Differential-algebraic equations. Control and Numerics III

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Outline

1. Recap from last lectures
2. Control systems
3. Feedback
4. Reinterpretation of variables
5. Control methods for DAEs
6. Optimal control
7. Optimal H_∞ control

Differential-algebraic equations (DAEs)

We have considered general nonsquare DAEs

$$0 = \mathcal{F}(t, x, \dot{x}),\ x(t_0) = x_0,$$

with $F \in C^0(\mathbb{R} \times \mathbb{D}_x \times \mathbb{D}_{\dot{x}}, \mathbb{R}^\ell)$,
In the linear case (linearization along non-stationary solutions) we get

$$E(t)\dot{x} = A(t)x + f(t),\ x(t_0) = x_0,$$

or

$$E\dot{x} = Ax + f(t),\ x(t_0) = x_0.$$

Theorem (Kunkel/M. 1994)

Let the s-index μ be well–defined for $(E(t), A(t))$ and let $f \in C^{\mu}(\mathbb{I}, \mathbb{C}^{l})$. Then the system is equivalent to a remodeled DAE in normal form

$$\begin{array}{rcll} \dot{x}_1(t) & = & A_{13}(t)x_3(t) + f_1(t), & d_\mu \text{ equations}, \\ 0 & = & x_2(t) + f_2(t), & a_\mu \text{ equations}, \\ 0 & = & f_3(t), & v_\mu \text{ equations}, \end{array}$$

where the inhomogeneity is determined by $f^{(0)}, \ldots, f^{(\mu)}$.

- *The problem is solvable if and only if $f_3(t) \equiv 0$.*
- *An initial condition is consistent if and only if in addition $x_2(t_0) = -f_2(t_0)$ holds.*
- *The problem is uniquely solvable if again in addition we have $u_\mu = n - d_\mu - a_\mu = 0$.*

We assume that derivatives of original functions are available or can be obtained via computer algebra or automatic differentiation.

Linear case: We put $E(t)\dot{x} = A(t)x + f(t)$ and its derivatives up to order μ into a large DAE

$$M_k(t)\dot{z}_k = N_k(t)z_k + g_k(t), \quad k \in \mathbb{N}_0$$

for $z_k = [x^T, \dot{x}^T, \dots, {x^{(k)}}^T]$.

$$M_2 = \begin{bmatrix} E & 0 & 0 \\ A - \dot{E} & E & 0 \\ \dot{A} - 2\ddot{E} & A - \dot{E} & E \end{bmatrix}, \; N_2 = \begin{bmatrix} A & 0 & 0 \\ \dot{A} & 0 & 0 \\ \ddot{A} & 0 & 0 \end{bmatrix}, \; z_2 = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix}.$$

Theorem (Kunkel/M. 1996)

Under some constant rank assumptions, for every linear DAE there exist integers μ, a, d and v such that:

1. $\operatorname{corank} M_{\mu+1}(t) - \operatorname{corank} M_{\mu}(t) = v$.
2. $\operatorname{rank} M_{\mu}(t) = (\mu + 1)m - a - v$ *on* $\mathbb{I}$, *and there exists a smooth matrix function* $Z_{2,3}$ *(left nullspace of* M_{μ}*) with* $Z_{2,3}^T M_{\mu}(t) = 0$.
3. *The projection* $Z_{2,3}$ *can be partitioned into two parts:* Z_2 *(left nullspace of* (M_{μ}, N_{μ})*) so that the first block column* $\hat{A}_2$ *of* $Z_2^* N_{\mu}(t)$ *has full rank* a *and* $Z_3^* N_{\mu}(t) = 0$. *Let* T_2 *be a smooth matrix function such that* $\hat{A}_2 T_2 = 0$, *(right nullspace of* $\hat{A}_2$*).*
4. $\operatorname{rank} E(t) T_2 = d = \ell - a - v$ *and there exists a smooth matrix function* Z_1 *of size* (n, d) *with* $\operatorname{rank} \hat{E}_1 = d$, *where* $\hat{E}_1 = Z_1^T E$.

Reduced problem, remodeled DAE

We obtain a numerically computable **strangeness-free condensed form**

$$\begin{array}{rcll} \hat{E}_1(t)\dot{x} & = & \hat{A}_1(t)x + \hat{f}_1(t), & d \text{ equations} \\ 0 & = & \hat{A}_2(t)x + \hat{f}_2(t), & a \text{ equations} \\ 0 & = & \hat{f}_3(t), & v \text{ equations} \end{array}$$

where $\hat{A}_1 = Z_1^T A$, $\hat{f}_1 = Z_1^T f$, and $\hat{f}_2 = Z_2^T g_\mu$, $\hat{f}_3 = Z_3^T g_\mu$.

