Differential-algebraic equations. Control and Numerics I

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Outline

1 Introduction
2 Applications
3 Research projects
4 Setting for this lecture
5 Linear constant coefficient DAEs
6 Linear variable coefficient DAEs

Theses

- Modern key technologies need modeling, simulation, control and optimization of complex dynamical systems.
- Simulation and control of systems form the third pillar of scientific development besides theory and experiment.
- Most complex systems in key technologies are multi-physics systems.
- We need mathematical techniques to analyze the dynamics of complex systems.
- Modeling, analysis, numerical methods and control/optimization techniques should go hand in hand.
- New levels of interdisciplinary cooperation and a new modeling paradigm is needed.
- Differential-Algebraic Equations (DAEs) equations provide the ideal framework for such a paradigm.

What are DAEs/descriptor systems ?

Differential-algebraic equations (DAEs), descriptor systems, singular differential eqns, algebro-differential eqns, ...
are implicit systems of differential equations of the form

$$\begin{array}{rcl} 0 & = & \mathcal{F}(t,\xi,u,\dot{\xi},p,\omega), \\ y_1 & = & \mathcal{G}_1(t,\xi,u,p,\omega), \\ y_2 & = & \mathcal{G}_2(t,\xi,u,p,\omega), \end{array}$$

with $F \in C^0(\mathbb{R} \times \mathbb{D}_\xi \times \mathbb{D}_u \times \mathbb{D}_{\dot{\xi}} \times \mathbb{D}_p \times \mathbb{D}_\omega, \mathbb{R}^\ell)$,
$G_i \in C^0(\mathbb{R} \times \mathbb{D}_\xi \times \mathbb{D}_u \times \mathbb{D}_p \times \mathbb{D}_\omega, \mathbb{R}^{p_i})$, $i = 1, 2$.

- $t \in \mathbb{I} \subset \mathbb{R}$ is the time,
- ξ denotes the state (finite or infinite dimensional), $\dot{\xi} = \frac{d}{dt}\xi$,
- u denotes control inputs, ω denotes uncertainties/disturbances,
- y_1 denotes controlled, y_2 measured outputs,
- p denotes parameters.

In the linear case (linearization along non-stationary solutions) we get

$$\begin{aligned} E(t,p)\dot{\xi} &= A(t,p)\xi + B_1(t,p)u + B_2(t,p)\omega + \phi(t,p), \\ y_1 &= C_1(t,p)\xi + D_{11}(t,p)u + D_{12}(t,p)\omega + \psi_1(t), \\ y_2 &= C_2(t,p)\xi + D_{21}(t,p)u + D_{22}(t,p)\omega + \psi_2(t). \end{aligned}$$

or (linearization along stationary solutions)

$$\begin{aligned} E(p)\dot{\xi} &= A(p)\xi + B_1(p)u + B_2(p)\omega, \\ y_1 &= C_1(p)\xi + D_{11}(p)u + D_{12}(p)\omega, \\ y_2 &= C_2(p)\xi + D_{21}(p)u + D_{22}(p)\omega. \end{aligned}$$

DAEs provide a unified framework for the analysis, simulation and control of (coupled) dynamical systems (continuous and discrete time).

- Automatic modeling leads to DAEs. (Constraints at interfaces).
- Conservation laws lead to DAEs. (Conservation of mass, energy, momentum).
- Coupling of solvers leads to DAEs (discrete time).
- Control problems are DAEs (behavior).

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Classical applications of DAE modeling.

- Electronic circuit simulation (Kirchhoff's laws).
- Simulation and control of mechanical multibody systems (position or velocity constraints).
- Flow simulation and flow control (mass conservation).
- Metabolic networks (balance equations).
- Simulation and control of systems from chemical engineering (mass balances).
- Simulation and control of traffic systems (mass conservation).
- …

