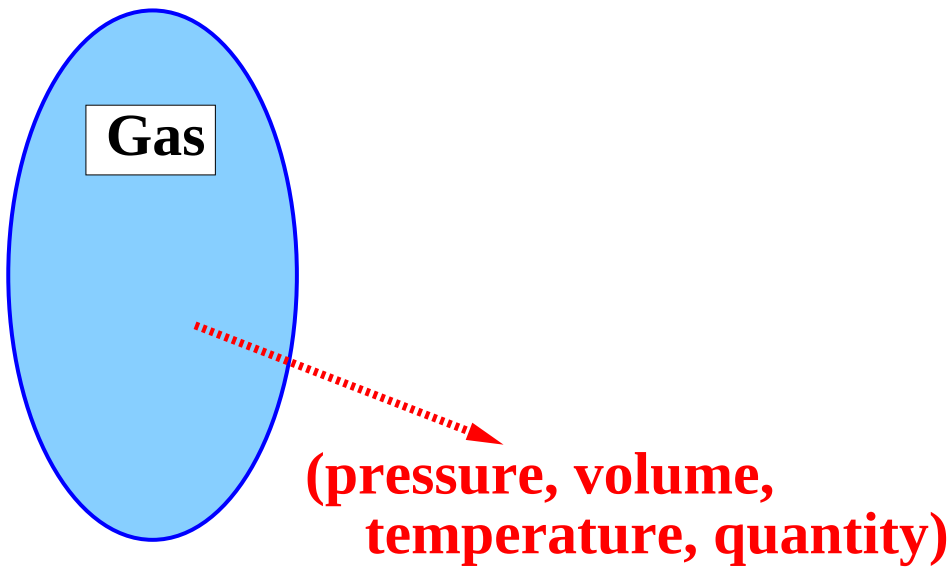
Examples:

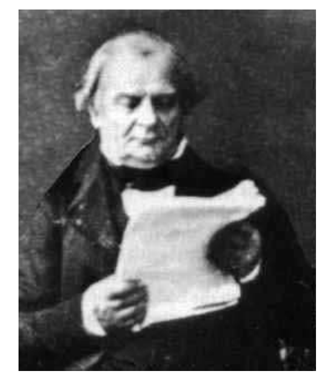
- **The gas law**
- **A spring**
- **The gravitational attraction of two bodies**
- **A resistor**

Continuous phenomena

- **The gas law**

Event: pressure, volume, temperature, quantity of a gas in a vessel.





Benoît Clapeyron
1799–1864

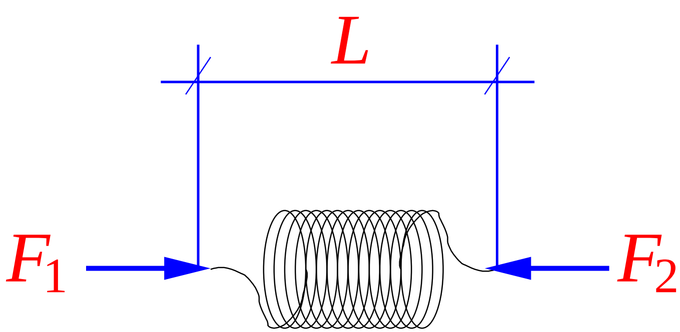
$$\mathscr{U} = [0,\infty)^4; \mathscr{B} = \{(P,V,T,N) \in [0,\infty)^4 \mid PV = NT\}.$$

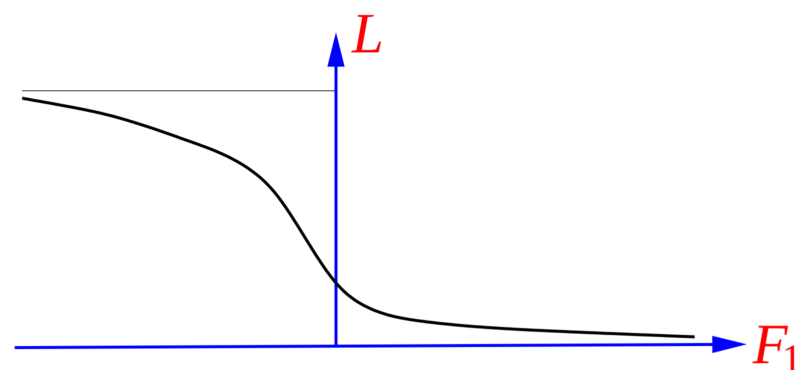
Occasionally in these lectures, we assume that the units are chosen so that certain constants, as the proportionality constant in this example, are equal to one.

Continuous phenomena

- **A spring**

  Event: (force F_1, force F_2, length L).





$$\mathscr{U} = \mathbb{R} \times \mathbb{R} \times [0, \infty);$$

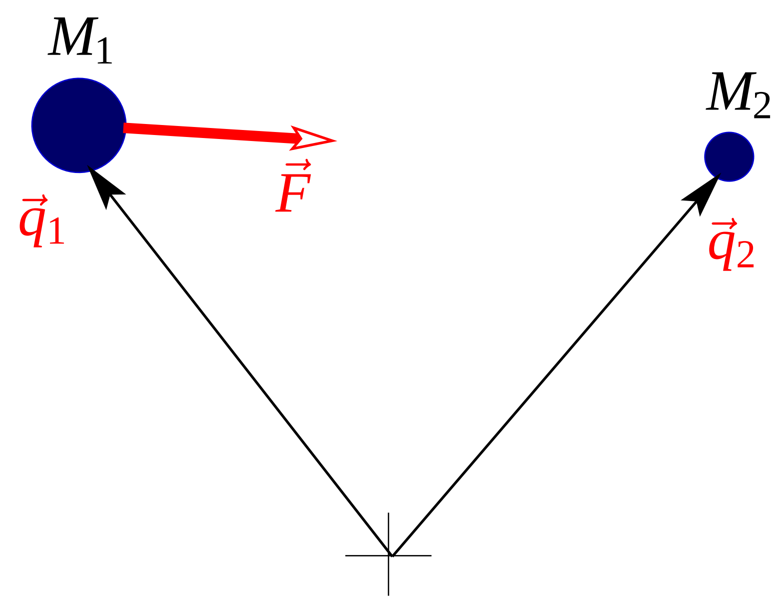
$$\mathscr{B} = \{(F_1, F_2, L) \in \mathbb{R} \times \mathbb{R} \times [0, \infty) \mid F_1 = F_2, L = \rho(F_1)\}.$$

Continuous phenomena

▶ The gravitational attraction of two bodies

Occasionally in these lectures, we assume that the units are chosen so that certain constants, as the universal gravitational constant in this example, are equal to one.

Event: (position $\vec{q}_1$, position $\vec{q}_2$, force $\vec{F}$).



Isaac Newton (1643–1727)

$$\mathscr{U} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3;$$

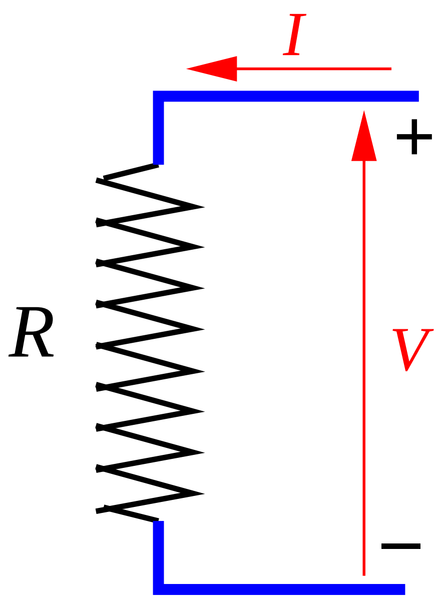
$$\mathscr{B} = \left\{ (\vec{q}_1, \vec{q}_2, \vec{F}) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \mid \vec{F} = \frac{M_1 M_2 \vec{1}_{(\vec{q}_2 - \vec{q}_1)}}{|\vec{q}_1 - \vec{q}_2|^2} \right\}.$$

Continuous phenomena

- **A resistor**

Event: (voltage V, current I).

Throughout, we take the current positive when it runs *into* the circuit, and we take the voltage positive when it goes *from higher to lower* potential.



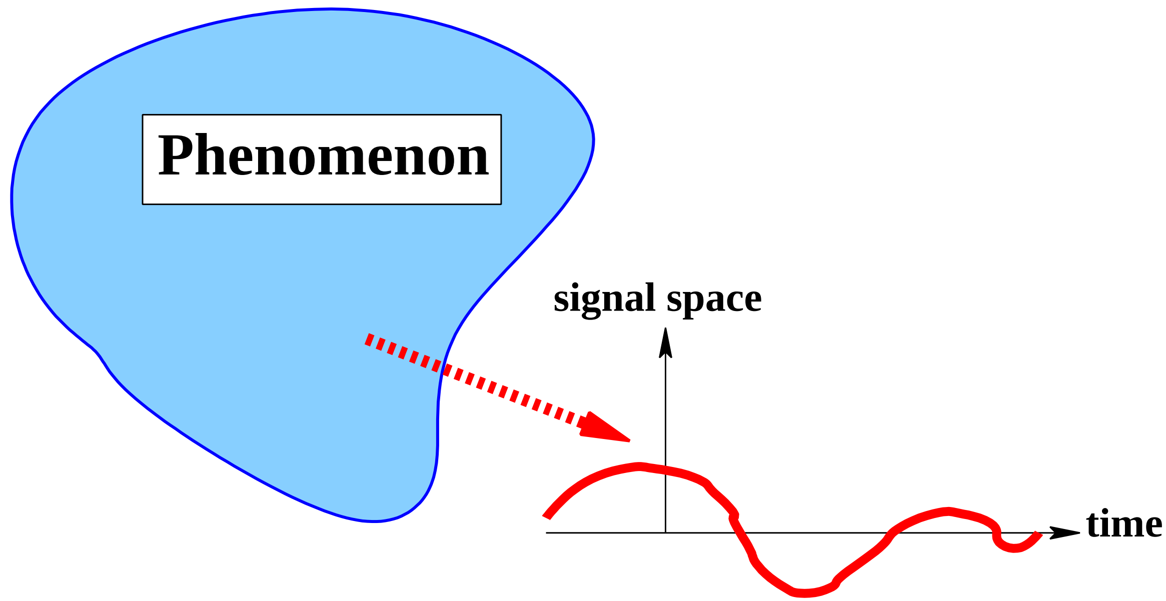
Georg Ohm
(1789–1854)

$$\mathscr{U} = \mathbb{R} \times \mathbb{R}$$

$$\mathscr{B} = \{(V, I) \in \mathbb{R} \times \mathbb{R} \mid V = RI\}$$ **(Ohm's law)**

Dynamical phenomena

If $\mathscr{U}$ is a set of functions of time, we speak about dynamical models.



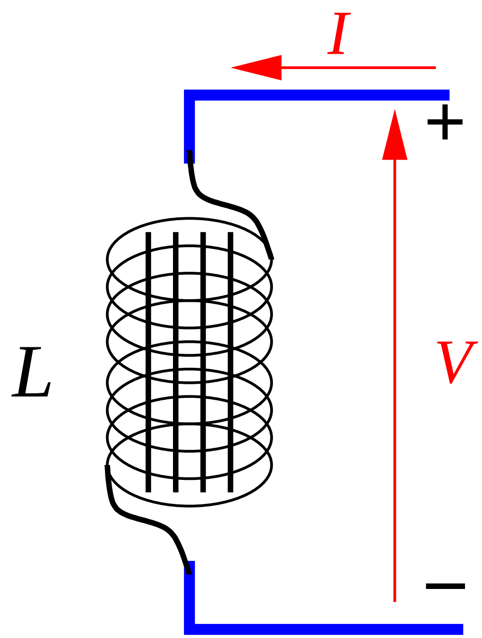
Examples:

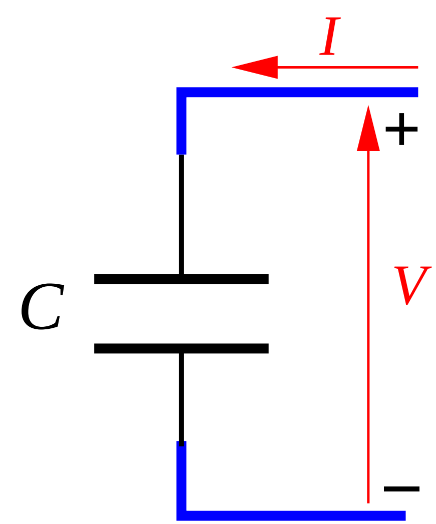
- **Inductors, capacitors**
- **Kepler's laws**
- **Newton's second law**

