DAEs - Control and Numerics

Exercise Sheat 1 - Solution theory

Solutions

Exercise 1 (Regularity)
Both matrix pairs are singular, because $\det(Es - A) \equiv 0$. Basic column and row transformations yield

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right), \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

Exercise 2 (Wong-sequences)
See also [1]. Some calculation gives

$$\mathcal{V}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathcal{W}_1 = \ker E = \operatorname{im} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

and

$$\mathcal{V}_2 = \operatorname{im} V, \quad \text{where } V := \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix}, \qquad \mathcal{W}_2 = \operatorname{im} W, \quad \text{where } W := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}$$

and $\mathcal{V}_{2+i} = \mathcal{V}_i$ and $\mathcal{W}_{2+i} = \mathcal{W}_i$ for all $i \in \mathbb{N}$. Hence

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \quad Q^{-1} = \begin{bmatrix} 4 & 1 & 4 & -1 \\ 0 & 3 & 2 & -2 \\ -4 & -1 & 4 & -1 \\ -2 & -2 & 0 & 3 \end{bmatrix}$$

and

$$PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}, \quad PAQ = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Observation:

$$PEQ = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix},$$

where $N \in \mathbb{R}^{2\times 2}$ is nilpotent. Solutions of $E\dot{x} = Ax + f$ is given by $Q\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right)$ with v, w solutions of

$$\dot{v} = Jv + f_1, \quad N\dot{w} = w + f_2,$$

where $Pf = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Exercise 3 (Drazin inverse)
From $(\lambda E - A)^{-1}(\lambda E - A) = I$ it follows that $\lambda\hat{E} - \hat{A} = I$, hence $\hat{A} = \lambda\hat{E} - I$. Therefore

$$\hat{E}\hat{A} = \hat{E}(\lambda\hat{E} - I) = (\lambda\hat{E} - I)\hat{E} = \hat{A}\hat{E}.$$

Some calculations show

$$V\hat{A}V^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ with } V^{-1} = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

and

$$W\hat{E}W^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ with } W^{-1} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 1 & 1 & -1 \end{bmatrix},$$

hence

$$\hat{E}^D = \begin{bmatrix} \frac{5}{6} & -\frac{1}{2} & -\frac{5}{6} & -\frac{1}{6} \\ -\frac{2}{3} & 2 & \frac{2}{3} & -\frac{2}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{3} & 1 & \frac{1}{3} & -\frac{1}{3} \end{bmatrix}, \quad \hat{A}^D = \begin{bmatrix} \frac{5}{6} & -\frac{5}{6} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{5}{3} & \frac{2}{3} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{4}{3} & -\frac{2}{3} & 0 & 2 \end{bmatrix}$$

Exercise 4 (A bad distribution)
Lemma 2.2.3 in [2].

References

[1] Thomas Berger, Achim Ilchmann, and Stephan Trenn. The quasi-Weierstraß form for regular matrix pencils. *Lin. Alg. Appl.*, 2010. Preprint available online, Institute for Mathematics, Ilmenau University of Technology, Preprint Number 09-21.

[2] Stephan Trenn. *Distributional differential algebraic equations.* PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany, 2009.