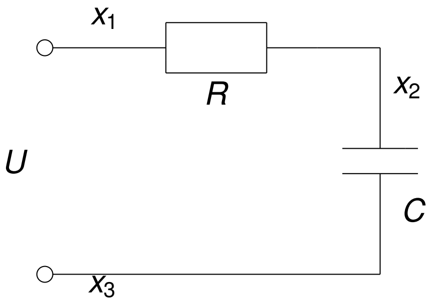


Figure: A simple electrical network

- Charging a capacitor via a resistor (ideal electronic units).
- Associate a potential x_i, $i = 1, 2, 3$, with each node of the circuit, zero potential $x_3 = 0$.
- The voltage source increases the potential x_3 to x_1 by U, i. e., $x_1 - x_3 - U = 0$.
- By Kirchhoff's first law the sum of the currents vanishes in each node.
- For the second node we obtain that $C(\dot{x}_3 - \dot{x}_2) + (x_1 - x_2)/R = 0$, R is resistance of the resistor and C is the capacity of the capacitor.
- We get the DAE:

$$\begin{aligned} C(\dot{x}_3 - \dot{x}_2) + (x_1 - x_2)/R &= 0, \\ x_1 - x_3 - U &= 0, \\ x_3 &= 0. \end{aligned}$$

A physical pendulum

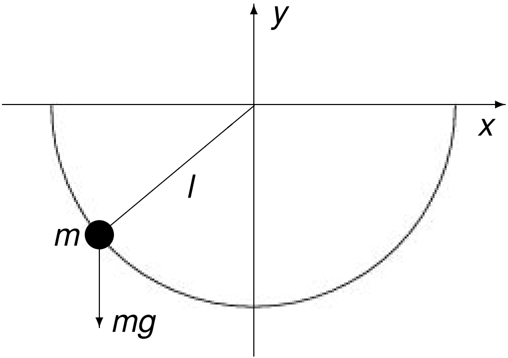


Figure: A mechanical multibody system

DAE modeling

- Mass point with mass m in Cartesian coordinates (x, y) moves under influence of gravity in a distance l around the origin.
- Kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$
- potential energy $U = mgy$, where g is the gravity constant,
- Constraint equation $x^2 + y^2 - l^2 = 0$,
- Lagrange function $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy - \lambda(x^2 + y^2 - l^2)$
- Equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

  for the variables $q = x, y, \lambda$, i. e.,
- DAE model:

$$\begin{aligned} m\ddot{x} + 2x\lambda &= 0, \\ m\ddot{y} + 2y\lambda + mg &= 0, \\ x^2 + y^2 - l^2 &= 0. \end{aligned}$$

A chemical reactor

Chemical reactor in which a first order isomerization reaction takes place and which is externally cooled.

- ▷ c_0 the given feed reactant concentration,
- ▷ T_0 the initial temperature,
- ▷ $c(t)$ and $T(t)$ the concentration and temperature at time t, and
- ▷ R the reaction rate per unit volume,
- ▷ DAE model

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{c} \\ \dot{T} \\ \dot{R} \end{bmatrix} = \begin{bmatrix} k_1(c_0 - c) - R \\ k_1(T_0 - T) + k_2 R - k_3(T - T_C) \\ R - k_3 \exp(-\frac{k_4}{T})c \end{bmatrix},$$

- ▷ T_C is the cooling temperature (control input),
- ▷ k_1, k_2, k_3, k_4 are constants.

Semi-discretized PDEs

- The non-stationary Stokes equation is a linear model for the laminar flow of a Newtonian fluid

$$u_t = \Delta u + \nabla p, \quad \nabla \cdot u = 0,$$

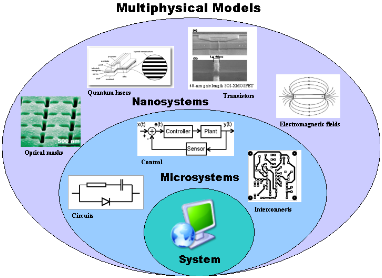
  together with initial and boundary conditions.
- u describes the velocity and p the pressure of the fluid.
- Discretizing first the space variables with finite element or finite difference methods gives the DAE

$$\dot{u}_h = Au_h + Bp_h, \quad B^T u_h = 0,$$

  where u_h and p_h are semi-discrete approximations for u and p.
- The non-uniqueness of a free constant in the pressure must be fixed by the discretization method.

Multi-physics systems

DAE modeling is standard in multi-physics systems.



Packages like MATLAB (SIMULINK, DYMOLA (MODELLICA) and SPICE like circuit simulators proceed as follows:

- Modularized modeling of uni-physics components.
- Network based connection of components.
- Identification of input and output parameters.
- Numerical simulation and control on full model.