Dynamical phenomena

- **Inductors and capacitors**

Event: voltage and current as a function of time.





$$\mathscr{U} = (\mathbb{R} \times \mathbb{R})^{\mathbb{R}};$$

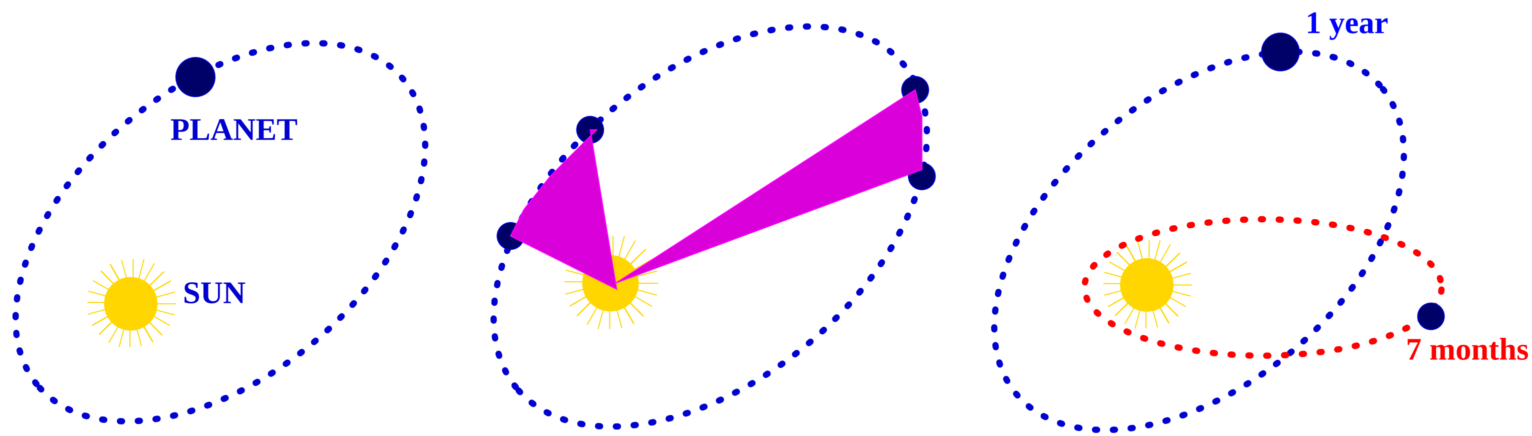
$$\mathscr{B} = \{(V,I) : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \mid L\frac{d}{dt}I = V\}$$ **(inductor),**

$$\mathscr{B} = \{(V,I) : \mathbb{R} \to \mathbb{R} \times \mathbb{R} \mid C\frac{d}{dt}V = I\}$$ **(capacitor).**

Dynamical phenomena

- **Kepler's laws**

Event: the position of a planet as a function of time.



K1: ellipse, sun in focus,
K2: equal areas in equal times,
K3: square of the period
= third power of major axis

$\mathscr{U} = (\mathbb{R}^3)^{\mathbb{R}}$;

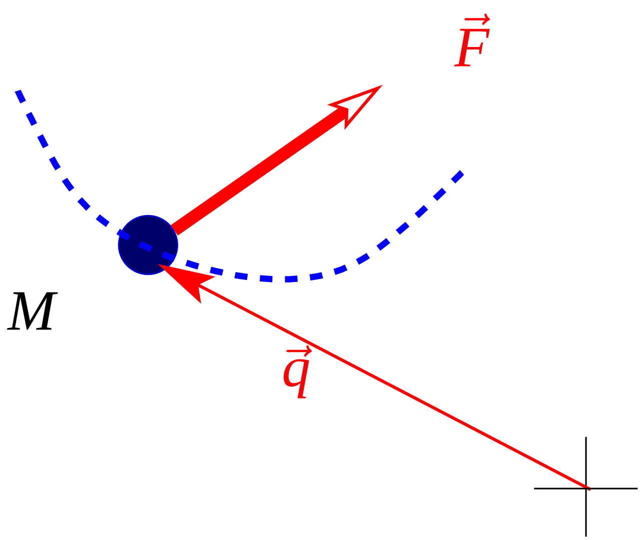
$\mathscr{B} = \{\vec{q} : \mathbb{R} \to \mathbb{R}^3 \mid$ **K1, K2, & K3 hold** $\}$.

Johannes Kepler (1571–1630)

Dynamical phenomena

- **Newton's second law**

Event: the position of a pointmass and the force acting on it, both as a function of time.



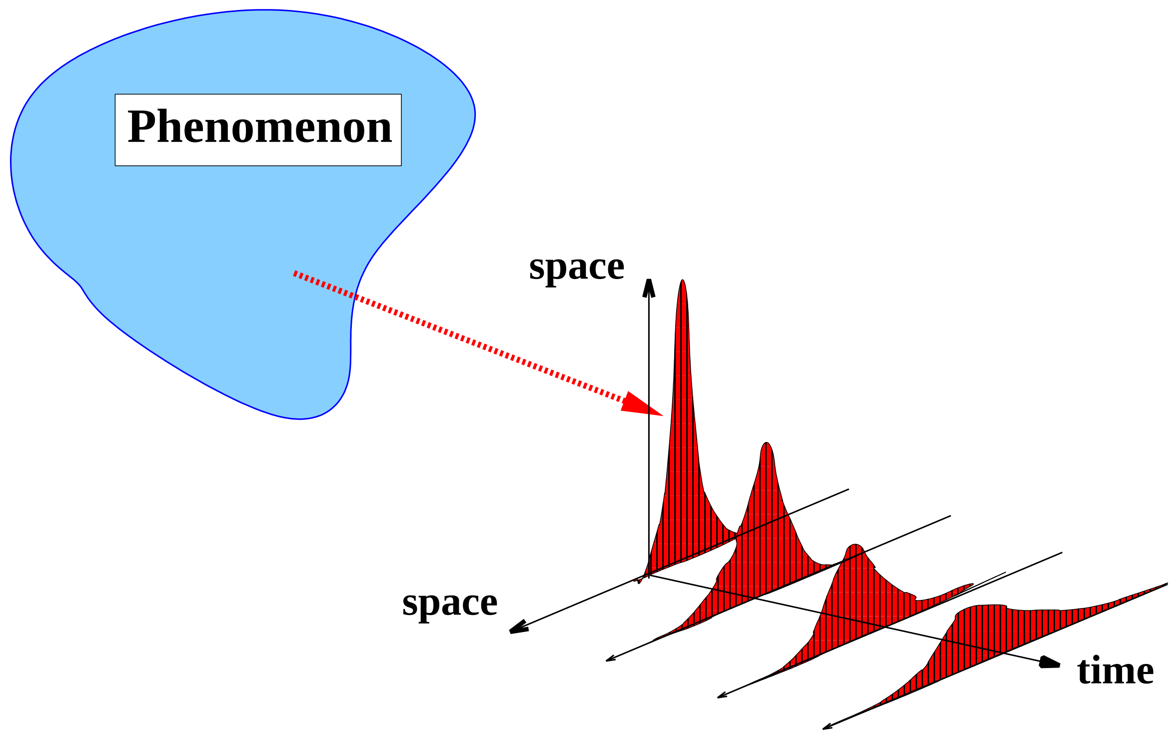
Newton painted by William Blake

$$\mathscr{U} = (\mathbb{R}^3 \times \mathbb{R}^3)^{\mathbb{R}};$$

$$\mathscr{B} = \{(\vec{q}, \vec{F}) : \mathbb{R} \to \mathbb{R}^3 \times \mathbb{R}^3 \mid \vec{F} = M\frac{d^2}{dt^2}\vec{q}\}.$$

Distributed phenomena

If $\mathscr{U}$ is a set of functions of space and time, we speak about distributed parameter systems.



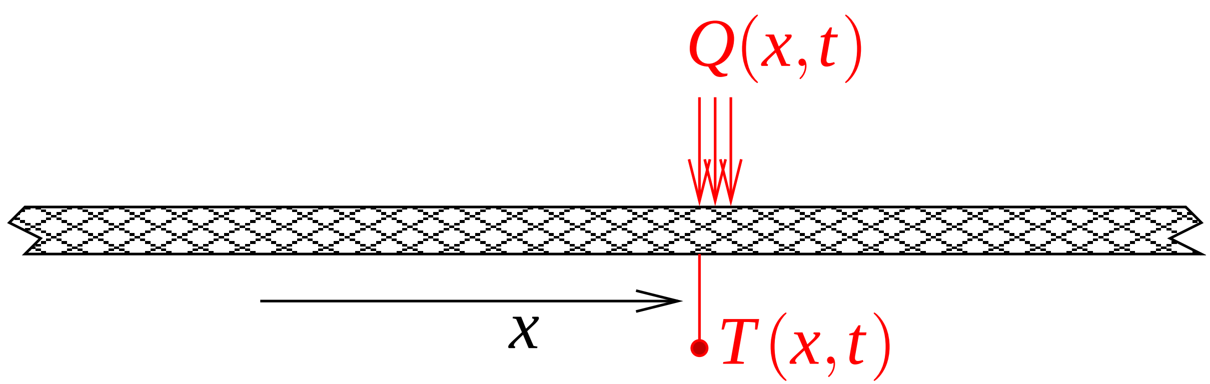
Examples:

- **Heat diffusion**
- **Maxwell's equations**

Distributed phenomena

- **Heat diffusion**

Event: temperature and heat flow as a function of time and space.



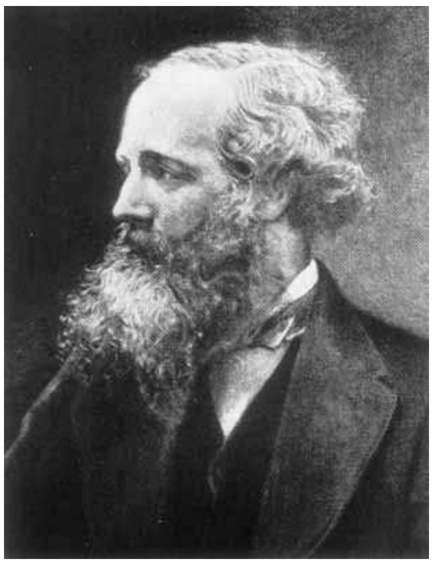
$$\mathscr{U} = ([0,\infty) \times \mathbb{R})^{\mathbb{R}^2};$$

$$\mathscr{B} = \left\{ (T,Q) : \mathbb{R}^2 \to [0,\infty) \times \mathbb{R} \mid \frac{\partial}{\partial t} T = \frac{\partial^2}{\partial x^2} T + Q \right\}.$$

Distributed phenomena

- **Maxwell's equations**

Event: electric field, magnetic field, current density, charge density as a function of time and space.



James Clerk Maxwell
(1831–1879)

$$\begin{aligned}
\nabla \cdot \vec{E} &= \frac{1}{\varepsilon_0}\rho \ , \\
\nabla \times \vec{E} &= -\frac{\partial}{\partial t}\vec{B}, \\
\nabla \cdot \vec{B} &= 0 \ , \\
c^2 \nabla \times \vec{B} &= \frac{1}{\varepsilon_0}\vec{j} + \frac{\partial}{\partial t}\vec{E}.
\end{aligned}$$

$$\mathscr{U} = (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})^{\mathbb{R}^4};$$

$$\mathscr{B} = \{(\vec{E}, \vec{B}, \vec{j}, \rho) : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mid \text{Maxwell's equations are satisfied}\}.$$

The behavior

Behavioral models

The behavior captures the essence of what a model is.

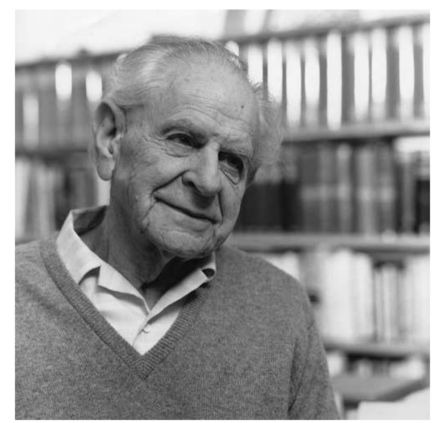
The behavior is all there is.
Equivalence of models, properties of models,
symmetries, system identification, etc.
must all refer to the behavior.

Behavioral models

The behavior captures the essence of what a model is.

The behavior is all there is.
Equivalence of models, properties of models,
symmetries, system identification, etc.
must all refer to the behavior.

Every 'good' scientific theory is prohibition:
it forbids certain things to happen.
The more it forbids, the better it is.