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$$\begin{aligned} 0 &= \mathcal{F}(t,\xi,u,\dot{\xi}), \\ y &= \mathcal{G}(t,\xi,u), \end{aligned}$$

with $F \in C^0(\mathbb{R} \times \mathbb{D}_\xi \times \mathbb{D}_u \times \mathbb{D}_{\dot{\xi}}, \mathbb{R}^\ell)$.
In the linear case (linearization along solutions)

$$\begin{aligned} E(t)\dot{\xi} &= A(t)\xi + B(t)u + f(t), \\ y &= C(t)\xi + D(t)u + g(t). \end{aligned}$$

- ξ denotes the state (finite or infinite dimensional),
- u denotes the control input,
- y denotes the output.

Forming $x(t) = \begin{bmatrix} \xi(t) \\ u(t) \\ y(t) \end{bmatrix}$ we obtain (even in the ODE case) a general non-square DAE

$$\mathcal{E}(t)\dot{x}(t) = \mathcal{A}(t)x(t) + \gamma(t),$$

where

$$\mathcal{E} = \begin{bmatrix} E(t) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A(t) & B(t) & 0 \\ C(t) & D(t) & -I_p \end{bmatrix}, \gamma = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}.$$

$$M_k(t)\dot{x}_k = N_k(t)x_k + \psi_k(t),\ k = 0, 1, \dots,$$

$$M_2 = \left[\begin{array}{ccc|ccc|ccc} E & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline A-\dot{E} & B & 0 & E & 0 & 0 & 0 & 0 & 0 \\ C & D & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \dot{A}-2\ddot{E} & \dot{B} & 0 & A-\dot{E} & B & 0 & E & 0 & 0 \\ \dot{C} & \dot{D} & 0 & C & D & I & 0 & 0 & 0 \end{array}\right],$$

$$N_2 = \left[\begin{array}{ccc|ccc|ccc} A & B & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ C & D & I & 0 & \dots & 0 & 0 & \dots & 0 \\ \hline \dot{A} & \dot{B} & 0 & \cdot & & & & & \cdot \\ \dot{C} & \dot{D} & 0 & \cdot & & & & & \cdot \\ \hline \ddot{A} & \ddot{B} & 0 & \cdot & & & & & \cdot \\ \ddot{C} & \ddot{D} & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{array}\right].$$

Theorem (Kunkel/M./Rath 2001)

If the strangeness-index is well-defined for the system given by $(\mathcal{E}, \mathcal{A})$, then there exist integers α, d and v such that from the associated derivative array (M_μ, N_μ) we compute the following strangeness-free system with the same solution set

$$\begin{array}{rcll} \hat{E}_1(t)\dot{x} & = & \hat{A}_1(t)x + \hat{f}_1(t), & d \text{ equations}, \\ 0 & = & \hat{A}_2(t)x + \hat{f}_2(t), & \alpha \text{ equations}, \\ 0 & = & \hat{f}_3(t), & v \text{ equations} \end{array}$$

where $\hat{A}_1 = Z_1^T \mathcal{A}$, $\hat{A}_1 = Z_1^T \mathcal{A}$, $\hat{f}_1 = Z_1^T f$, $\hat{f}_i = Z_i^T \psi_\mu$ for $i = 2, 3$.

In general α is different from a, when considering the free system with $u = 0$.

Observations

- The output equation will occur (essentially) unchanged in the extracted system.
- The partitioning in states ξ and controls u is not mixed up.

$$
\begin{array}{llcl}
(a) & E_1(t)\dot{\xi} & = & A_1(t)\xi + B_1(t)u + \hat{f}_1(t), \\
(b) & 0 & = & A_2(t)\xi + B_2(t)u + \hat{f}_2(t), \\
(c) & 0 & = & \hat{f}_3(t), \\
(d) & y & = & C(t)\xi + D(t)u + g(t), \\
(e) & \xi(t_0) & = & \xi_0,
\end{array}
$$

where $E_1 = \hat{E}_1 \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, $A_i = \hat{A}_i \begin{bmatrix} I_n \\ 0 \end{bmatrix}$, $B_i = \hat{A}_i \begin{bmatrix} 0 \\ I_m \end{bmatrix}$, $i = 1, 2$.

- No derivatives of original inputs are needed.

Theorem (Kunkel/M. 2001)

Consider a linear variable coefficient control system and suppose that the strangeness-index μ of the system in behavior form is well defined. Then the characteristic quantities d, α and v are invariant under proportional state feedback $u = F(t)\xi + w$, and proportional output feedback $u = F(t)y + w$.