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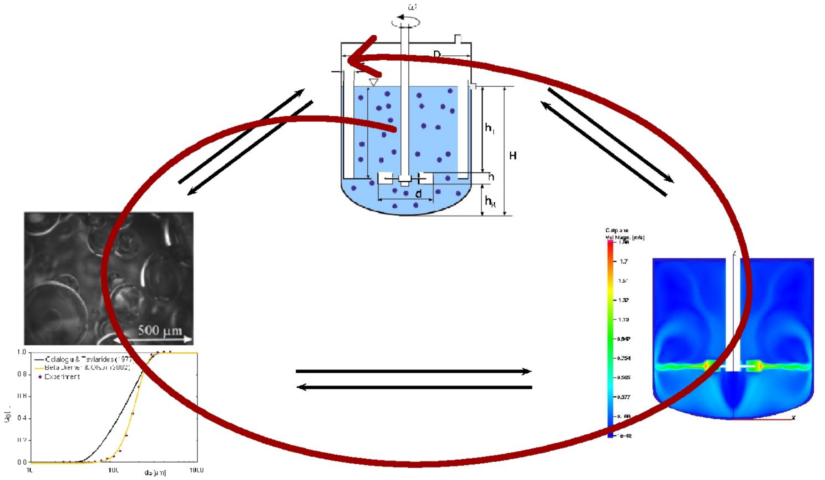
Modeling, simulation and control of automatic gearboxes.
Project with Daimler AG (Peter Hamann) $\rightarrow$ film.

Technological Application

- Modeling of multi-model: multibody-system, including elasticity, hydraulics.
- Development of control methods for coupled system.
- Real time control of gearbox.

Goal: Decrease full consumption, improve smoothness of switching

with M. Kraume (Chemical Eng., TU Berlin), M. Schäfer (Mech. Eng. TU Darmstadt)



Technological Application, Tasks

Chemical industry: pearl polymerization and extraction processes

- Modeling of coalescence and breakage in turbulent flow.
- Numerical methods for simulation of coupled system of population balance equations/fluid flow equations. $\rightarrow$ film.
- Development of optimal control methods for large scale coupled systems
- Model reduction and observer design.
- Feedback control of real configurations via stirrer speed.

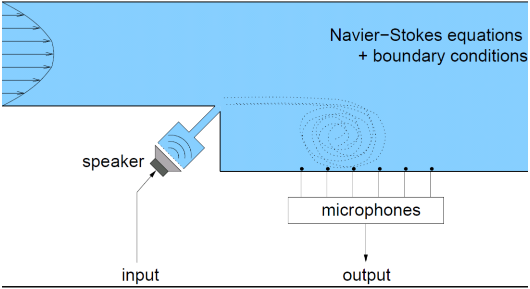
Goal: Achieve specified average drop diameter and small standard deviation for distribution by real time-control of stirrer-speed.

Mathematical system components

- Navier Stokes equation (flow field)
- Population balance equation (drop size distribution).
- One or two way coupling.
- Initial and boundary conditions.

Space discretization leads to an extremely large control system of nonlinear DAEs.

Project in Sfb 557 Control of complex shear flows, with F. Tröltzsch, M. Schmidt



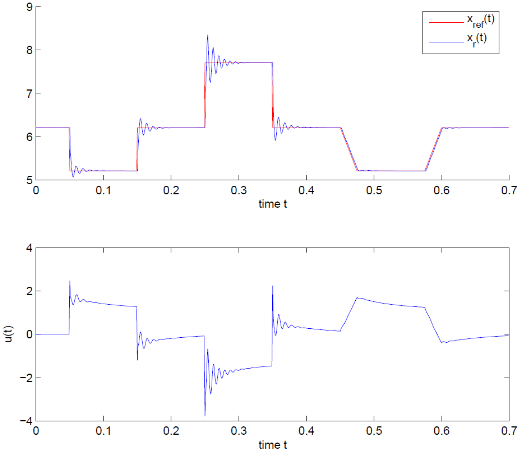
Technological Application, Tasks

Control of detached turbulent flow on airline wing

- Test case (backward step to compare experiment/numerics.)
- modeling of turbulent flow.
- Development of control methods for large scale coupled systems.
- Model reduction and observer design.
- Optimal feedback control of real configurations via blowing and sucking of air in wing.

Ultimate goal: Force detached flow back to wing.

Movement of recirculation bubble following reference curve.