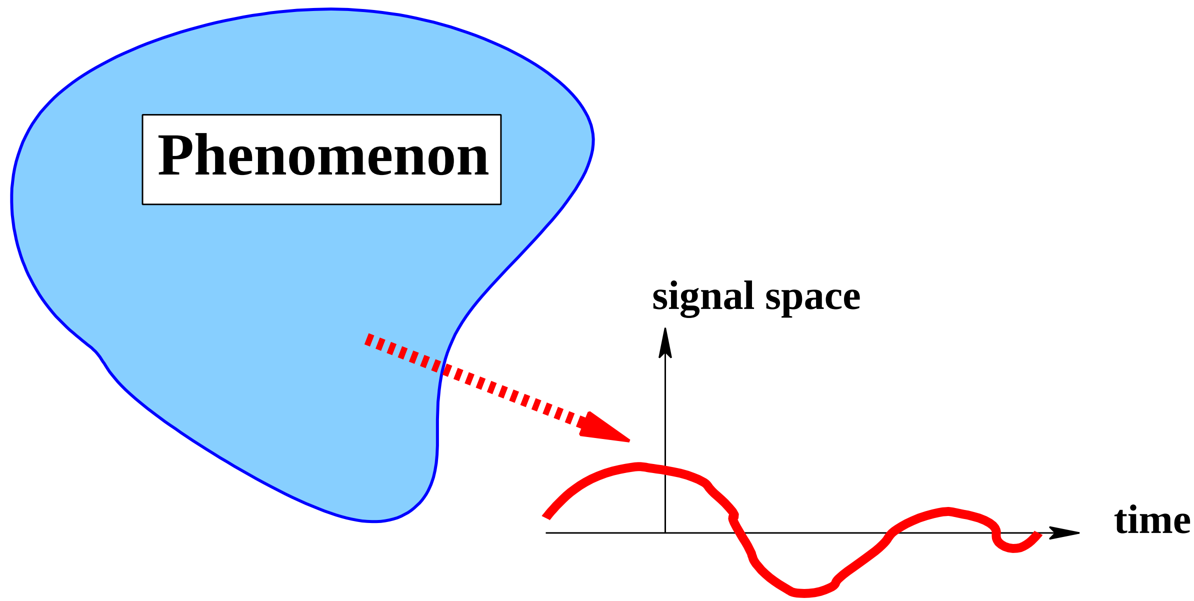
Karl Popper (1902-1994)

Replace 'scientific theory' by 'mathematical model'.

Dynamical systems

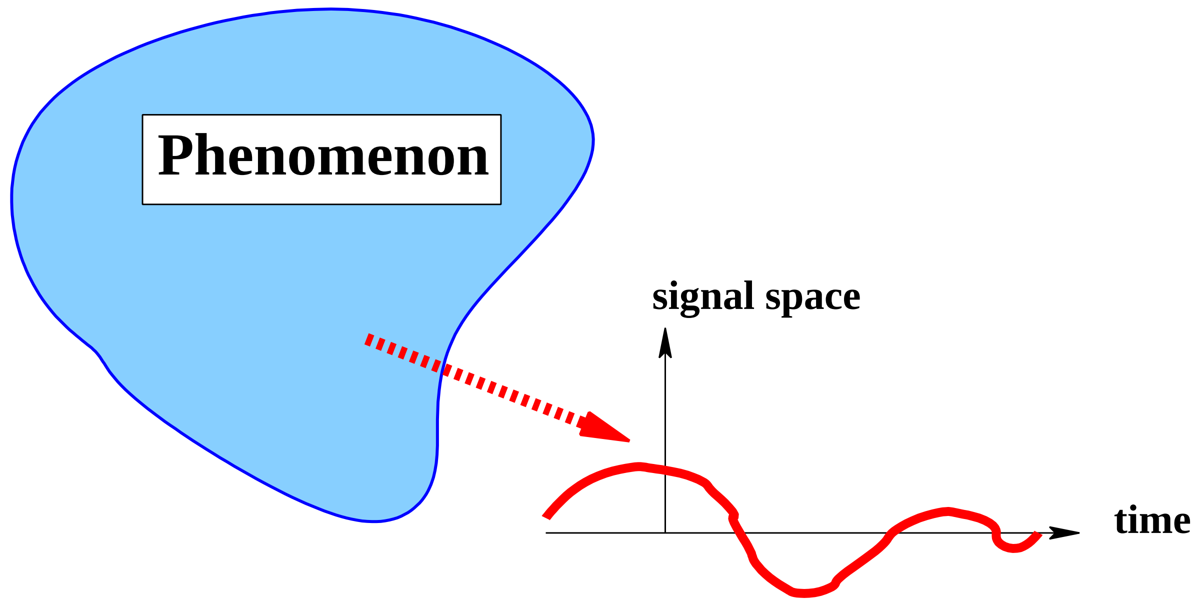
The dynamic behavior

In dynamical systems, the ‘events’ are maps, with the time-axis as domain. The events are functions of time.



The dynamic behavior

In dynamical systems, the 'events' are maps, with the time-axis as domain. The events are functions of time.



It is convenient to distinguish, in the notation,

the domain of the event maps, the **time set,**

and the codomain, the **signal space,**

that is, the set where the functions take on their values.

The dynamic behavior

Definition: A dynamical system $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with**

- $\mathbb{T} \subseteq \mathbb{R}$ **the time set,**
- $\mathbb{W}$ **the signal space,**
- $\mathscr{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ **the behavior,**
  that is, $\mathscr{B}$ **is a family of maps from** $\mathbb{T}$ **to** $\mathbb{W}$**.**

$w : \mathbb{T} \to \mathbb{W} \in \mathscr{B}$ **means: the model allows the trajectory** w**,**

$w : \mathbb{T} \to \mathbb{W} \notin \mathscr{B}$ **means: the model forbids the trajectory** w**.**

The dynamic behavior

Definition: A dynamical system $:\Leftrightarrow$ $(\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with**

- $\mathbb{T} \subseteq \mathbb{R}$ **the time set,**
- $\mathbb{W}$ **the signal space,**
- $\mathscr{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ **the behavior,**
  that is, $\mathscr{B}$ **is a family of maps from** $\mathbb{T}$ **to** $\mathbb{W}$**.**

$w : \mathbb{T} \to \mathbb{W} \in \mathscr{B}$ **means: the model allows the trajectory** w**,**

$w : \mathbb{T} \to \mathbb{W} \notin \mathscr{B}$ **means: the model forbids the trajectory** w**.**

Mostly, $\mathbb{T} = \mathbb{R}, \mathbb{R}_+ := [0, \infty), \mathbb{Z}$**, or** $\mathbb{N} := \{0, 1, 2, \ldots\}$**,**
$\mathbb{W} =$ **(a subset of)** $\mathbb{R}^{\mathtt{w}}$**, for some** $\mathtt{w} \in \mathbb{N}$**,**
$\mathscr{B}$ **is then a family of trajectories taking values in a finite-dimensional real vector space.**

$\mathbb{T} = \mathbb{R}$ **or** $\mathbb{R}_+ \rightsquigarrow$ **'continuous-time' systems,**
$\mathbb{T} = \mathbb{Z}$ **or** $\mathbb{N} \rightsquigarrow$ **'discrete-time' systems.**

Dynamical systems described by differential equations

Consider the ODE

$$f\left(w, \frac{d}{dt}w, \frac{d^2}{dt^2}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w\right) = 0, \qquad (*)$$

with

$$f : \mathbb{W} \times \underbrace{\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{w}} \times \cdots \times \mathbb{R}^{\mathtt{w}}}_{\mathtt{n}\ \textbf{times}} \to \mathbb{R}^{\bullet}, \qquad \mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}.$$

Some may prefer to write

$$f \circ \left(w, \frac{d}{dt}w, \frac{d^2}{dt^2}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w\right) = 0,$$

instead of $(*)$, but we leave the $\circ$ notation to puritans.

Dynamical systems described by differential equations

Consider the ODE

$$f\left(w, \frac{d}{dt}w, \frac{d^2}{dt^2}w, \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w\right) = 0, \qquad (*)$$

with

$$f : \mathbb{W} \times \underbrace{\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{\mathtt{w}} \times \cdots \times \mathbb{R}^{\mathtt{w}}}_{\mathtt{n}\ \textbf{times}} \to \mathbb{R}^{\bullet}, \qquad \mathbb{W} \subseteq \mathbb{R}^{\mathtt{w}}.$$

This ODE defines the dynamical system $(\mathbb{R}, \mathbb{W}, \mathscr{B})$**, with**

$$\mathscr{B} = \{w : \mathbb{R} \to \mathbb{W}, \textbf{sufficiently smooth} \mid$$
$$f\left(w(t), \frac{d}{dt}w(t), \frac{d^2}{dt^2}w(t), \ldots, \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w(t)\right) = 0 \qquad \forall\, t \in \mathbb{R}\}.$$

'Sufficiently smooth': for example $\mathscr{C}^{\infty}(\mathbb{R}, \mathbb{W})$**,**
but other solution concepts may be appropriate ...

Examples

- **Inductor:** $\mathbb{W} = \mathbb{R}^2$, $f : (V, I, \frac{d}{dt}V, \frac{d}{dt}I) \mapsto V - L\frac{d}{dt}I$.
- **Capacitor:** $\mathbb{W} = \mathbb{R}^2$, $f : (V, I, \frac{d}{dt}V, \frac{d}{dt}I) \mapsto C\frac{d}{dt}V - I$.
- **Newton's second law:**

$$\mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3,$$

$$f : (\vec{F}, \vec{q}, \frac{d}{dt}\vec{F}, \frac{d}{dt}\vec{q}, \frac{d^2}{dt^2}\vec{F}, \frac{d^2}{dt^2}\vec{q}) \mapsto \vec{F} - M\frac{d^2}{dt^2}\vec{q}.$$

Properties of dynamical systems

Linearity and time-invariance

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ **is said to be**

linear $:\Leftrightarrow$

$\mathbb{W}$ **is a vector space (over the field** $\mathbb{F}$**) and**

$[\![w_1, w_2 \in \mathscr{B}$ **and** $\alpha \in \mathbb{F}]\!] \Rightarrow [\![w_1 + \alpha w_2 \in \mathscr{B}]\!]$**.**

Linearity $\Leftrightarrow$ **the 'superposition principle' holds.**

Linearity and time-invariance

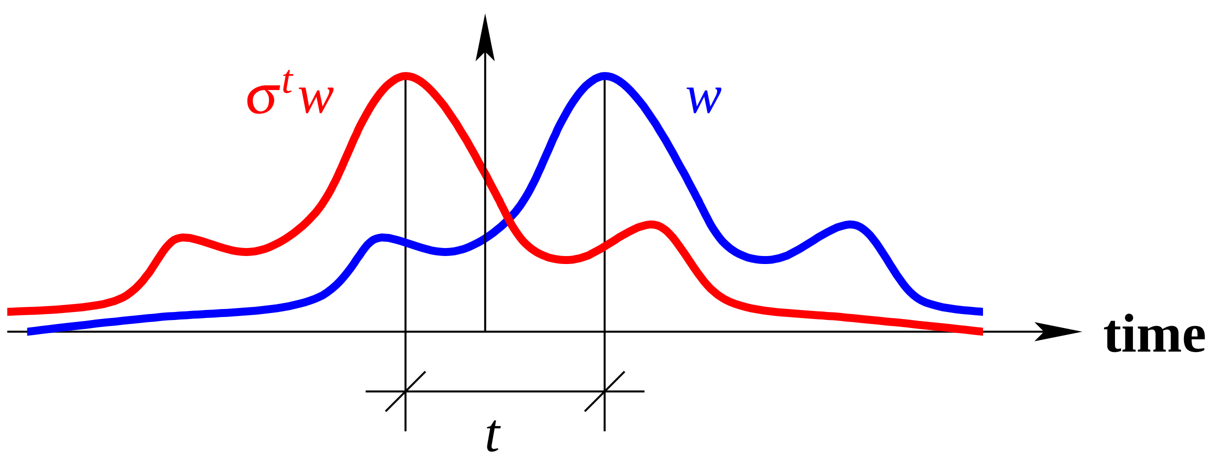
The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ is said to be

time-invariant $:\Leftrightarrow \mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z},$ **or** $\mathbb{N}$**, and**

$$[\![w \in \mathscr{B} \text{ and } t \in \mathbb{T}]\!] \Rightarrow [\![\sigma^t w \in \mathscr{B}]\!].$$

σ^t **denotes the backwards t-shift, defined as**

$$\sigma^t w : \mathbb{T} \to \mathbb{W}, \qquad \sigma^t w(t') := w(t' + t).$$



Shift-invariance $\Leftrightarrow$ shifts of 'legal' trajectories are 'legal'.