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Feedback is just a standard equivalence transformation

$$\begin{bmatrix} A & B & 0 \\ C & D & -I_p \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ F & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A+BF & B & 0 \\ C+DF & D & -I_p \end{bmatrix}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & -I_p \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ FC & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A+BFC & B & 0 \\ C+DFC & D & -I_p \end{bmatrix}$$

$$
\begin{array}{lrcl}
(a) & E_1(t)\dot{\xi} & = & A_1(t)\xi + B_1(t)u + \hat{f}_1(t), \\
(b) & 0 & = & A_2(t)\xi + B_2(t)u + \hat{f}_2(t), \\
(c) & 0 & = & \hat{f}_3(t), \\
(d) & y & = & C(t)\xi + D(t)u + g(t), \\
(e) & \xi(t_0) & = & \xi_0,
\end{array}
$$

- Transform equation (b) to

$$
\begin{array}{lrcl}
(b.1) & 0 & = & A_{2,1}(t)\xi + B_{2,1}(t)u + \hat{f}_{2,1}(t), \\
(b.2) & 0 & = & A_{2,2}(t)\xi + \hat{f}_{2,2}(t),
\end{array}
$$

  where $B_{2,1}$ has full row rank. (Impulse controllable part)
- Index reduction/remodeling can be done via differentiation (on equation (b.2)) (the part that is not impulse controllable) and feedback in (b.1).

- If $\hat{f}_3(t)$ is not 0 then the system has no solution, return to user or set $\hat{f}_3(t) \to 0$.
- (b.2) $0 = A_{2,2}(t)\xi + \hat{f}_{2,2}(t)$ is an algebraic constraint that cannot be influenced by feedback. It describes the solution manifold for the system.
- In (b.1) $0 = A_{2,1}(t)\xi + B_{2,1}(t)u + \hat{f}_{2,1}(t)$, we have choice, we can move the algebraic constraint via state feedback and make the free system $u = 0$ strangeness-free.

$$\begin{array}{lrcl}
(a) & E_1(t)\dot{\xi} & = & A_1(t)\xi + B_1(t)u + \hat{f}_1(t), \\
(b.1) & 0 & = & \tilde{A}_{2,1}(t)\xi + B_{2,1}(t)u + \hat{f}_{2,1}(t), \\
(b.2) & 0 & = & A_{2,2}(t)\xi + \hat{f}_{2,2}(t), \\
(d) & y & = & C(t)\xi + D(t)u + g(t), \\
(e) & \xi(t_0) & = & \xi_0,
\end{array}$$

Output feedback regularization

- Index reduction and regularization via output feedback needs further refinement via changes of bases.
- Systems that are not impulse observable i.e. where derivative of the input or inhomogeneity are not observed are ill-posed in the sense that nothing can be done to change this property by feedback.
- We can only change the interpretation of variables.

At any fixed point $\hat{t} \in \mathbb{I}$ we proceed as follows.

1: Determine a unitary matrix $Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}$ of size (n, n) such that

$$E_1(\hat{t}) \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} =: \begin{bmatrix} E_{11} & 0 \end{bmatrix},$$

where E_{11} has size (d, d) and is nonsingular.

2: Determine unitary matrices $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, of size $(l - d, l - d)$ and $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ of size $(n - d, n - d)$ with

$$U^T A_2(\hat{t}) Q_2 V = \begin{bmatrix} A_{22} & 0 \\ 0 & 0 \end{bmatrix},$$

where A_{22} is nonsingular of size (a, a), set $\phi = \alpha - a$.

3: Determine the column rank o of $C(\hat{t}) Q_2 V_2$, i.e., determine a unitary $W = \begin{bmatrix} W_1 & W_2 \end{bmatrix}$, of size $(n - a - d, n - a - d)$ with

$$C(\hat{t}) Q_2 V_2 W = \begin{bmatrix} C_3 & 0 \end{bmatrix}.$$

Condensed form at fixed point $\hat{t}$.

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 & 0 \end{bmatrix}$$

with nonsingular A_{22} of size $a \times a$, full row rank B_3 with ϕ rows, and full column rank C_3 with o columns.

Properties of the control problem

Definition

- A control problem of the given form is called consistent if there exists a (sufficiently smooth) input function w for which there exists a solution.
- It is called regular if it has a unique solution for every sufficiently smooth input function w and every initial value that is consistent for the system with this input function.