Modeling becomes extremely convenient, but:

- Numerical simulation does not always work, instabilities and convergence problems occur (e.g. SIMULINK) !
- Consistent initialization is difficult.
- The discretized system may be unsolvable even if the DAE is solvable and vice versa.
- Numerical drift-off phenomenon.
- Model reduction is difficult.
- Classical control is difficult (non-proper transfer functions).

Black-box DAE modeling pushes all difficulties into the numerics. In general the methods cannot handle this!

Today several packages (e.g. Dymola) use computer algebra to turn back to ODE, this is bad.

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Setting for this lecture

- We assume that space discretization has been done, i.e., we only discuss differential-algebraic systems (DAEs).
- This is justified for the analysis.
- However, for the numerical solution methods typically space and time discretization have to be considered together.
- We ignore the dependence on parameters and disturbances and consider only one type of outputs.
- We will discuss first, constant coefficient, then variable coefficient and then nonlinear systems.

DAE theory in behavior framework

We first take a mathematical (behavioral) point of view: In

$$\begin{aligned} 0 &= F(t, \xi, u, \dot{\xi}), \ t \in \mathbb{I} \\ y &= G(t, \xi, u), \end{aligned}$$

with $F \in C^0(\mathbb{R} \times \mathbb{D}_\xi \times \mathbb{D}_u \times \mathbb{D}_{\dot{\xi}}, \mathbb{R}^\ell)$, or

$$\begin{aligned} E(t)\dot{x} &= A(t)\xi + B(t)u + \phi(t), \ t \in \mathbb{I} \\ y &= C(t)\xi + D(t)u, \end{aligned}$$

we introduce $x = [y^T, \xi^T, u^T]^T$ and obtain an over- and under-determined DAE system

$$\begin{aligned} 0 &= \mathcal{F}(t, x, \dot{x}), \ t \in \mathbb{I} \\ \mathcal{E}(t)\dot{x} &= \mathcal{A}(t)x + f(t), \ t \in \mathbb{I}. \end{aligned}$$

In practice and computation, we keep variables separate.

Definition

Consider an initial value problem for general DAEs

$$F(\dot{x}, x, t) = 0, \quad x(t_0) = x_0.$$

- A function x is called *(classical) solution* of the DAE if x is one times continuously differentiable and x satisfies the equation pointwise.
- It is called solution of the initial value problem if it is a solution and satisfies the initial condition.
- An initial condition is called consistent if the corresponding initial value problem is solvable, i.e. has at least one solution.

Other solvability concepts, weak or distributional solutions.

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Linear DAEs with constant coefficients

Weierstraß/Kronecker 1890-1896 Consider

$$E\dot{x} = Ax + f(t), \quad x(t_0) = x_0,$$

where $E, A \in \mathbb{C}^{\ell,n}$ and $f \in C(\mathbb{I}, \mathbb{C}^{\ell})$.
Scaling from the left and changes of basis with nonsingular matrices.

$$PEQ\dot{\tilde{x}} = PAQ\tilde{x} + Pf(t), \ \tilde{x}(t_0) = \tilde{x}_0.$$

Definition

Two pairs of matrices $(E_i, A_i), \ i = 1, 2$, are called *(strongly) equivalent* if there exist invertible matrices $P \in \mathbb{C}^{\ell,\ell}$, $Q \in \mathbb{C}^{n,n}$ with $E_2 = PE_1Q, \ A_2 = PA_1Q$. Write as:

$$[E_2, A_2] = P[E_1, A_1]\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix}.$$

Kronecker canonical form (KCF)

Theorem (Kronecker 1896)

For every pair $E, A \in \mathbb{C}^{\ell,n}$ there exist nonsingular $P \in \mathbb{C}^{\ell,\ell}, Q \in \mathbb{C}^{n,n}$ such that $P(\lambda E - A)Q =$
$\mathrm{Diag}\,(L_{\epsilon_1}, \ldots, L_{\epsilon_p}, M_{\eta_1}, \ldots, M_{\eta_q}, J_{\rho_1}, \ldots, J_{\rho_v}, N_{\sigma_1}, \ldots, N_{\sigma_w})$,

$$L_{\epsilon_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},$$

$$J_{\rho_j} = \lambda \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \quad M_{\eta_j} = \lambda \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix},$$

$$N_{\sigma_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \quad L_{\epsilon_j} = M_{\epsilon_j}^T$$

Definition

A matrix pencil $\lambda E - A,\ E, A \in \mathbb{C}^{\ell,n}$, is called regular if $\ell = n$ and if

$$P(\lambda) = \det(\lambda E - A)$$

is not identically 0, otherwise singular. The size ν of the largest N-block in the KCF is the (differentiation) d-index of $\lambda E - A$.