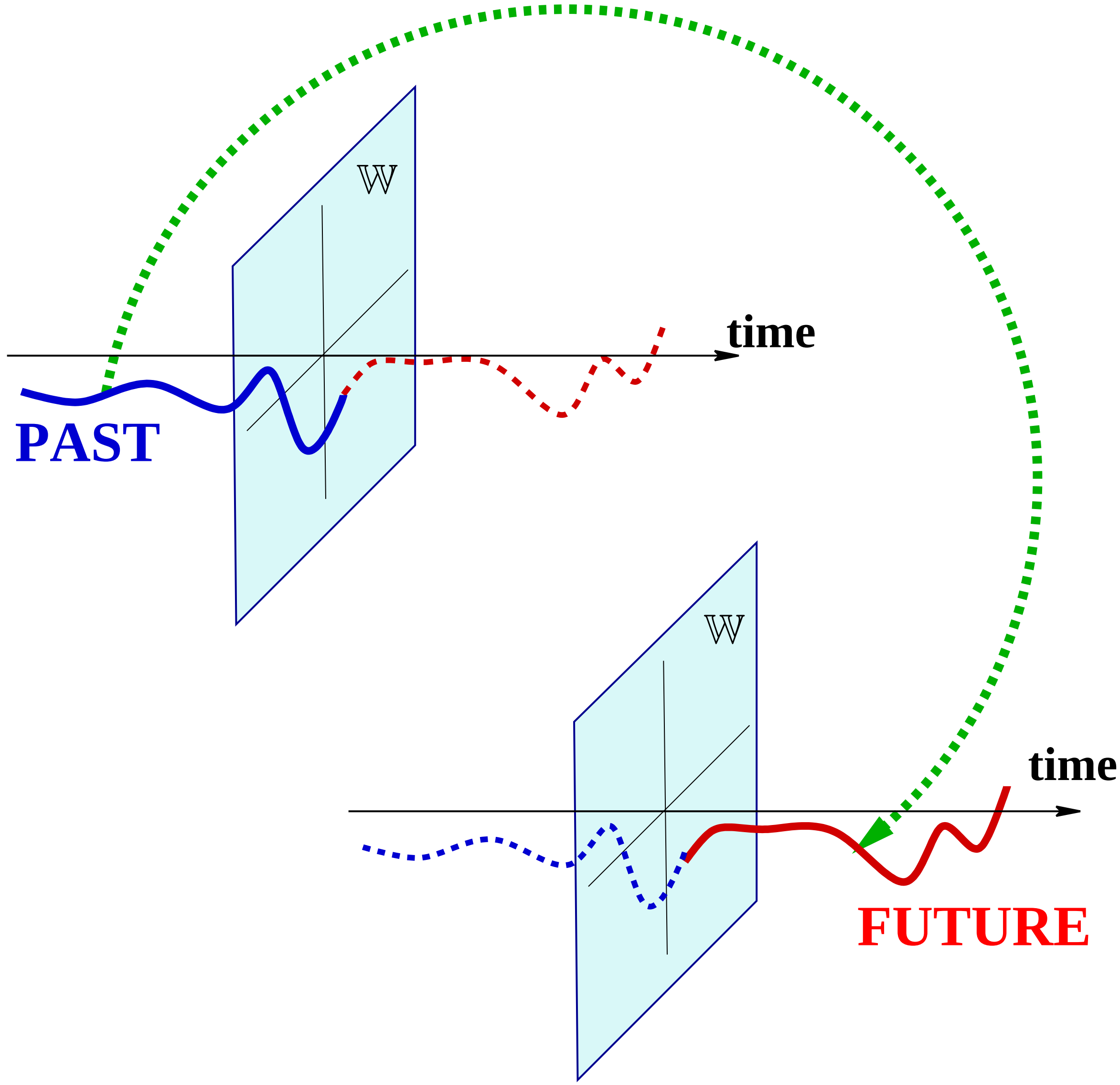
Autonomous systems

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with** $\mathbb{T} = \mathbb{R}$ **or** $\mathbb{Z}$**, is said to be**

autonomous $:\Leftrightarrow$

$$[\![w_1, w_2 \in \mathscr{B}, \textbf{ and } w_1(t) = w_2(t) \textbf{ for } t < 0]\!] \Rightarrow [\![w_1 = w_2]\!].$$

Autonomous in a picture



autonomous :⇔ the past implies the future.

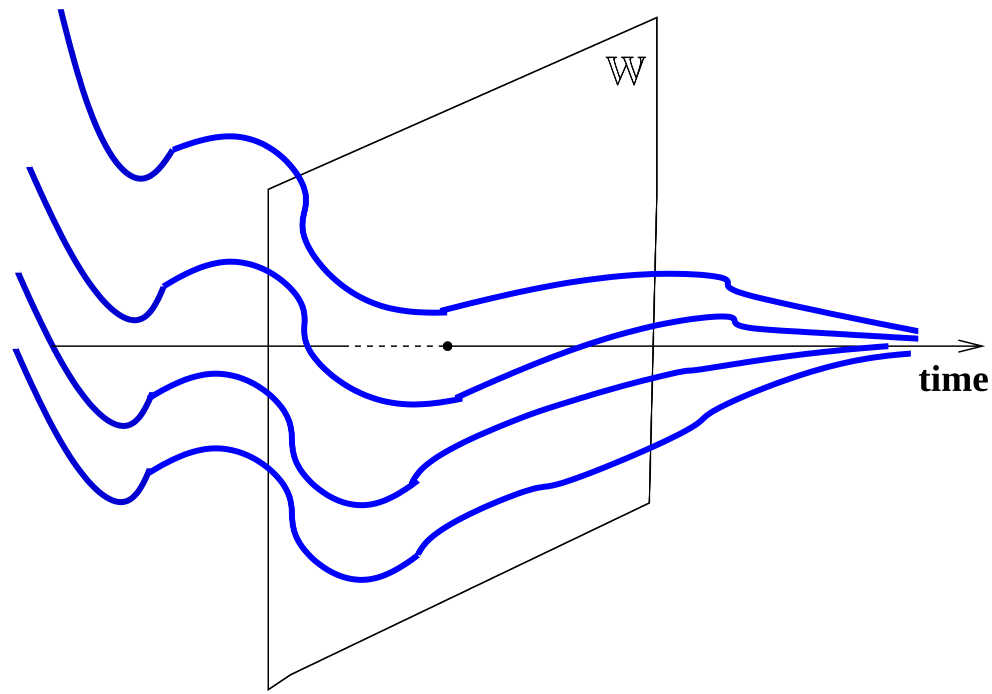
Stability

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with** $\mathbb{T} = \mathbb{R}, [0, \infty), \mathbb{Z},$ **or** $\mathbb{N}$**, and** $\mathbb{W}$ **a normed vector space (for simplicity), is said to be stable** $:\Leftrightarrow [\![w \in \mathscr{B}]\!] \Rightarrow [\![w(t) \to 0$ **for** $t \to \infty]\!]$.

Stability

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$**, with** $\mathbb{T} = \mathbb{R}, [0, \infty),$ $\mathbb{Z},$ **or** $\mathbb{N}$**, and** $\mathbb{W}$ **a normed vector space (for simplicity), is said to be stable** $:\Leftrightarrow [\![w \in \mathscr{B}]\!] \Rightarrow [\![w(t) \to 0$ **for** $t \to \infty]\!]$.

In a picture



stability $:\Leftrightarrow$ **all trajectories go to** 0**.**

Sometimes this is referred to as ‘asymptotic stability’.

Controllability

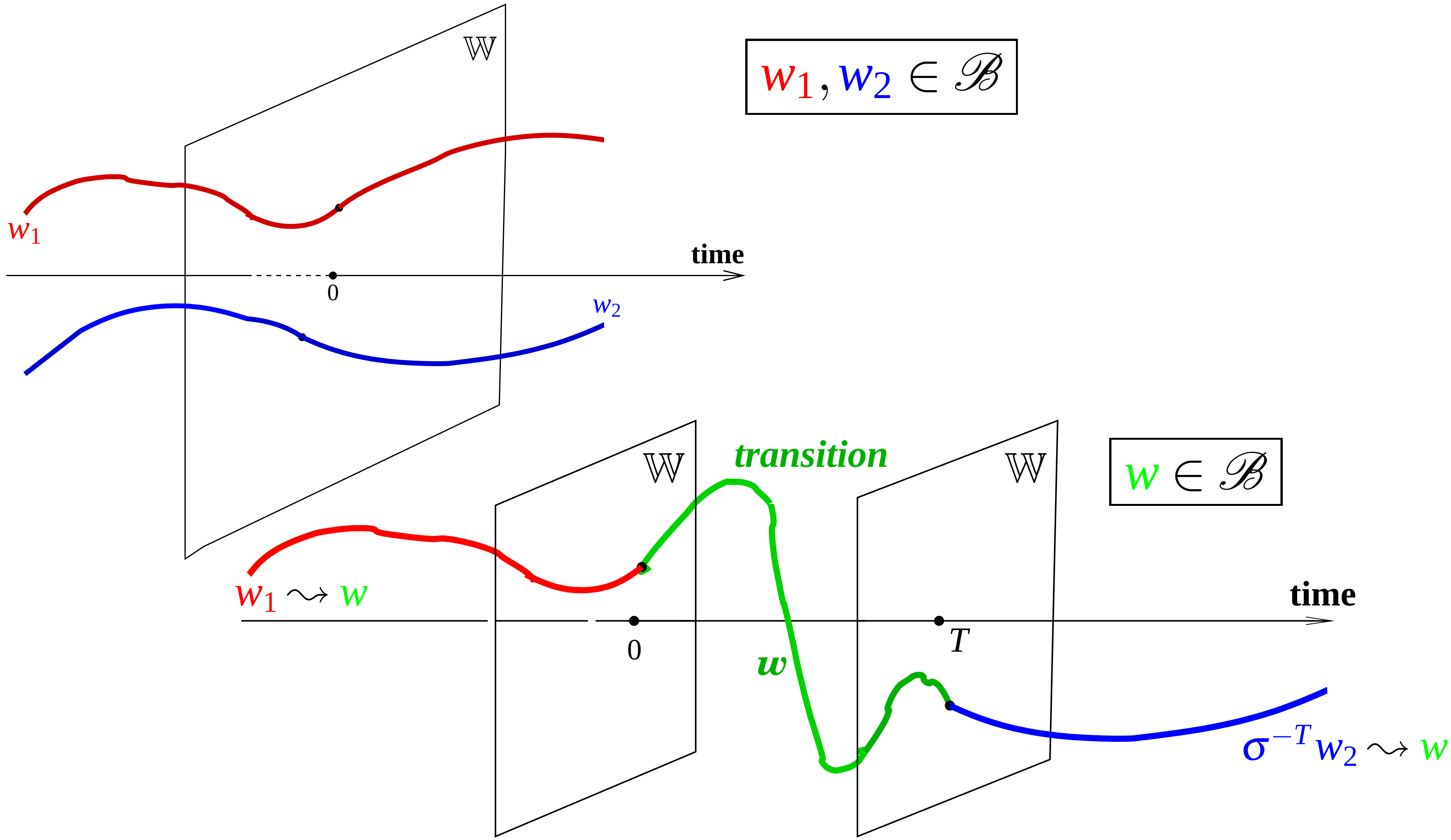
The time-invariant (to avoid irrelevant complications) dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, is said to be

controllable $:\Leftrightarrow$

for all $w_1, w_2 \in \mathscr{B}$, there exist $T \in \mathbb{T}, T \geq 0$, and $w \in \mathscr{B}$, such that

$$w(t) = \begin{cases} w_1(t) & \textbf{for } t < 0; \\ w_2(t-T) & \textbf{for } t \geq T. \end{cases}$$

Controllability in a picture



controllability :⇔ concatenability of trajectories after a delay

Stabilizability

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$, with $\mathbb{T} = \mathbb{R}$ or $\mathbb{Z}$, and $\mathbb{W}$ a normed vector space (for simplicity), is said to be