Corollary

If for a linear descriptor system the s-index is well defined and d, α, ϕ, o, u and v are constant, then

1. *The system is consistent if and only if either $v = 0$ or $\hat{f}_3 \equiv 0$. If $v \neq 0$ and $\hat{f}_3 \equiv 0$ then the equations in part (c) describe redundancies in the system that can be omitted* **(at least in exact arithmetic).**
2. *If the system is consistent and if $\phi = 0$, then for a given input function w, an initial condition is consistent if and only if it satisfies part (b). Solutions of the corresponding initial value problem will in general not be unique.*
3. *The system is regular if and only if it is consistent and $o = \phi = u = 0$.*

Corollary

Consider a descriptor system for which the s-index is well defined and let o, ϕ, a *be the characteristic quantities.*
There exists a state feedback $w = F(t)x$ *such that the closed loop system*

$$E(t)\dot{x} = (A(t) + B(t)F(t))x + f(t)$$

is regular and has s-index 0 *if and only if the system is consistent and* $u + o = \phi$ *(impulse controllability).*

Note that consistency is only needed if an inhomogeneity is present.

Corollary

Given a descriptor system for which the s-index is well defined. There exists an output feedback $w = F(t)y$ such that the closed loop system

$$E(t)\dot{x} = (A(t) + B(t)F(t)C(t))x + f(t)$$

*is regular and has s-index 0 (as a free system) if and only if the system is consistent, $u^r = 0$ and $\phi = o$ (***Impulse controllability and impulse observability***).*

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Condensed form at fixed point $\hat{t}$.

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 & C_3 & 0 \end{bmatrix}$$

with nonsingular A_{22} of size $a \times a$, full row rank B_3 with ϕ rows, and full column rank C_3 with o columns.

At $\hat{t} \in \mathbb{I}$ we obtain the variables that are free variables (controls) and fixed variables (states). Perform the compression

$$B_3(\hat{t})P_1 = \begin{bmatrix} B_{31} & 0 \end{bmatrix}, P_2C_3(\hat{t}) = \begin{bmatrix} C_{13} \\ 0 \end{bmatrix}$$

with nonsingular B_{31}, C_{13}. We obtain

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & 0 & 0 \end{bmatrix}$$

Set $T := \begin{bmatrix} Q_1 & Q_2V_1 & Q_2V_2W_1 & Q_2V_2W_2 \end{bmatrix}$,

$$\begin{aligned} \xi(\hat{t}) &= T\tilde{\xi} =: T \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \\ \tilde{\xi}_4 \end{bmatrix} \\ u(\hat{t}) &= P_1\tilde{u} =: \begin{bmatrix} P_{11} & P_{12} \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}, \\ y(\hat{t}) &= P_2\tilde{y} =: \begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix}. \end{aligned}$$

$$\begin{bmatrix} E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \\ \tilde{\xi}_4 \end{bmatrix}^{\cdot} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \\ \tilde{\xi}_4 \end{bmatrix}$$

$$+ \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ C_{21} & C_{22} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{\xi}_3 \\ \tilde{\xi}_4 \end{bmatrix}$$

$$\begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{u}_1 \end{bmatrix}^{\cdot} = \begin{bmatrix} A_{11} & A_{12} & B_{11} \\ A_{21} & A_{22} & B_{21} \\ 0 & 0 & B_{31} \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{u}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} A_{13} & A_{14} & B_{12} \\ 0 & 0 & B_{22} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_3 \\ \tilde{\xi}_4 \tilde{u}_2 \end{bmatrix}$$

$$\begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_1 \\ \tilde{\xi}_2 \\ \tilde{u}_1 \end{bmatrix} + \begin{bmatrix} C_{13} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi}_3 \\ \tilde{\xi}_4 \tilde{u}_2 \end{bmatrix}$$

Then $\tilde{\xi}_1, \tilde{\xi}_2, \tilde{u}_1$ correspond to the ‘real’ state variables, while $\tilde{u}_2, \tilde{\xi}_3, \tilde{\xi}_4$ correspond to the ‘real’ input variables.

Consequences

- Redundant equations can be removed and the consistency of the model can be verified.
- There may be original input variables that are constrained (states).
- There may be original state variables that are free (controls).
- We can re-interpret the meaning of variables, so that the resulting system is strangeness-free (this is not unique).
- The behavior approach is the only reasonable way to treat general DAE control problems, Ilchmann/M. 2005
- Similar results are available for the general nonlinear case, but much more technical, Kunkel/M. 2001.

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We discuss descriptor systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t), \qquad x(t_0) = x^0, \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t), \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t), \end{aligned}$$

where $E, A \in \mathbb{R}^{\ell,n}$, $B_i \in \mathbb{R}^{\ell,m_i}$, $C_i \in \mathbb{R}^{p_i,n}$, and $D_{ij} \in \mathbb{R}^{p_i,m_j}$ for $i, j = 1, 2$.