Control systems in behavior form have singular pencils.

Weierstraß canonical form (WCF)

Theorem (Weierstraß 1867)

For every regular pair $E, A \in \mathbb{C}^{n,n}$ *there exist nonsingular* $P, Q \in \mathbb{C}^{n,n}$ *such that*

$$P(\lambda E - A)Q = \lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix},$$

where $J = \operatorname{diag}(J_{\rho_1}, \ldots, J_{\rho_v})$ *and* $N = \operatorname{diag}(N_{\sigma_1}, \ldots, N_{\sigma_w})$,

$$J_{\rho_j} = \lambda \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} - \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}, \quad N_{\sigma_j} = \lambda \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} - \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix},$$

Lemma

Consider a regular constant coefficient DAE in WCF

$$\lambda \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \end{bmatrix} = \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix}$$

with d-index ν. The solution is

$$\begin{aligned} x_1(t) &= e^{J(t-t_0)} x_{1,0} + \int_{t_0}^{t} e^{J(t-s)} f_1(s) ds \\ x_2(t) &= -\sum_{i=0}^{\nu-1} f_2^{(i)}(t) \end{aligned}$$

Consistent initial values have to satisfy $x_2(t_0) = -\sum_{i=0}^{\nu-1} f^{(i)}(t_0)$.

Theorem (Campbell 1982)

Consider a linear constant coefficient system with regular $\lambda E - A$ and let $f \in C^{\nu}(\mathbb{I}, \mathbb{C}^n)$.
Then the system is solvable and every consistent initial condition fixes a unique solution.

Note that this is not an -if and only if- result.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

is not regular but has unique solution $x = 1$.

Let

$$TET^{-1} = \begin{bmatrix} J_1 & \\ & J_0 \end{bmatrix}$$

be the Jordan form of E with J_0 nilpotent of nilpotency index ν, then

$$E^D = T^{-1} \begin{bmatrix} J_1^{-1} & \\ & 0 \end{bmatrix} T$$

is the *unique Drazin inverse* satisfying

$$E^D E E^D = E^D, \; E^D E = E E^D, \; E^D E^{\nu+1} = E^\nu$$

Lemma (Campbell 1982)

Let (E, A) be a regular pair. Then for all $\hat{\lambda} \in \mathbb{R}$ such that $(\hat{\lambda}E - A)^{-1}$ exist, $\hat{E} = (\hat{\lambda}E - A)^{-1}E$ and $\hat{A} = (\hat{\lambda}E - A)^{-1}A$ commute.

Theorem (Campbell 1982, Kunkel/M. 2006)

Consider the regular DAE $E\dot{x} = Ax + f$, $x(t_0) = x_0$, *let* ν *be the d-be index of* (E, A) *and let* $\hat{\lambda} \in \mathbb{R}$ *such that* $(\hat{\lambda}E - A)^{-1}$ *exist. If* $\hat{f} = (\hat{\lambda}E - A)^{-1}f$ *is sufficiently smooth, then the solution is*

$$x(t) = e^{\hat{E}^D\hat{A}(t-t_0)}\hat{E}^D\hat{E}v + \int_{t_0}^{t} e^{\hat{E}^D\hat{A}(t-s)}\hat{E}^D\hat{f}(s)\,ds - \\ - (I - \hat{E}^D\hat{E})\sum_{i=0}^{\nu-1}(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{f}^{(i)}(t).$$

- *The formula is independent of the choice of* $\hat{\lambda}$.
- *An initial condition is consistent iff there is a vector* v *such that*

$$x_0 = \hat{E}^D\hat{E}v - (I - \hat{E}^D\hat{E})\sum_{i=0}^{\nu-1}(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{f}^{(i)}(t_0).$$

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$$E(t)\dot{x}(t) = A(t)x(t) + f(t), \; x(t_0) = x_0.$$

Scaling from left and basis changes

$$PEQ\dot{\tilde{x}} = (PAQ - PE\dot{Q})\tilde{x} + Pf, \; \tilde{x}(t_0) = \tilde{x}_0.$$

Definition

Two pairs of matrix functions $(E_i(t), A_i(t))$ in $\mathbb{C}^{\ell,n}$ are called *globally equivalent* if there exist $P \in C(\mathbb{I}, \mathbb{C}^{\ell,\ell})$ and $Q \in C^1(\mathbb{I}, \mathbb{C}^{n,n})$, $P(t), Q(t)$ nonsingular for all $t \in \mathbb{I}$ such that

$$[E_2(t), A_2(t)] = P(t)[E_1(t), A_1(t)] \begin{bmatrix} Q(t) & -\dot{Q}(t) \\ 0 & Q(t) \end{bmatrix}.$$

Regularity and d-index at time *t* are not invariant.

