DAEs - Control and Numerics

Exercise Sheet 4 - Stability and optimal control

Solutions

Exercise 12 (Re-interpretation of variables continued)
Calculation the ranks and coranks of $\mathcal{M}_i$ we obtain

$$\begin{aligned}
\operatorname{rank} M_0 &= 2 &\Rightarrow&\quad \operatorname{corank} M_0 = 8,\\
\operatorname{rank} M_1 &= 11 &\Rightarrow&\quad \operatorname{corank} M_1 = 9,\\
\operatorname{rank} M_2 &= 20 &\Rightarrow&\quad \operatorname{corank} M_2 = 10,\\
\operatorname{rank} M_3 &= 30 &\Rightarrow&\quad \operatorname{corank} M_3 = 10,
\end{aligned}$$

hence $v_0 = 1$, $v_1 = 1$, $v_2 = 0$ and $a_0 = 7$, $a_1 = 8$, $a_2 = 10$. Let Z_{23i} be the kernel of $M_i^\top$ with $i = 0, 1, 2$. Then

$$\begin{aligned}
\operatorname{rank} Z_{230}^\top N_0 &= 8 &\neq&\quad a_0 + v_0 = 9,\\
\operatorname{rank} Z_{231}^\top N_1 &= 9 &\neq&\quad a_1 + v_1 = 10,\\
\operatorname{rank} Z_{232}^\top N_2 &= 10 &=&\quad a_2 + v_2 = 10,
\end{aligned}$$

there $\mu = 0$ and $\mu = 1$ is not possible. Let

$$\hat{A}_2 = Z_{232}^\top N_2 = \begin{bmatrix}
R & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & R_L & -1 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{R_L}{R} & \dot{R}_L & 0 & 0 & 0\\
0 & 0 & \frac{R_L}{CL} & 0 & 0 & \frac{L\dot{R}_L - R_L^2}{L^2} & \frac{L\ddot{R}_L - R_L\dot{R}_L}{L} & 0 & 0 & 0
\end{bmatrix},$$

then rank $\hat{A}_2 = 10$. Let $\hat{Z} \in \mathbb{R}^{2\times 10}$ be the left kernel of the first eight columns of $\hat{A}_2$, e.g.

$$\hat{Z} = \begin{bmatrix} R_L^2 & CR_L(R_L^2 - L\dot{R}_L) \\ L\dot{R}_L & L(R_L - CL\ddot{R}_L + CR_L\dot{R}_L) \\ -RR_L^2 & CLR\dot{R}_L R_L - LR_L^2 - CRR_L^3 \\ 0 & LR_L^2 \\ 0 & LR_L^2 \\ R_L^2 & CR_L(R_L^2 - L\dot{R}_L) \\ R_L^2 & CR_L(R_L^2 - L\dot{R}_L) \\ L\dot{R}_L - R_L^2 & LR_L - CR_L^3 - CL^2\dot{R}_L + 2CLR_L\dot{R}_L \\ -LR_L & 0 \\ 0 & CL^2R_L \end{bmatrix}^\top,$$

then $\begin{bmatrix} I & 0 \\ & \hat{Z} \end{bmatrix} \hat{A}_2 = \begin{bmatrix} A_{22} & B_2 \\ 0 & B_3 \end{bmatrix}$ with invertible $A_{22} \in \mathbb{R}^{8\times 8}$ and invertible $B_3 \in \mathbb{R}^{2\times 2}$ where

$$B_3 = \begin{bmatrix} RR_L^2 & R_L^2 \\ CRR_L^3 + LR_L^2 - CLR\dot{R}_L R_L & CR_L(R_L^2 - L\dot{R}_L) \end{bmatrix},$$

hence the two input variables u_1 and u_2 must be reinterpreted as a state-variable which have to fulfill the algebraic equation

$$0 = B_3 u + \hat{Z}^\top Z_{232}^\top \begin{pmatrix} f \\ f' \\ f'' \end{pmatrix}.$$