Definition

- A subspace $\mathcal{L} \subset \mathbb{R}^n$ is called deflating subspace for $\lambda E - A$ if for $X_\mathcal{L} \in \mathbb{R}^{n,k}$ with full column rank and $\mathrm{Im} X_\mathcal{L} = \mathcal{L}$ there exist matrices $Y_\mathcal{L} \in \mathbb{R}^{n,k}$, $R_\mathcal{L} \in \mathbb{R}^{k,k}$, $U_\mathcal{L} \in \mathbb{R}^{k,k}$ such that

$$EX_\mathcal{L} = Y_\mathcal{L} R_\mathcal{L}, \quad AX_\mathcal{L} = Y_\mathcal{L} U_\mathcal{L}.$$

- A deflating subspace $\mathcal{L}$ of $\lambda E - A$ is called stable (semi-stable) if all finite eigenvalues of $\lambda R_\mathcal{L} - U_\mathcal{L}$ are in the open (closed) left half plane.
- Let $\mathcal{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, where I_n is the $n \times n$ identity matrix. A subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ is called isotropic if $x^T \mathcal{J} y = 0$ for all $x, y \in \mathcal{L}$.
- An isotropic subspace with $\dim \mathcal{L} = n$ is called Lagrangian.

Necessary properties of the system

Definition

Let $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n,m}$ and $C \in \mathbb{R}^{p,n}$. Further, let T_∞, S_∞ be matrices with $\mathrm{Im}\, T_\infty = \ker E^T$ and $\mathrm{Im} S_\infty = \ker E$.

i) The triple (E, A, B) is called *finite dynamics stabilizable* if $\mathrm{rank}[\lambda E - A, B] = n$ for all $\lambda \in \mathbb{C}^+$;

ii) (E, A, B) is *impulse controllable* if $\mathrm{rank}[E, AS_\infty, B] = n$;

iii) (E, A, B) is *strongly stabilizable* if it is both finite dynamics stabilizable and impulse controllable;

iv) The triple (E, A, C) is *finite dynamics detectable* if $\mathrm{rank}[\lambda E^T - A^T, C^T] = n$ for all $\lambda \in \mathbb{C}^+$;

v) (E, A, C) is *impulse observable* if $\mathrm{rank}[E^T, A^T T_\infty, C^T] = n$;

vi) $(\lambda E - A, C)$ is *strongly detectable* if it is both finite dynamics detectable and impulse observable.

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Special case $w = 0,\ B_2 = B,\ C_2 = C,\ D = 0$.
Minimize the cost functional

$$\int_0^\infty x^T Q x + 2u^T S x + u^T R u \; dt$$

(with $Q = Q^T$ and $R = R^T$) subject to

$$\begin{aligned} E\dot{x} &= Ax + Bu,\ x(t_0) = x^0 \\ y &= Cx \end{aligned}$$

Theorem (M. 91)

Suppose that the system is strongly stabilizable and detectable. Let u_* define the minimal solution, let x_* be the solution of

$$E\dot{x}(t) = Ax(t) + Bu_*(t), \;\; x(t_0) = x^0$$

and let $y_*(t) = Cx_*(t)$. Then there exists a $\mu(t) \in \mathbb{R}^n$, such that $x_*(t), \mu(t), u_*(t)$ satisfy the DAE boundary value problem:

$$\begin{aligned} \mathbf{L}(x,\mu,u) &= \begin{bmatrix} 0 & -E^T & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{u}(t) \end{bmatrix} \\ &- \begin{bmatrix} C^TQC & A^T & C^TS \\ A & 0 & B \\ S^TC & B^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ \mu(t) \\ u(t) \end{bmatrix} = 0 \end{aligned}$$

with boundary conditions $x(t_0) = x^0$, $E^T\mu(T) = 0$.

The associated matrix pencil

$$L(\lambda) = \lambda N - M = \lambda \begin{bmatrix} 0 & -E^T & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} C^TQC & A^T & C^TS \\ A & 0 & B \\ S^TC & B^T & R \end{bmatrix}$$

is called an even pencil since $P(\lambda) = P^T(-\lambda)$.
The (formal) differential-algebraic operator $\mathcal{L}(\xi) = N\dot{\xi} - M\xi$ is self-adjoint in the inner product

$$(\xi, \phi) = \int_{t_0}^{T} \xi^T \phi dt$$

with appropriate spaces.

- If the even pencil $L(\lambda) = \lambda N - M$ is regular and of index 1, if $\operatorname{rank} E = r$, and if there exists a generalized Lagrangian subspace associated with the closed left half plane eigenvalues, then the optimal control is a feedback control acting only on the dynamical part.
- If E is invertible and R is singular, then the index of $L(\lambda)$ is bigger than 1. Then either a reformulation/remodelling of the system (index reduction) is necessary or one can also use a modified cost function.
- If E is singular then the cost function itself is not important but the part of M in the kernel of N.
- In general there is no Riccati equation analogue.