The system

$$\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x(t), \quad t \in \mathbb{R}$$

is uniformly regular and of uniform d-index $\nu = 2$ but

$$x(t) = c(t) \begin{bmatrix} t \\ 1 \end{bmatrix}$$

is a solution for all $c \in C^1(\mathbb{R}, \mathbb{C})$.

Example 2

The system

$$\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix} \dot{x}(t) = \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix},$$

is uniformly singular, because the pencil is singular for all t. But the system has the unique solution

$$\begin{bmatrix} f_1 + tf_2 - t\dot{f}_1 \\ f_2 - \dot{f}_1 \end{bmatrix}$$

independent of any initial condition.

Local version of global equivalence

Definition

Two pairs of matrices

$$(E_i, A_i),\ E_i, A_i \in \mathbb{R}^{\ell,n}, \quad i = 1, 2$$

are called *locally equivalent* if there exist matrices $P \in \mathbb{C}^{\ell,\ell}$, $Q, R \in \mathbb{C}^{n,n}$ with P, Q nonsingular such that

$$[E_2, A_2] = P[E_1, A_1] \begin{bmatrix} Q & -R \\ 0 & Q \end{bmatrix}.$$

By Hermite interpolation there always exists a function $Q(t)$ such that at any point $\hat{t}$ one has $Q(\hat{t}) = Q$ and $\dot{Q}(\hat{t}) = R$.

Theorem (Kunkel/M. 1994)

Let $E, A \in \mathbb{C}^{\ell,n}$ and

$$\begin{array}{ll}(a) & T \text{ basis of kernel } E\\ (b) & Z \text{ basis of Co-range } E = \text{kernel } E^*\\ (c) & T' \text{ basis of Co-kernel } E = \text{kernel } E^*\\ (d) & V \text{ basis of Co-range } (Z^*AT).\end{array}$$

Then, the quantities (convention rank $\emptyset = 0$)

$$\begin{array}{lll}(a) & r = \text{rank } E & \text{(rank)}\\ (b) & a = \text{rank } (Z^*AT) & \text{(algebraic part)}\\ (c) & s = \text{rank } (V^*Z^*AT') & \text{(strangeness)}\\ (d) & d = r - s & \text{(differential part)}\\ (e) & v = \ell - r - a - s & \text{(redundant part)}\end{array}$$

are invariant under the local equivalence transformation.

Local canonical form

Theorem (Kunkel/M. 1994)

(E, A) *is locally equivalent to the canonical form:*

$$\begin{matrix} s \\ d \\ a \\ s \\ v \end{matrix} \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

Eigenvalues are not invariants of this canonical form.

Applying the local canonical form for all t we get integer functions $r(t), a(t), s(t)$.

Theorem (Kunkel/M. 1994)

Let E, A be sufficiently smooth and let r, a, s be constant in $\mathbb{I}$. Then $(E(t), A(t))$ is globally equivalent to a pair of matrix functions of the form

$$\begin{matrix} s \\ d \\ a \\ s \\ v \end{matrix} \left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right).$$

More equivalence transformations

$$\begin{array}{lll} (a) & \dot{x}_1 & = A_{12}(t)x_2 + A_{14}(t)x_4 + g_1(t) \\ (b) & \dot{x}_2 & = A_{24}(t)x_4 + g_2(t) \\ (c) & 0 & = x_3 + g_3(t) \\ (d) & 0 & = x_1 + g_4(t) \\ (e) & 0 & = g_5(t). \end{array}$$

Insert the derivative of (d) in (a), which becomes an algebraic equation. This gives

$$\begin{array}{c} s \\ d \\ a \\ s \\ v \end{array} \left(\begin{bmatrix} \color{red}{0} & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12}(t) & 0 & A_{14}(t) \\ 0 & 0 & 0 & A_{24}(t) \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right),$$

for which we can again compute characteristic values r, a, s, d, v.