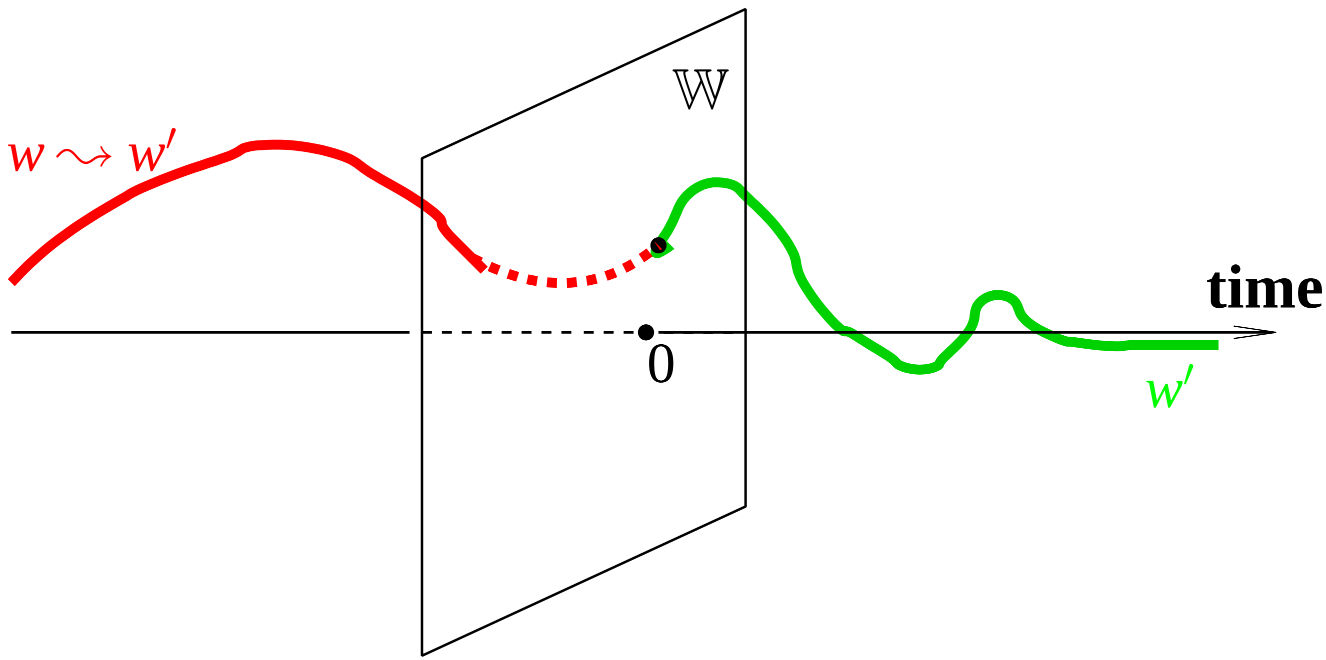
stabilizable $:\Leftrightarrow$ **for all** $w \in \mathscr{B}$**, there exist** $w' \in \mathscr{B}$**, such that**

$$w'(t) = w(t) \quad \textbf{for } t < 0,$$

and

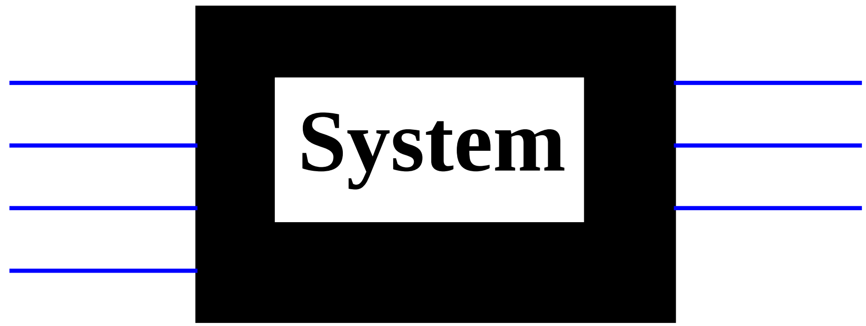
$$w'(t) \to 0 \quad \textbf{for } t \to \infty.$$

Stabilizability in a picture



stabilizability :⇔ all trajectories can be steered to 0.

Observability



Consider the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathscr{B})$**.**

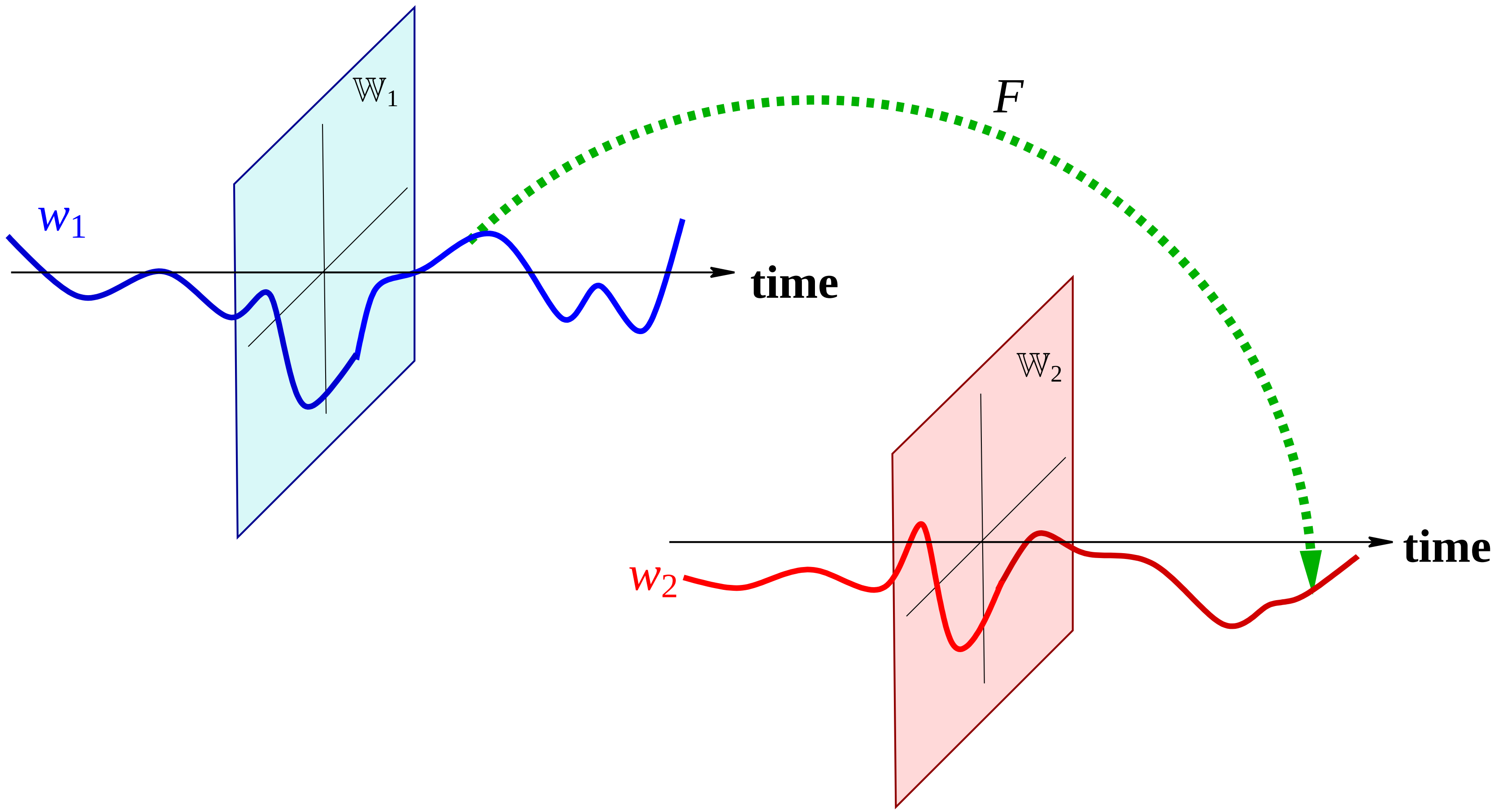
w_2 **is said to be observable from** w_1 **in** $\Sigma :\Leftrightarrow$

$$[\![(w_1, w_2), (w_1', w_2') \in \mathscr{B} \text{ and } w_1 = w_1']\!] \Rightarrow [\![w_2 = w_2']\!].$$

observability $:\Leftrightarrow$ w_2 **may be deduced from** w_1**.**

!!! Knowing the laws of the system !!!

Observability in a picture



Equivalently, there exists a map $F : \mathbb{W}_1^{\mathbb{T}} \to \mathbb{W}_2^{\mathbb{T}}$, **such that**

$$[\![(w_1, w_2) \in \mathscr{B}]\!] \Rightarrow [\![w_2 = F(w_1)]\!].$$

Detectability

Consider the dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathscr{B})$, with $\mathbb{T} = \mathbb{R}, \mathbb{R}_+, \mathbb{Z}$, or $\mathbb{N}$, and $\mathbb{W}$ a normed vector space (for simplicity).

w_2 **is said to be detectable from** w_1 **in** Σ $:\Leftrightarrow$

$$[\![(w_1, w_2), (w_1', w_2') \in \mathscr{B} \text{ and } w_1 = w_1']\!] \Rightarrow [\![w_2(t) - w_2'(t) \to 0 \quad \text{for } t \to \infty]\!].$$

Detectability $:\Leftrightarrow$ w_2 **can be asymptotically deduced from** w_1**.**

Examples

- **All these properties will be discussed in detail for linear time-invariant differential systems.**
- **Resistors, inductors, capacitors, Newton's second law: linear.**
- **All the examples given are time-invariant.**
- **Newton's second law: controllable, hence stabilizable, not stable, $\vec{F}$ observable from $\vec{q}$, $\vec{q}$ not observable and not detectable from $\vec{F}$.**
- **Kepler's laws define an autonomous system. So does**

$$\frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}}w = f\left(w, \frac{d}{dt}w, \ldots, \frac{d^{\mathtt{n}-1}}{dt^{\mathtt{n}-1}}w\right).$$

In particular, $\frac{d}{dt}x = f(x)$, **and** $x(t+1) = f(x(t))$.

Representations of behaviors

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as the set of solutions of equations,**
- **as the image of a map,**
- **as a projection.**

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as the set of solutions of equations:**

$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid f(w) = 0\},$$

- **as the image of a map:**

$$f : \bullet \to \mathscr{U}, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \text{ such that } w = f(\ell)\},$$

- **as a projection:**

$$\mathscr{B}_{\text{extended}} \subseteq \mathscr{U} \times \mathscr{L},$$

$$\mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\}.$$

Kernels, images, and projections

A model $\mathscr{B}$ is a subset of $\mathscr{U}$.
There are many ways to specify a subset. For example,

- **as solutions of equations:** **kernel representation**

$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid f(w) = 0\},$$

- **as the image of a map:** **image representation**

$$f : \bullet \to \mathscr{U}, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \text{ such that } w = f(\ell)\},$$

- **as a projection:** **latent variable representation**

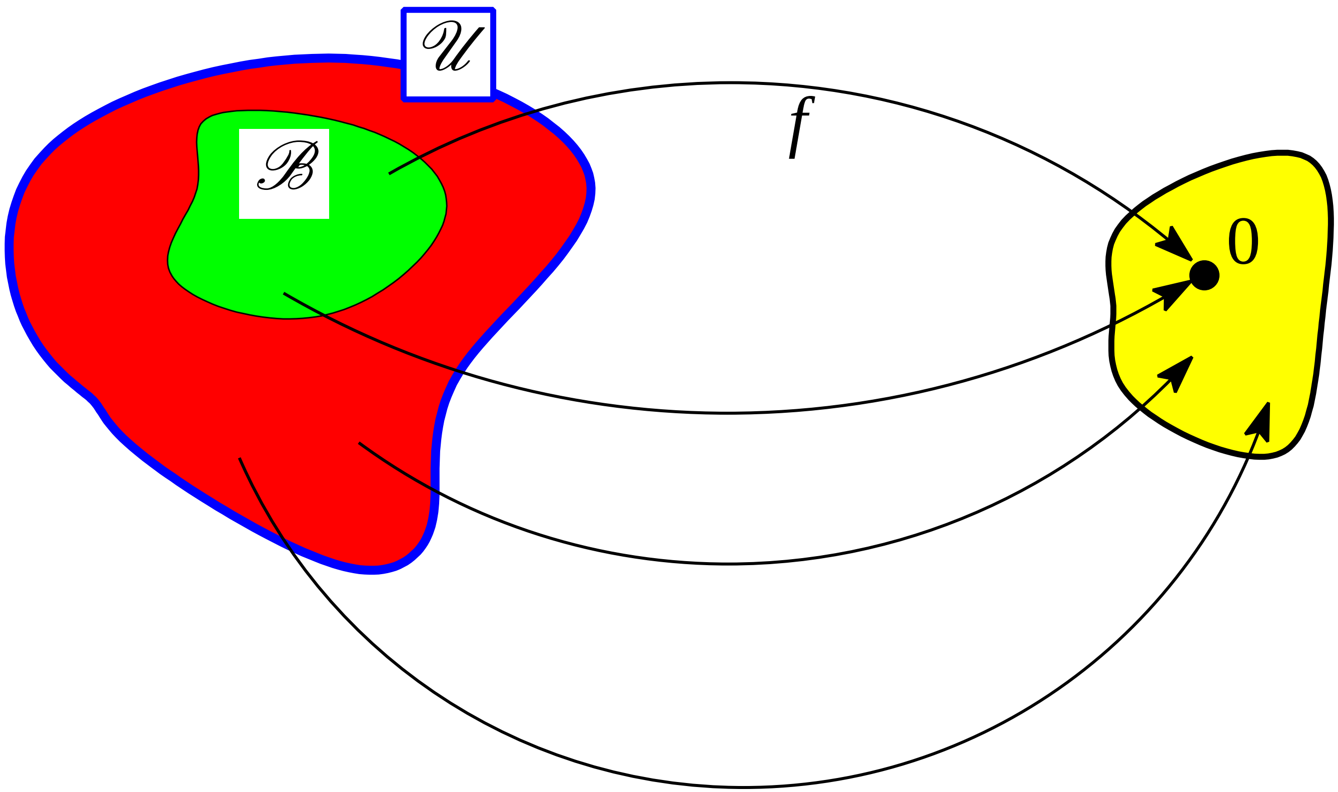
$$\mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \text{ such that } (w, \ell) \in \mathscr{B}_{\text{extended}}\},$$

w's 'manifest' variables: the variables the model aims at,
ℓ's 'latent' variables: auxiliary variables.