Reduced Euler Lagrange equations

If E and R are invertible then we obtain the equivalent reduced Euler-Lagrange system

$$\begin{bmatrix} \dot{x} \\ \dot{\mu} \end{bmatrix} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} \begin{bmatrix} x \\ \mu \end{bmatrix}, \; x(t_0) = x^0, \mu(\infty) = 0,$$

with the Hamiltonian matrix

$$\begin{bmatrix} F & G \\ H & -F^T \end{bmatrix} := \begin{bmatrix} E^{-1}(A - BR^{-1}S) & E^{-1}BR^{-1}B^T E^{-T} \\ Q - SR^{-1}S^T & -(E^{-1}(A - BR^{-1}S))^T \end{bmatrix}.$$

- In this case, theoretically one can go via Riccati equations but in finite precision arithmetic it is better to use the deflating subspace associated with the left half plane of $L(\lambda)$.
- Even pencils generalize Hamiltonian matrices.
- Analysis and methods for even pencils, Mehl 1998, Benner/Byers/M./Xu 2000

Outline

1. Recap from last lectures
2. Control systems
3. Feedback
4. Reinterpretation of variables
5. Control methods for DAEs
6. Optimal control
7. Optimal H_∞ control

Joint work with P. Losse, L. Poppe, T. Reis

- The space $\mathcal{H}_\infty^{p,m}$ consists of all $\mathbb{C}^{p,m}$-valued functions that are analytic and bounded in the complex half plane $\mathbb{C}^+ = \{s \in \mathbb{C} \,:\, \mathbf{R}(s) > 0\}$.
- For $F \in \mathcal{H}_\infty^{p,m}$ the $\mathcal{H}_\infty$-norm is given by

$$\|F\|_\infty = \sup_{s\in\mathbb{C}^+} \sigma_{\max}(F(s)),$$

  where $\sigma_{\max}(F(s))$ denotes the maximal singular value of $F(s)$.
- $\|F\|_\infty$ is used as a measure of the worst case influence of the disturbances w on the output z, where F is the transfer function mapping noise or disturbance inputs to error signals.
- Here only the square case $\ell = n$.

Definition (The optimal $\mathcal{H}_\infty$ control problem)

Determine a controller (dynamic compensator)

$$\mathbf{K}: \quad \begin{aligned} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\ u(t) &= \hat{C}\hat{x}(t) + \hat{D}y(t) \end{aligned}$$

with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$, and transfer function $K(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ such that

$$\begin{aligned} E\dot{x}(t) &= (A + B_2\hat{D}Z_1C_2)x(t) + (B_2Z_2\hat{C})\hat{x}(t) + (B_1 + B_2\hat{D}Z_1D_{21})w(t), \\ \hat{E}\dot{\hat{x}}(t) &= \hat{B}Z_1C_2x(t) + (\hat{A} + \hat{B}Z_1D_{22}\hat{C})\hat{x}(t) + \hat{B}Z_1D_{21}w(t), \\ z(t) &= (C_1 + D_{12}Z_2\hat{D}C_2)x(t) + D_{12}Z_2\hat{C}\hat{x}(t) + (D_{11} + D_{12}\hat{D}Z_1D_{21})w(t) \end{aligned}$$

with $Z_1 = (I_{p_2} - D_{22}\hat{D})^{-1}$ and $Z_2 = (I_{m_2} - \hat{D}D_{22})^{-1}$, satisfies

1.) The closed loop system is internally stable, that is, the solution $\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$ of the system with $w \equiv 0$ is asymptotically stable, i.e. $\lim_{t\to\infty} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = 0$.

2.) The closed-loop transfer function $T_{zw}(s)$ from w to z satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ and is minimized in the $\mathcal{H}_\infty$-norm.

Definition (The modified optimal $\mathcal{H}_\infty$ control problem.)

let Γ be the set of positive real numbers γ for which there exists an internally stabilizing dynamic controller of the form so that the transfer function $T_{zw}(s)$ of the closed loop system satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$. Determine $\gamma_{mo} = \inf \Gamma$. If no internally stabilizing dynamic controller exits, we set $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$.

It is possible that there is no internally stabilizing dynamic controller with the property $\|T_{zw}\|_\infty = \gamma_{mo}$. In this case one solves the suboptimal $\mathcal{H}_\infty$ control problem.

Definition (The suboptimal $\mathcal{H}_\infty$ control problem.)

For the given descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$, determine an internally stabilizing dynamic controller of the form such that the closed loop transfer function satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$. We call such a controller γ-suboptimal controller.

A1) The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable.

A2) $\operatorname{rank} \begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$.