Inductive procedure

Proceeding inductively we get a sequence of pairs of matrix functions $(E_i(t), A_i(t))$ and integers $r_i, a_i, s_i, d_i, v_i,\ i \in \mathbb{N}_0$, which we assume to be constant in $\mathbb{I}$.

We start with $(E_0(t), A_0(t)) = (E(t), A(t))$, and then $(E_{i+1}(t), A_{i+1}(t))$ is derived from $(E_i(t), A_i(t))$ by bringing it into canonical form and inserting the derivative of (d) into (a). The procedure stops after finitely many steps.

Definition

The number μ of steps is called the *strangeness-index or s-index* μ. If $\mu = 0$, then the system is called *strangeness-free*.

Existence, uniqueness, consistency

Theorem (Kunkel/M. 1994)

Let the s-index μ be well–defined for $(E(t), A(t))$ and let $f \in C^{\mu}(\mathbb{I}, \mathbb{C}^{l})$. Then the system is equivalent to a remodeled DAE in normal form

$$\begin{array}{rcll} \dot{x}_1(t) & = & A_{13}(t)x_3(t) + f_1(t), & d_\mu \text{ equations}, \\ 0 & = & x_2(t) + f_2(t), & a_\mu \text{ equations}, \\ 0 & = & f_3(t), & v_\mu \text{ equations}, \end{array}$$

where the inhomogeneity is determined by $f^{(0)}, \ldots, f^{(\mu)}$.

- *The problem is solvable if and only if $f_3(t) \equiv 0$.*
- *An initial condition is consistent if and only if in addition $x_2(t_0) = -f_2(t_0)$ holds.*
- *The problem is uniquely solvable if again in addition we have $u_\mu = n - d_\mu - a_\mu = 0$.*

We get the canonical form

$$\left(\begin{bmatrix} -t & t^2 \\ -1 & t \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right) \sim \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

We have $\mu = 1$ with

$$r_0 = 1, a_0 = 0, s_0 = 1, d_0 = 0, u_0 = 0,$$

$$r_1 = 0, a_1 = 1, s_1 = 0, d_1 = 0, u_1 = 1.$$

The problem is solvable, since $f(t) = 0$, but not uniquely solvable, since $u_\mu \neq 0$. The general solution is given by

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \hat{x}_2(t) \end{bmatrix} = \hat{x}_2(t) \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

Example 2 cont.

$$\left(\begin{bmatrix} 0 & 0 \\ 1 & -t \end{bmatrix}, \begin{bmatrix} -1 & t \\ 0 & 0 \end{bmatrix}, f\right) \sim \left(0, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} f_1 \\ f_2 - \dot{f}_1 \end{bmatrix}\right)$$

We have $\mu = 1$ with

$$r_0 = 1, a_0 = 0, s_0 = 1, d_0 = 0, u_0 = 0,$$

$$r_1 = 0, a_1 = 2, s_1 = 0, d_1 = 0, u_1 = 0.$$

The problem is uniquely solvable for every consistent initial condition with

$$x(t) = \begin{bmatrix} f_1(t) + tf_2(t) - t\dot{f}_1(t) \\ f_2(t) - \dot{f}_1(t) \end{bmatrix}.$$

$$
\begin{aligned}
C(\dot{x}_3 - \dot{x}_2) + (x_1 - x_2)/R &= 0, \\
x_1 - x_3 - U &= 0, \\
x_3 &= 0.
\end{aligned}
$$

This system has d-index 1, s-index 0.

$$\dot{u}_h = Au_h + Bp_h, \quad B^T u_h = 0,$$

where u_h and p_h are semi-discrete approximations for u and p. If the non-uniqueness in p is not fixed then the d-index is not defined and the s-index is 1. If it is fixed then the d-index is 2.

Evaluation of the algebraic approach

- The algebraic approach is essential for the theoretical understanding of DAEs.
- It can be used to study controllable, observable, autonomous behavior in the sense of Willems, see Ilchmann/M. 2005.
- The approach allows to do bifurcation analysis, the points where ranks change are a superset of the set of *critical points*.
- But, it cannot be used for the nonlinear case, for numerical methods or the design of controllers, since one would need derivatives of computed transformation matrices.

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