Kernel

- **as solutions of equations:** **kernel representation**

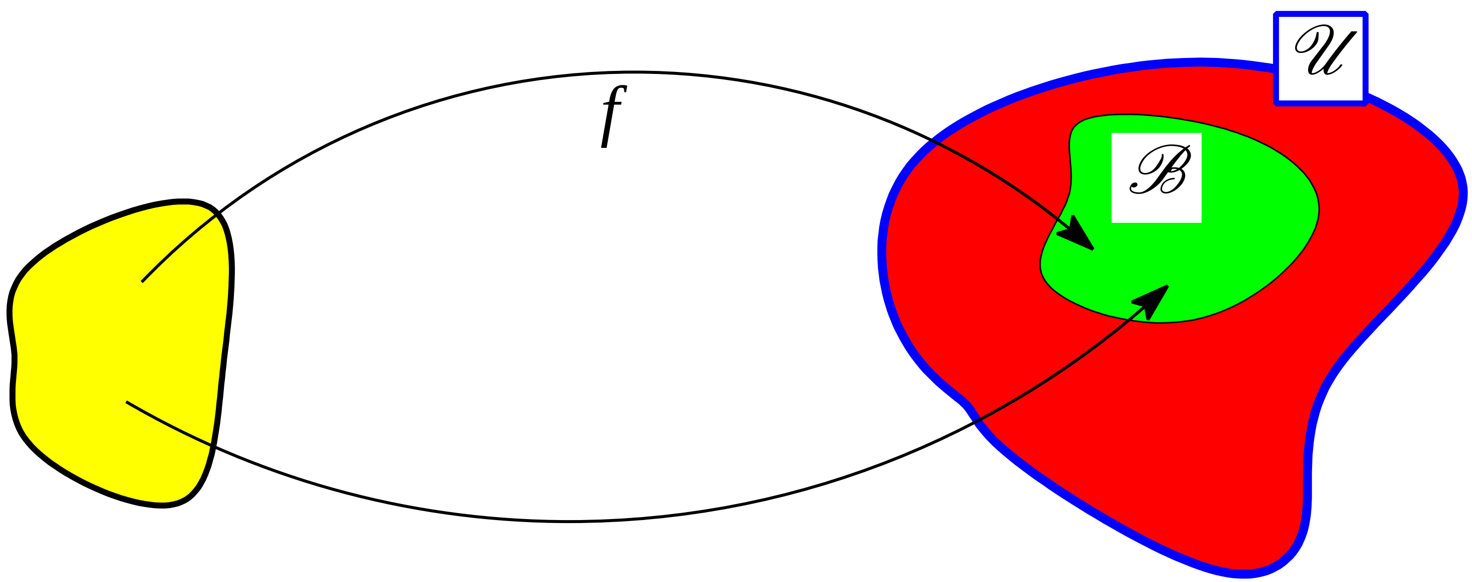
$$f : \mathscr{U} \to \bullet, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid f(w) = 0\}.$$



Image

- **as the image of a map:** **image representation**

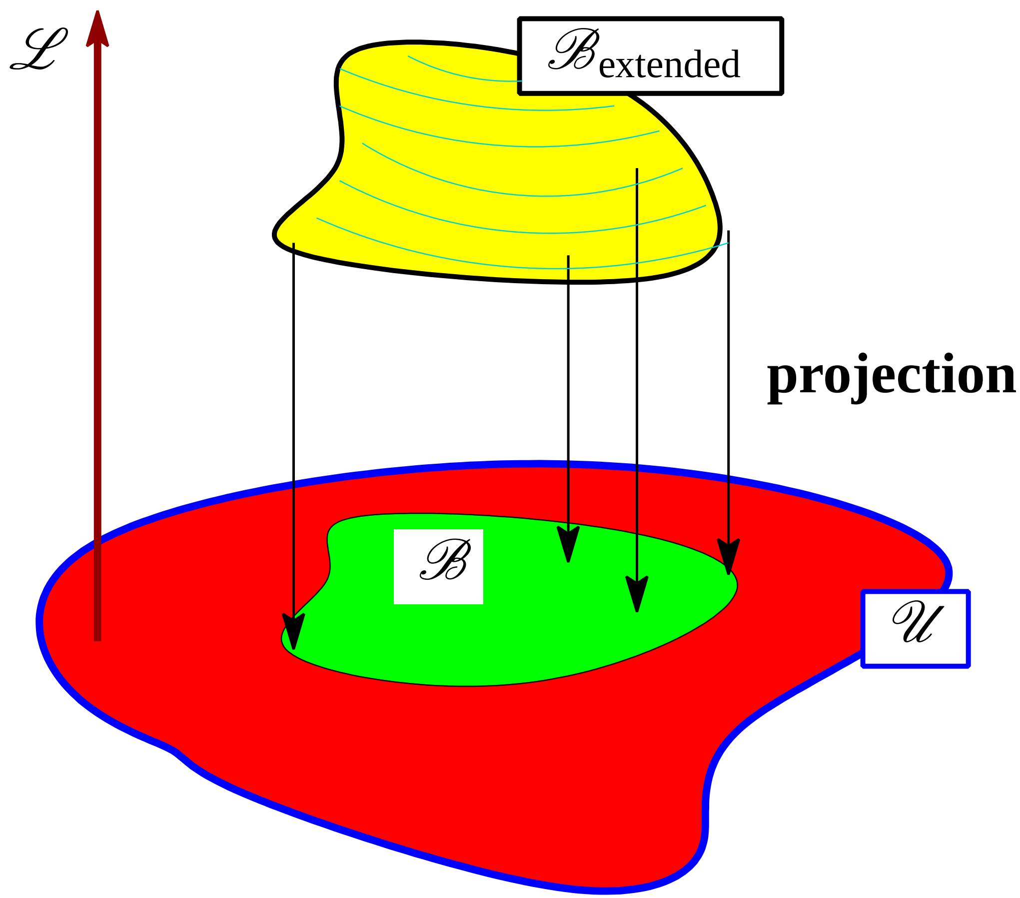
$$f : \bullet \rightarrow \mathscr{U}, \qquad \mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \textbf{ such that } w = f(\ell)\}.$$



Projection

- **as a projection:** **latent variable representation**

$$\mathscr{B} = \{w \in \mathscr{U} \mid \exists\, \ell \in \mathscr{L} \text{ such that } (w,\ell) \in \mathscr{B}_{\text{extended}}\},$$



Latent variable representations

Combining equations with latent variables $\rightsquigarrow$

$$\mathscr{B}_{\text{extended}} = \{(w,\ell) \mid f(w,\ell)=0\},$$

$$\mathscr{B} = \{w \mid \exists\, \ell \textbf{ such that } f(w,\ell)=0\}.$$

w's 'manifest' variables: the variables the model aims at,
ℓ's 'latent' variables: auxiliary variables.

First principles models usually contain latent variables.
See Lecture III.
Latent variables naturally emerge from interconnections.
See Lecture IV.

Latent variable representations

Combining equations with latent variables $\rightsquigarrow$

$$\mathscr{B}_{\text{extended}} = \{(w,\ell) \mid f(w,\ell) = 0\},$$

$$\mathscr{B} = \{w \mid \exists\, \ell \text{ such that } f(w,\ell) = 0\}.$$

***w*'s 'manifest' variables: the variables the model aims at,**
ℓ's 'latent' variables: auxiliary variables.

First principles models usually contain latent variables.

See Lecture III.

Latent variables naturally emerge from interconnections.

See Lecture IV.

FAQ: Does $\mathscr{B}$ inherit the structure of $\mathscr{B}_{\text{extended}}$?

State models

State equations

We now discuss how state models fit in.

$$\frac{d}{dt}x = f(x,u), \quad y = h(x,u), \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \qquad (\spadesuit)$$

with $u : \mathbb{R} \to \mathbb{U}$ **the input, $y : \mathbb{R} \to \mathbb{Y}$ the output, and** $x : \mathbb{R} \to \mathbb{X}$ **the state.**

In particular, the linear case, these systems are parametrized by the 4 matrices $(A,B,C,D) \leadsto$

$$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ y \end{bmatrix},$$

with $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}, B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}, C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}, D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$**.**
These models have dominated linear system theory since the 1960's.

State equations

We now discuss how state models fit in.

$$\frac{d}{dt}x = f(x,u), \;\; y = h(x,u), \;\; w = \begin{bmatrix} u \\ y \end{bmatrix}, \qquad (\spadesuit)$$

with $u : \mathbb{R} \to \mathbb{U}$ **the input**, $y : \mathbb{R} \to \mathbb{Y}$ **the output, and** $x : \mathbb{R} \to \mathbb{X}$ **the state.**

It is common to view state space systems as models to describe the input/output behavior by means of input/state/output equations, with the state as latent variable. Define

$$\mathscr{B}_{\text{extended}} := \{(u,y,x) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \times \mathbb{X} \mid (\spadesuit) \textbf{ holds}\},$$

$$\mathscr{B} := \{(u,y) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \mid \exists\, x : \mathbb{R} \to \mathbb{X} \textbf{ such that } (\spadesuit) \textbf{ holds}\}.$$

State controllability

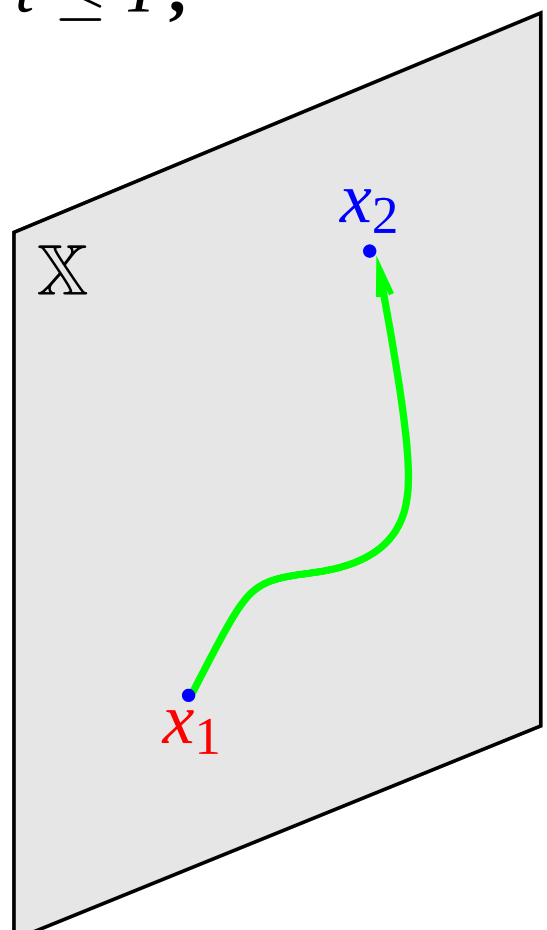
State models propagated under the influence of R.E. Kalman. Especially important in this development were the notions of state controllability and state observability.