A3) $\operatorname{rank} \begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$.

A4) For matrices $T_\infty, S_\infty \in \mathbb{R}^{n,n-r}$ with $\mathrm{Im}\, S_\infty = \ker E$ and $\mathrm{Im}\, T_\infty = \ker E^T$ we have

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} &= n + m_2 - r, \\ \operatorname{rank} \begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} &= n + p_1 - r. \end{aligned}$$

$$\lambda N_H + M_H(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E^T - A^T & 0 & 0 & -C_1^T \\ \lambda E - A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I_{m_1} & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I_{p_1} \end{array}\right]$$

$$\lambda N_J + M_J(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E - A & 0 & 0 & -B_1 \\ \lambda E^T - A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I_{p_1} & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{21} \\ -B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I_{m_1} \end{array}\right].$$

We compute deflating subspaces

$$X_H(\gamma) = \begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \\ X_{H,3}(\gamma) \\ X_{H,4}(\gamma) \\ X_{H,5}(\gamma) \end{bmatrix}, \quad X_J(\gamma) = \begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \\ X_{J,3}(\gamma) \\ X_{J,4}(\gamma) \\ X_{J,5}(\gamma) \end{bmatrix},$$

with

$$X_{H,1}(\gamma),\ X_{H,2}(\gamma), X_{J,1}(\gamma),\ X_{J,2}(\gamma) \in \mathbb{R}^{n,r}, X_{H,4}(\gamma) \in \mathbb{R}^{m_2,r},$$
$$X_{J,4}(\gamma) \in \mathbb{R}^{p_2,r}, \quad X_{H,3}(\gamma), X_{J,5}(\gamma) \in \mathbb{R}^{m_1,r}, X_{H,5}(\gamma), X_{J,3}(\gamma) \in \mathbb{R}^{p_1,r}.$$

C1) The index of the two even pencils is at most one.

C2) There exists a matrix $X_H(\gamma)$ such that

C2.a) the space $\mathrm{Im} X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H(\gamma)$ and $\mathrm{Im} \begin{bmatrix} EX_{H,1} \\ X_{H,2} \end{bmatrix}$ is an r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;

C2.b) $\mathrm{rank}\, EX_{H,1}(\gamma) = r$.

C3) There exists a matrix $X_J(\gamma)$ such that

C3.a) the space $\mathrm{Im} X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J(\gamma)$ and $\mathrm{Im} \begin{bmatrix} E^T X_{J,1} \\ X_{J,2} \end{bmatrix}$ is an r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;

C3.b) $\mathrm{rank}\, E^T X_{J,1}(\gamma) = r$.

Definition

Consider the even pencils
$\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Define the sets

$$\Gamma_H = \{\gamma \in \mathbb{R}^+ \mid \text{the index of } \lambda N_H + M_H(\gamma) \text{ is greater than one}\},$$
$$\Gamma_J = \{\gamma \in \mathbb{R}^+ \mid \text{the index of } \lambda N_J + M_J(\gamma) \text{ is greater than one}\},$$

and set $\hat{\gamma}_H = \sup \Gamma_H$, $\hat{\gamma}_J = \sup \Gamma_J$ and $\hat{\gamma} = \max\{\hat{\gamma}_H, \hat{\gamma}_J\}$.

- In general the sets Γ_H and Γ_J may be all of $\mathbb{R}^+$, but it follows from the assumptions **A1) – A4)** that $\hat{\gamma}_H$ and $\hat{\gamma}_J$ and therefore also $\hat{\gamma}$ are finite.
- If $\gamma > \hat{\gamma}$ then these pencils have $2r$ finite eigenvalues, where $r = \operatorname{rank} E$.

Definition

Consider the even pencils $\lambda N_H + M_H(\gamma)$ in and $\lambda N_J + M_J(\gamma)$. Define

$$
\begin{aligned}
\Gamma_H^L &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_H + M_H(\gamma) \text{ satisfies } \textbf{C2.a)}\}, \\
\Gamma_J^L &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_J + M_J(\gamma) \text{ satisfies } \textbf{C3.a)}\}, \\
\Gamma^L &= \Gamma_J^L \cap \Gamma_H^L, \\
\Gamma_H^R &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_H + M_H(\gamma) \text{ satisfies condition } \textbf{C2)}\}, \\
\Gamma_J^R &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_J + M_J(\gamma) \text{ satisfies condition } \textbf{C3)}\}, \\
\Gamma^R &= \Gamma_J^R \cap \Gamma_H^R
\end{aligned}
$$

$$
\begin{aligned}
\hat{\gamma}_H^L &= \inf \Gamma_H^L, \hat{\gamma}_J^L = \inf \Gamma_J^L, \ \hat{\gamma}^L = \inf \Gamma^L, \\
\hat{\gamma}_H^R &= \inf \Gamma_H^R, \ \hat{\gamma}_J^R = \inf \Gamma_J^R, \hat{\gamma}^R = \inf \Gamma^R.
\end{aligned}
$$