In fact,

$$\begin{aligned} u(t) &= B_3^{-1}\hat{Z} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ f_{10}(t) \\ f_{10}'(t) \\ f_{10}''(t) \end{pmatrix} \\ &= \begin{pmatrix} \frac{R_L(t)^2 - CL\ddot{R}_L(t)R_L(t) + CL\dot{R}_L(t)^2}{R_L(t)^3} \\ -\frac{RR_L(t)^2 - R_L(t)(L\dot{R}_L(t) + CLR\ddot{R}_L(t)) + CLR\dot{R}_L(t)^2}{R_L(t)^3} - 1 \end{pmatrix} f_{10}(t) \\ &\quad + \begin{pmatrix} C - \frac{CL\dot{R}_L(t)}{R_L(t)^2} \\ -CR - \frac{L(R_L(t) - CR\dot{R}_L(t))}{R_L(t)^2} \end{pmatrix} f_{10}'(t) + \begin{pmatrix} \frac{CL}{R_L(t)} \\ -\frac{CLR}{R_L(t)} \end{pmatrix} f_{10}''(t) \end{aligned}$$

Exercise 13 (Stability of DAEs and higher index)
The solution of the homogeneous DAE is given by

$$x(t) = \begin{pmatrix} e^{-t}c \\ 0 \\ e^{-t}c \end{pmatrix}, \quad c \in \mathbb{R},$$

which converges to zero for all $c \in \mathbb{R}$. The solution to the inhomogeneous DAE is given by

$$x(t) = \begin{pmatrix} e^{-t}c + \sin(t^2) \\ 2t\cos(t) + \sin(t^2) \\ e^{-t}c + 2t\cos(t) + \sin(t^2) \end{pmatrix}, \quad c \in \mathbb{R},$$

which is unbounded for all $c \in \mathbb{R}$.

Exercise 14 (Optimal control)
In the notation of the lecture, $Q = S = R = 1$, the necessary optimality condition reads as

$$\begin{bmatrix} 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & -1 & \varepsilon & 0 \\ -\varepsilon & 1 & 0 & 0 & 0 \\ 0 & -\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{\mu} \\ \dot{u} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ \mu \\ u \end{pmatrix}$$

with boundary condition $x(0) = x_0$ and $\begin{bmatrix} \varepsilon & 0 \\ -1 & \varepsilon \end{bmatrix} \mu(1) = 0$, i.e. $\mu(1) = 0$. The last row of the DAE yields

$$u(t) = -x_2(t) - \mu_2(t).$$

Plugging this into the rest of the equations yields

$$\begin{bmatrix} \varepsilon & 0 \\ -1 & \varepsilon \end{bmatrix} \dot{\mu} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which together with $\mu(1) = 0$ yields $\mu(t) = 0$ for all $t \in \mathbb{R}$. Hence the optimal control is given by $u(t) = -x_2(t) = -y(t)$ independently of ε.
For $\varepsilon = 0$ we get the same control law

$$u(t) = -x_2 + \mu_2(t),$$

but now μ is a solution of

$$\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \dot{\mu} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \mu(1) = 0,$$

which implies $\mu_1(t) = 0$ and $\mu_2(t)$ can be arbitrary. Plugging this into the necessary condition for x yields

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix},$$

hence $\mu_2(t) = 0$ and $\dot{x}_2 = x_1$. Therefore the optimal control is given by $u(t) = -\int_0^t x_1 - x_{2,0}$, where $x_1(t)$ can be arbitrary with $x_1(0) = x_{1,0}$.

Exercise 15 (Lyapunov regularity)
The underlying ODE for the first DAE is $\dot{x}_1 = -x_1$ which is Lyapunov-regular because its solutions are given by $e^{-t}c$, $c \in \mathbb{R}$. However the DAE has the solution $x_1(t) = e^{-t}c$ and $x_2(t) = x_1 e^{t - t\sin(t)} = e^{-t\sin(t)}c$ for $c \in \mathbb{R}$. Since

$$\lambda^u = \limsup_{t\to\infty} \tfrac{1}{t} \ln(|e^{-t}| + |e^{-t\sin(t)}|) \geq \limsup \tfrac{1}{t} \ln |e^{-t\sin(t)}| = 1$$

and

$$\lambda^\ell = \liminf \tfrac{1}{t}\ln(|e^{-t}| + |e^{-t\sin(t)}|) \le \liminf \tfrac{1}{t}\ln(2|e^{-t\sin(t)}|) = -1,$$

hence the DAE is not Lyapunov regular.
The adjoint DAE is given by

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \dot{z} = \begin{bmatrix} 1 & -1 \\ 0 & e^{-t+t\sin(t)} \end{bmatrix},$$

which has the solution $z_2(t) = 0$ and $z_1(t) = e^{-t}c$, $c > 0$. Hence the adjoint DAE is Lyapunov regular.