Rudolf Kalman (1930–)

State controllability

(♠) is said to be **state controllable**
if for all $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{X}$**, there exists** $T \geq 0$, $u : \mathbb{R} \to \mathbb{U}$**, and** $x : \mathbb{R} \to \mathbb{X}$
such that

1. $\frac{d}{dt}x(t) = f(x(t), u(t))$ **for** $0 \leq t \leq T$**,**
2. $x(0) = \mathrm{x}_1$**,**
3. $x(T) = \mathrm{x}_2$**.**



State controllability

(♠) is said to be state controllable if for all $\mathrm{x}_1, \mathrm{x}_2 \in \mathbb{X}$, there exists $T \geq 0$, $u : \mathbb{R} \to \mathbb{U}$, and $x : \mathbb{R} \to \mathbb{X}$ such that

1. $\frac{d}{dt}x(t) = f(x(t), u(t))$ **for** $0 \leq t \leq T$**,**
2. $x(0) = \mathrm{x}_1$**,**
3. $x(T) = \mathrm{x}_2$**.**

It is easy to prove that

$[\![$state controllability$]\!]$ $\Leftrightarrow$ $[\![$behavioral controllability of $\mathscr{B}_{\text{extended}}$$]\!]$.

$[\![$state controllability$]\!]$ $\Rightarrow$ $[\![$behavioral controllability of $\mathscr{B}$$]\!]$.

Behavioral controllability makes controllability into a genuine, an intrinsic, system property.

State observability

(♠) is said to be state observable if

$$[\![(u, y, x_1), (u, y, x_2) \in \mathscr{B}_{\text{extended}}]\!] \Rightarrow [\![x_1(0) = x_2(0)]\!].$$

State observability

(♠) is said to be state observable if

$$[\![(u,y,x_1),(u,y,x_2) \in \mathscr{B}_{\text{extended}}]\!] \Rightarrow [\![x_1(0) = x_2(0)]\!].$$

It is easy to prove that
$[\![$state observability$]\!]$ $\Leftrightarrow$ $[\![$behavioral observability of $\mathscr{B}_{\text{extended}}$$]\!]$, with (u,y) as 'observed' variables, and x as 'to-be-deduced' variable.

State observability

(♠) is said to be state observable if

$$[\![(u,y,x_1),(u,y,x_2)\in\mathscr{B}_{\text{extended}}]\!] \Rightarrow [\![x_1(0)=x_2(0)]\!].$$

It is easy to prove that
$[\![$state observability$]\!] \Leftrightarrow [\![$behavioral observability of $\mathscr{B}_{\text{extended}}]\!]$, with (u,y) as 'observed' variables, and x as 'to-be-deduced' variable.

Behavioral controllability and observability are meaningful generalizations of state controllability and observability.

Why should we be so interested in the state?

LTIDSs

LTIDSs

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is said to be a**

linear time-invariant differential system (LTIDS) $:\Leftrightarrow$

the behavior $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}}$ **consists of the set of solutions of a system of linear constant-coefficient ODEs**

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $R_0, R_1, \ldots, R_{\mathtt{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$ **real matrices that parametrize the system, and** $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$**.**

LTIDSs

The dynamical system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ is said to be a linear time-invariant differential system (LTIDS) $:\Leftrightarrow$

the behavior $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}}$ consists of the set of solutions of a system of linear constant-coefficient ODEs

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

with $R_0, R_1, \ldots, R_{\mathtt{n}} \in \mathbb{R}^{\bullet \times \mathtt{w}}$ real matrices that parametrize the system, and $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$. In polynomial matrix notation

$$R\left(\tfrac{d}{dt}\right) w = 0,$$

with $R(\xi) = R_0 + R_1 \xi + \cdots + R_{\mathtt{n}} \xi^{\mathtt{n}} \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}$.

Examples

- $M\frac{d^2}{dt^2}\vec{q} = \vec{F}, w = \begin{bmatrix} \vec{F} \\ \vec{q} \end{bmatrix}, \quad \rightsquigarrow R(\xi) = \begin{bmatrix} I_{3\times 3} & \vdots & -I_{3\times 3}\xi^2 \end{bmatrix}.$

Examples

- $M\frac{d^2}{dt^2}\vec{q} = \vec{F}, w = \begin{bmatrix} \vec{F} \\ \vec{q} \end{bmatrix}, \quad \rightsquigarrow R(\xi) = \begin{bmatrix} I_{3\times 3} & \vdots & -I_{3\times 3}\xi^2 \end{bmatrix}.$

- $\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ x \\ y \end{bmatrix},$

  $u : \mathbb{R} \to \mathbb{R}^{\mathtt{n}}, y : \mathbb{R} \to \mathbb{R}^{\mathtt{m}}, x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}}.$

$$\rightsquigarrow R(\xi) = \begin{bmatrix} A - I_{\mathtt{n}\times\mathtt{n}}\xi & B & 0 \\ C & D & -I_{\mathtt{p}\times\mathtt{p}} \end{bmatrix}.$$

Examples

- $M\frac{d^2}{dt^2}\vec{q} = \vec{F}, w = \begin{bmatrix} \vec{F} \\ \vec{q} \end{bmatrix}, \quad \rightsquigarrow R(\xi) = \begin{bmatrix} I_{3\times 3} & \vdots & -I_{3\times 3}\xi^2 \end{bmatrix}.$

- $\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du, \quad w = \begin{bmatrix} u \\ x \\ y \end{bmatrix},$
  $u : \mathbb{R} \to \mathbb{R}^{\mathtt{n}}, y : \mathbb{R} \to \mathbb{R}^{\mathtt{m}}, x : \mathbb{R} \to \mathbb{R}^{\mathtt{n}}.$

$$\rightsquigarrow R(\xi) = \begin{bmatrix} A - I_{\mathtt{n}\times\mathtt{n}}\xi & B & 0 \\ C & D & -I_{\mathtt{p}\times\mathtt{p}} \end{bmatrix}.$$

- $p_0, p_1, \ldots, p_{\mathtt{n}} \in \mathbb{R}, w : \mathbb{R} \to \mathbb{R}$

$$p_0 w + p_1 \frac{d}{dt} w + \cdots + p_{\mathtt{n}} \frac{d^{\mathtt{n}}}{dt^{\mathtt{n}}} w = 0,$$

$\rightsquigarrow R = p,$ **with** $p(\xi) = p_0 + p_1\xi + \cdots + p_{\mathtt{n}}\xi^{\mathtt{n}}.$

The solution set

We should define what we take to be the solution set.
For ease of exposition, we take $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ solutions. Hence

$$\mathscr{B} = \{w \in \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right) w = 0\}.$$

The solution set

We should define what we take to be the solution set.
For ease of exposition, we take $\mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ solutions. Hence

$$\mathscr{B} = \{w \in \mathscr{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R\left(\frac{d}{dt}\right)w = 0\}.$$

There are other possibilities.

- **Strong solutions: w as many times differentiable as derivatives appear in the ODE, and $R\left(\frac{d}{dt}\right)w = 0$.**
  Has very few 'invariance' properties.
- **Weak solutions: $w \in \mathscr{L}^{\text{local}}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$, with $R\left(\frac{d}{dt}\right)w = 0$ in the sense of distributions. Includes steps, ramps, etc.**
- **Distributional solutions: w is a distribution, and $R\left(\frac{d}{dt}\right)w = 0$ as a distribution.**
  Includes also impulses, doublets, and such frivolities.

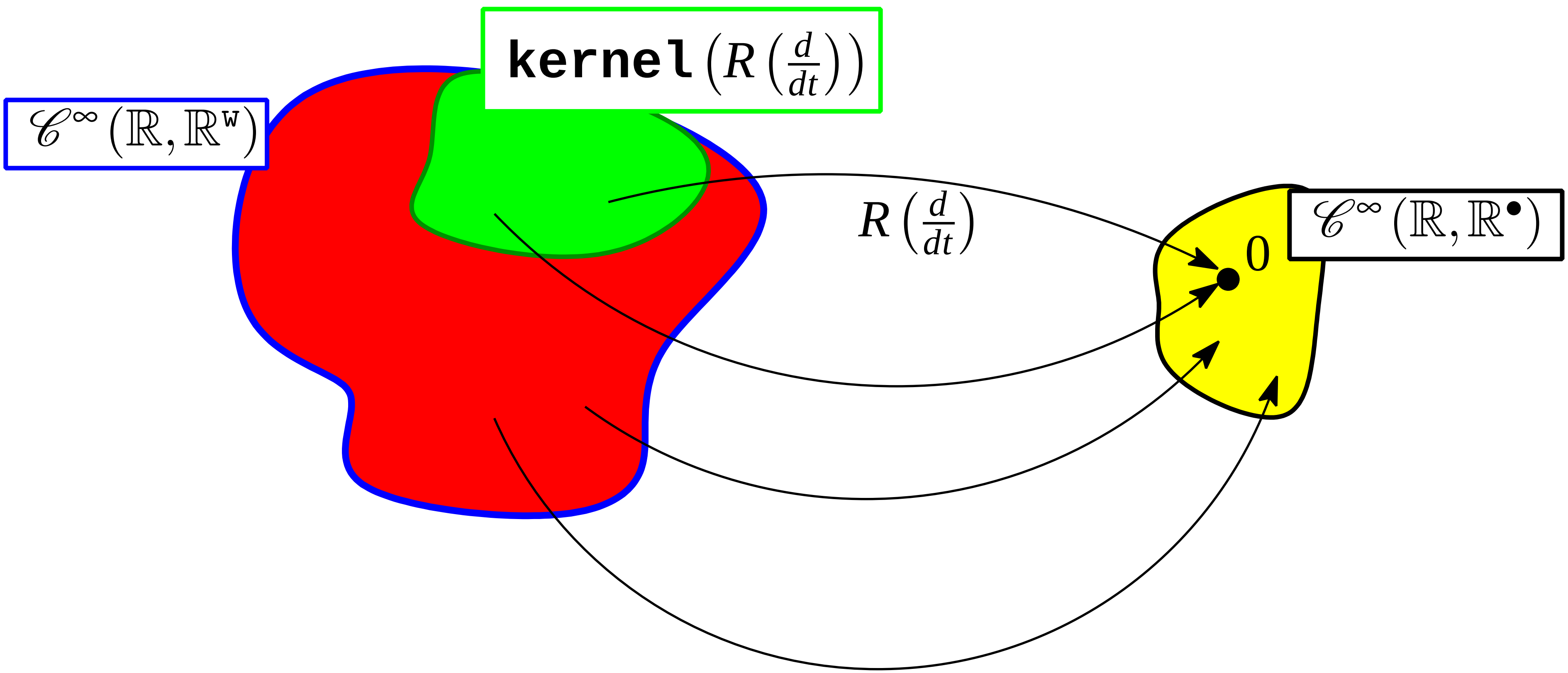
Weak and distributional: very sensible alternatives to $\mathscr{C}^\infty$!

Notation

$\mathscr{B} = \texttt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$,

$R\left(\frac{d}{dt}\right) w = 0$: **a kernel representation of** $\mathscr{B}$.

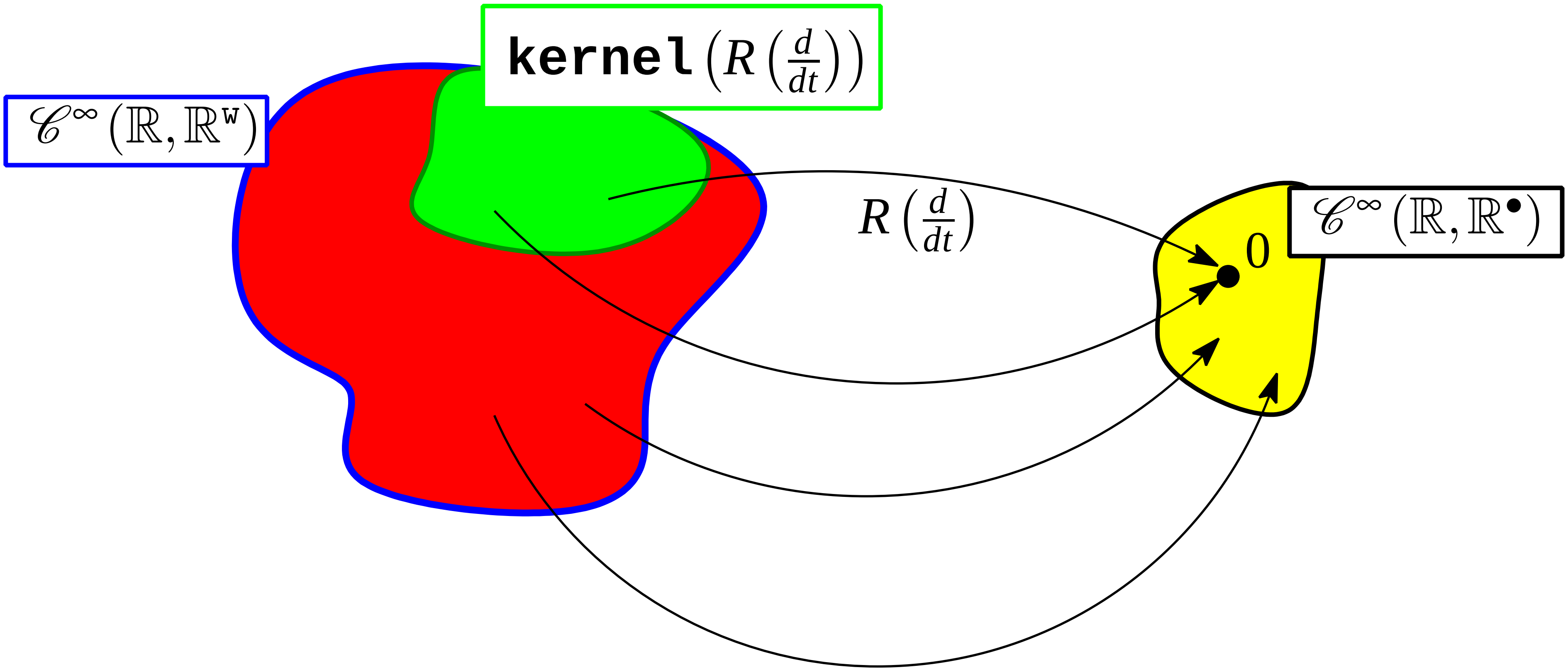
We will meet other representations later.