Definition

Consider a system satisfying **A1**-**A4** and the associated even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Define the sets

$$\begin{aligned}\Gamma_H^I &= \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \text{the pencil } \lambda N_H + M_H(\gamma) \text{ has at least one finite eigenvalue on the imaginary axis} \right\}, \\ \Gamma_J^I &= \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \text{the pencil } \lambda N_J + M_J(\gamma) \text{ has at least one finite eigenvalue on the imaginary axis} \right\}, \\ \Gamma^I &= \Gamma_J^I \cap \Gamma_H^I\end{aligned}$$

and set

$$\hat{\gamma}_H^I = \inf \Gamma_H^I, \quad \hat{\gamma}_J^I = \inf \Gamma_J^I, \quad \hat{\gamma}^I = \inf \Gamma^I.$$

Theorem

Consider a system of the form satisfying assumptions **A1) – A4)**. *Let $X_H(\gamma)$ and $X_J(\gamma)$ be deflating subspace matrices of the form that satisfy conditions* **C2)** *and* **C3)**), *respectively. Then there exist parameters $\hat{\gamma}_H^k \geq \hat{\gamma}_H^L$, $\hat{\gamma}_J^k \geq \hat{\gamma}_J^L$ and $\hat{k}_H, \hat{k}_J \in \mathbb{N}$ with the property that for all $\gamma_{H,1}, \gamma_{H,2} > \hat{\gamma}_H^k$, $\gamma_{J,1}, \gamma_{J,2} > \hat{\gamma}_J^k$ the rank conditions hold:*

$$\begin{aligned} \operatorname{rank} E^T X_{H,2}(\gamma_{H,1}) &= \operatorname{rank} E^T X_{H,2}(\gamma_{H,2}) = \hat{k}_H, \\ \operatorname{rank} E X_{J,2}(\gamma_{J,1}) &= \operatorname{rank} E X_{J,2}(\gamma_{J,2}) = \hat{k}_J. \end{aligned}$$

C4) The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} -\gamma X_{H,2}^T(\gamma) E X_{H,1}(\gamma) & X_{H,2}^T(\gamma) E X_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) E^T X_{H,2}(\gamma) & -\gamma X_{J,2}^T(\gamma) E^T X_{J,1}(\gamma) \end{bmatrix} \qquad (1)$$

is symmetric, positive semi-definite and satisfies $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.

Definition

Consider a system that satisfies assumptions **A1) – A4)**. Then we define

$$\Gamma^\rho = \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l} \text{the matrix } \mathcal{Y}(\gamma) \text{ is positive semi-} \\ \text{definite} \\ \text{with } \operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J \end{array} \right\}$$

and we set $\hat{\gamma}^\rho := \inf \Gamma^\rho$.

Optimality Theorem

Theorem (Losse, M., Poppe, Reis 2008)

Consider a descriptor system the associated even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Suppose that assumptions **A1) – A4)** *hold.*
Then there exists an internally stabilizing controller such that the transfer function from w to z satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$ if and only if γ is such that the conditions **C1) – C4)** *hold.*
Furthermore, the set of γ satisfying the conditions **C1) – C4)** *is nonempty.*

Classification of optimal γ

Algorithm **Input:** Data of system, value $\gamma \geq 0$.
Output: Decision whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

1. Form the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$.
2. Compute the deflating subspace matrices X_H and X_J associated with the eigenvalues in the closed left half plane.
3. IF the dimension of one/both of these subspaces is less than r, then $\gamma < \gamma_{mo}$,

   ELSE

   IF the rank of $EX_{H,1}$ and/or $E^T X_{J,1}$ is less than r, then $\gamma < \gamma_{mo}$,
   ELSE

   Form the matrix $\hat{\mathcal{Y}}$.
   IF $\hat{\mathcal{Y}}$ is not positive semi-definite and/or $\operatorname{rank} \hat{\mathcal{Y}} < \hat{k}_H + \hat{k}_J$, then $\gamma < \gamma_{mo}$,
   ELSE $\gamma \geq \gamma_{mo}$.

To determine γ_{mo} one then uses a bisection method.

Remarks

- There exist explicit formulas for the optimal controllers, Losse 2010.
- Extensions of H_∞ control to time-varying or nonlinear DAEs are open.
- Other classical control techniques, like stabilization, tracking control, etc.
- Extensions of optimal control to time-varying or nonlinear DAEs, next lecture.