Notation

$\mathscr{B} = \texttt{kernel}\left(R\left(\frac{d}{dt}\right)\right)$,

$R\left(\frac{d}{dt}\right) w = 0$: **a kernel representation of** $\mathscr{B}$**.**



$\mathscr{L}^{\mathtt{w}}$: **the LTIDSs with** $\mathtt{w}$ **variables,** $\mathscr{B} \in \mathscr{L}^{\mathtt{w}}$,

$\mathscr{L}^{\bullet}$: **the LTIDSs,** $\mathscr{B} \in \mathscr{L}^{\bullet}$.

Other sets of independent variables

Discrete-time systems

We have concentrated on continuous-time dynamical systems with time set $\mathbb{T} = \mathbb{R}$. Notions like controllability and stabilizability require appropriate changes for $\mathbb{T} = [0, \infty)$, but the development remains basically the same.

Discrete-time systems with $\mathbb{T} = \mathbb{N}$ are often described by difference equations

$$f(w, \sigma w, \ldots, \sigma^{\mathtt{n}} w) = 0,$$

leading to the behavior

$$\mathscr{B} = \{w : \mathbb{Z} \to \mathbb{W} \mid f(w(t), w(t+1), \ldots, w(t+\mathtt{n})) = 0 \;\forall\, t \in \mathbb{N}\}.$$

In the case $\mathbb{T} = \mathbb{Z}$, it is useful to have negative as well as positive lags, leading to

$$f(\sigma^{\mathtt{n}_-} w, \sigma^{\mathtt{n}_- + 1} w, \ldots, \sigma^{\mathtt{n}_+ - 1} w, \sigma^{\mathtt{n}_+})w = 0.$$

Systems described by constant-coefficient difference equations

The dynamical system $(\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **is said to be a**

linear time-invariant difference system $:\Leftrightarrow$

the behavior $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ **consists of the set of solutions of the system of difference equations**

$$R_{\mathtt{n}_-}\sigma^{\mathtt{n}_-}w + R_{\mathtt{n}_-+1}\sigma^{\mathtt{n}_-+1}w + \cdots + R_{\mathtt{n}_+}\sigma^{\mathtt{n}_+}w = 0,$$

with $R_{\mathtt{n}_-}, R_{\mathtt{n}_-+1}, \ldots, R_{\mathtt{n}_+} \in \mathbb{R}^{\bullet \times \mathtt{w}}$ **real matrices that parametrize the system,** $\mathtt{n}_- \leq \mathtt{n}_+ \in \mathbb{Z}$ **the minimal and maximal** ***lags*****, and** $w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}}$**.**

Systems described by constant-coefficient difference equations

The dynamical system $(\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ is said to be a

linear time-invariant difference system $:\Leftrightarrow$

the behavior $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ consists of the set of solutions of the system of difference equations

$$R_{\mathtt{n}_-}\sigma^{\mathtt{n}_-}w + R_{\mathtt{n}_-+1}\sigma^{\mathtt{n}_-+1}w + \cdots + R_{\mathtt{n}_+}\sigma^{\mathtt{n}_+}w = 0,$$

In polynomial matrix notation

$$R(\sigma, \sigma^{-1})w = 0,$$

$$R(\xi, \xi^{-1}) = R_{\mathtt{n}_-}\xi^{\mathtt{n}_-} + R_{\mathtt{n}_-+1}\xi^{\mathtt{n}_-+1} + \cdots + R_{\mathtt{n}_+}\xi^{\mathtt{n}_+} \in \mathbb{R}[\xi]^{\bullet \times \mathtt{w}}.$$

$$\mathscr{B} = \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \mid R(\sigma, \sigma^{-1})w = 0\}.$$

<u>Example</u>: The moving average system

$$w_1(t) = \frac{1}{2N+1} \sum_{t'=-N}^{N} w_2(t+t')$$

Example: **The moving average system**

$$w_1(t) = \frac{1}{2N+1} \sum_{t'=-N}^{N} w_2(t+t')$$

$\rightsquigarrow w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$**, and**

$$R(\xi, \xi^{-1}) = \left[1 \;\; \vdots \;\; -\frac{1}{2N+1} \left(\xi^{-N} + \cdots + 1 + \cdots + \xi^{N} \right) \right].$$

Completeness

It is undesirable to define system properties in terms of a representation, as we did for LTIDSs.

For the discrete-time case, it is possible to circumvent this disadvantage. There is indeed a very nice characterization of discrete-time LTIDSs purely in terms of the behavior.

Completeness

It is undesirable to define system properties in terms of a representation, as we did for LTIDSs.

For the discrete-time case, it is possible to circumvent this disadvantage. There is indeed a very nice characterization of discrete-time LTIDSs purely in terms of the behavior.

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ **is said to be**

$$[\![\ \textbf{complete}\ ]\!] :\Leftrightarrow [\![[\![w \in \mathscr{B}]\!] \Leftrightarrow [\![w|_{[t_1,t_2]} \in \mathscr{B}|_{[t_1,t_2]} \ \forall\ t_1, t_2 \in \mathbb{T}]\!]]\!].$$

Completeness

It is undesirable to define system properties in terms of a representation, as we did for LTIDSs.

For the discrete-time case, it is possible to circumvent this disadvantage. There is indeed a very nice characterization of discrete-time LTIDSs purely in terms of the behavior.

The dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ **is said to be**

$$[\![\, \textbf{complete} \,]\!] :\Leftrightarrow [\![[\![w \in \mathscr{B}]\!] \Leftrightarrow [\![w|_{[t_1,t_2]} \in \mathscr{B}|_{[t_1,t_2]} \; \forall\, t_1, t_2 \in \mathbb{T}]\!]]\!].$$

<u>Examples</u>: Systems described by differential or difference equations.

<u>Non-examples:</u> $\mathscr{B} = \mathscr{L}_2(\mathbb{R} : \mathbb{R}^{\mathtt{w}}), \ell_2(\mathbb{Z} : \mathbb{R}^{\mathtt{w}}),$ **or behaviors involving compact support conditions.**

An intrinsic definition of LTI difference systems

Theorem

The following conditions are equivalent for $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$.

1. $\exists\, R \in \mathbb{R}[\xi, \xi^{-1}]$ **such that** $\mathscr{B} = \mathtt{kernel}\left(R\left(\sigma, \sigma^{-1}\right)\right)$**,**
2. Σ **is linear, time-invariant, and complete,**
3. $\mathscr{B}$ **is a linear, shift-invariant, closed (in the topology of pointwise convergence) subspace of** $\left(\mathbb{R}^{\mathtt{w}}\right)^{\mathbb{Z}}$**.**

What a 'nice' analogue of this theorem is for differential equations is an open problem.

Distributed systems

The notion of a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ with $\mathbb{T} \subseteq \mathbb{R}$ can be generalized in a very meaningful way by considering a general set of 'independent' variables for $\mathbb{T}$.

Distributed systems

The notion of a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathscr{B})$ with $\mathbb{T} \subseteq \mathbb{R}$ can be generalized in a very meaningful way by considering a general set of 'independent' variables for $\mathbb{T}$.

In particular, it is possible to capture this way distributed parameter systems described by PDEs,

$$f\left(\cdots, \frac{\partial^{\mathtt{k}_1}\partial^{\mathtt{k}_2}\dots\partial^{\mathtt{k}_\mathtt{m}}}{\partial^{\mathtt{k}_1+\mathtt{k}_2+\cdots+\mathtt{k}_\mathtt{m}}}w, \cdots\right) = 0.$$

This leads to a behavior that consists of maps $w : \mathbb{R}^{\mathtt{n}} \to \mathbb{W}$.

For Maxwell's equations, for example, we have $\mathbb{T} = \mathbb{R}^4$, with $\mathscr{B}$ consisting of all maps

$$(\vec{E}, \vec{B}, \vec{j}, \rho) : \mathbb{R}^4 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$$

that satisfy Maxwell's PDEs.

PDEs

The analogue of LTIDSs are systems described by linear constant-coefficient PDEs. These behavioral equations can be written in terms of matrices of polynomials in many variables.

Assume $\mathbb{T} = \mathbb{R}^{\mathtt{n}}, \mathbb{W} = \mathbb{R}^{\mathtt{w}}$**, then** $R \in \mathbb{R}[\xi_1, \xi_2, \ldots, \xi_{\mathtt{n}}]^{\bullet \times \mathtt{w}}$ **leads to the system of PDEs**

$$R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0$$

This defines a system $\Sigma = (\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}, \mathscr{B})$ **with**

$$\mathscr{B} = \{w : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{\mathtt{w}} \mid R\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right) w = 0\}.$$

Example: Maxwell's equations (see Exercise I.6).

Locally specified

Dynamical systems described by ODEs, in particular, LTIDSs describe a behavior $\mathscr{B}$ that is 'locally specified', meaning that

$$[\![w \in \mathscr{B}]\!] \Leftrightarrow [\![w_{[t-\varepsilon,t+\varepsilon]} \in \mathscr{B}_{[t-\varepsilon,t+\varepsilon]} \ \forall\, \varepsilon > 0 \ \textbf{and}\ t \in \mathbb{R}]\!].$$

Thus the 'legality' of a trajectory can be verified by checking if it is 'locally' legal.

A similar property holds for systems described by PDEs.

The analogue property for discrete-time systems is completeness.

Recapitulation

Summary

- **A phenomenon produces ‘events’, ‘outcomes’.**
  $\leadsto$ the universum of events $\mathscr{U}$.
- **A *mathematical model* specifies a subset $\mathscr{B}$ of $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior and specifies which events can occur, according to the model.

Summary

- **A phenomenon produces 'events', 'outcomes'.**
  $\rightsquigarrow$ the universum of events $\mathscr{U}$.
- **A *mathematical model* specifies a subset $\mathscr{B}$ of $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior and specifies which events can occur, according to the model.
- **In dynamical systems, the events are maps from the time set to the signal space.**

Summary

- **A phenomenon produces ‘events’, ‘outcomes’.**
  $\rightsquigarrow$ the universum of events $\mathscr{U}$.
- **A *mathematical model* specifies a subset $\mathscr{B}$ of $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior and specifies which events can occur, according to the model.
- **In dynamical systems, the events are maps from the time set to the signal space.**
- **Controllability, observability, and similar properties can be nicely defined within this setting.**

Summary

- **A phenomenon produces ‘events’, ‘outcomes’.**
  $\rightsquigarrow$ the universum of events $\mathscr{U}$.
- **A *mathematical model* specifies a subset $\mathscr{B}$ of $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior and specifies which events can occur, according to the model.
- **In dynamical systems, the events are maps from the time set to the signal space.**
- **Controllability, observability, and similar properties can be nicely defined within this setting.**
- **LTIDSs are described by linear constant-coefficient differential equations. They can be represented in terms of polynomial matrices.**

Summary

- **A phenomenon produces 'events', 'outcomes'.**
  $\rightsquigarrow$ the universum of events $\mathscr{U}$.
- **A *mathematical model* specifies a subset $\mathscr{B}$ of $\mathscr{U}$.**
  $\mathscr{B}$ is the behavior and specifies which events can occur, according to the model.
- **In dynamical systems, the events are maps from the time set to the signal space.**
- **Controllability, observability, and similar properties can be nicely defined within this setting.**
- **LTIDSs are described by linear constant-coefficient differential equations. They can be represented in terms of polynomial matrices.**
- **Discrete-time LTIDSs admit an elegant characterization purely in terms of the behavior.**

End of Lecture I