Differential-algebraic equations. Control and Numerics IV

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Optimal control problem:

$$\mathcal{J}(x,u) = \mathcal{M}(x(\bar{t})) + \int_{\underline{t}}^{\bar{t}} \mathcal{K}(t,x,u)\, dt = \text{min!}$$

subject to a DAE constraint

$$F(t,x,u,\dot{x}) = 0,\ \ x(\underline{t}) = \underline{x}.$$

x–state, u–input, y–output.

Linear quadratic optimal control

Cost functional:

$$\mathcal{J}(x,u) = \frac{1}{2}x(\bar{t})^T M x(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} (x^T W x + 2x^T S u + u^T R u)\, dt,$$

$W = W^T \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $S \in C^0(\mathbb{I}, \mathbb{R}^{n,l})$, $R = R^T \in C^0(\mathbb{I}, \mathbb{R}^{l,l})$, $M = M^T \in \mathbb{R}^{n,n}$.

Constraint:

$$E(t)\dot{x} = A(t)x + B(t)u + f, \quad x(\underline{t}) = \underline{x},$$

$E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $A \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B \in C^0(\mathbb{I}, \mathbb{R}^{n,l})$, $f \in C^0(\mathbb{I}, \mathbb{R}^{n})$, $\underline{x} \in \mathbb{R}^n$.

Here: Determine optimal controls $u \in \mathbb{U} = C^0(\mathbb{I}, \mathbb{R}^l)$.,

More general spaces and also nonsquare E, A are possible.

Classical linear quadratic optimal control

Special case $w = 0,\ B_2 = B,\ C_2 = C,\ D = 0$.
Minimize the cost functional

$$\int_0^\infty y^T Q y + 2u^T S y + u^T R u \, dt$$

(with $Q = Q^T$ and $R = R^T$) subject to

$$\begin{aligned} E\dot{x} &= Ax + Bu,\ x(t_0) = x^0 \\ y &= Cx \end{aligned}$$

$$
\mathbf{L}(x,\mu,u) = \begin{bmatrix} 0 & -E^T & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\mu}(t) \\ \dot{u}(t) \end{bmatrix} - \begin{bmatrix} C^TQC & A^T & C^TS \\ A & 0 & B \\ S^TC & B^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ \mu(t) \\ u(t) \end{bmatrix} = 0
$$

with boundary conditions $x(t_0) = x^0$, $E^T\mu(T) = 0$.

The associated matrix pencil

$$L(\lambda) = \lambda N - M = \lambda \begin{bmatrix} 0 & -E^T & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} C^TQC & A^T & C^TS \\ A & 0 & B \\ S^TC & B^T & R \end{bmatrix}$$

is called an even pencil since $P(\lambda) = P^T(-\lambda)$.
The (formal) differential-algebraic operator $\mathcal{L}(\xi) = N\dot{\xi} - M\xi$ is self-adjoint in the inner product

$$(\xi, \phi) = \int_{t_0}^{T} \xi^T \phi dt$$

with appropriate spaces.

Outline

How does one do this today?

- Simplified models.
- Space discretization with very coarse meshes.
- Identification and realization of black box models.
- Model reduction (mostly based on heuristic methods).
- Coupling of simulation packages.
- Use of standard optimal control techniques for simplified mathematical model.
- But do they work for these models?

Why not just apply the Pontryagin maximum principle?

- For linear ODEs the initial value problem has a unique solution $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ for every $u \in \mathbb{U}$, every $f \in C^0(\mathbb{I}, \mathbb{R}^n)$, and every initial value $\underline{x} \in \mathbb{R}^n$.
- DAEs, where $E(t)$ may be singular, may not be (uniquely) solvable for any $u \in \mathbb{U}$ and also the initial conditons may be restricted.
- Furthermore, we need solutions $x \in \mathbb{X}$, where $\mathbb{X}$ usually is a larger space than $C^1(\mathbb{I}, \mathbb{R}^n)$.

Previous work

- Linear constant coefficient index 1 case, Bender/Laub 87, Campbell 87 M. 91, Geerts 93.
- Regularization to index 1, Bunse-Gerstner/M./Nichols 92, 94, Byers/Geerts/M. 97, Byers/Kunkel/M. 97.
- Linear variable coefficients index 1 case, Kunkel./M. 97.
- Semi-explicit nonlinear index 1 case, maximum principle, De Pinho/Vinter 97, Devdariani/Ledyaev 99.
- Semi-explicit index $2, 3$ case Roubicek/Valasek 02.
- Linear index $1, 2$ case with properly stated leading term, Balla/März, 02,04, Balla/Linh 05, Kurina/März 04, Backes 06.
- Multibody systems (structured index 3), Büskens/Gerdts 00, Gerdts 03,06.

Introduce Lagrange multiplier function $\lambda(t)$ and couple constraint into cost function, i.e. minimize

$$\begin{aligned}\tilde{\mathcal{J}}(x,u,\lambda) &= \frac{1}{2}x(\bar{t})^T Mx(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} (x^T Wx + 2x^T Su + u^T Ru) \\ &+ \lambda^T(\dot{x} - Ax + Bu + f)\, dt.\end{aligned}$$

Consider $x + \delta x$, $u + \delta u$ and $\lambda + \delta\lambda$.
For a minimum the cost function has to go up in the neighborhood, so we get optimality conditions (Euler-Lagrange equations):

Optimality system

Theorem If (x, u) is a solution to the optimal control problem, then there exists a Lagrange multiplier function $\lambda \in C^1(\mathbb{I}, \mathbb{R}^n)$, such that (x, λ, u) satisfy the *optimality boundary value problem*

$$
\begin{aligned}
&\text{(a)} \quad \dot{x} = Ax + Bu + f, \;\; x(\underline{t}) = \underline{x}, \\
&\text{(b)} \quad \dot{\lambda} = Wx + Su - A^T\lambda, \;\; \lambda(\bar{t}) = -Mx(\bar{t}), \\
&\text{(c)} \quad 0 = S^T x + Ru - B^T\lambda.
\end{aligned}
$$

Replace the identity in front of x by $E(t)$ and then do the analysis in the same way.
For DAEs the formal optimality system then could be

$$\begin{array}{lrl} \text{(a)} & E\dot{x} & = Ax + Bu + f,\ x(\underline{t}) = \underline{x} \\ \text{(b)} & \frac{d}{dt}(E^T\lambda) & = Wx + Su - A^T\lambda,\ (E^T\lambda)(\bar{t}) = -Mx(\bar{t}), \\ \text{(b)} & 0 & = S^Tx + Ru - B^T\lambda. \end{array}$$

This works if the system has strangeness-index $\mu = 0$ as a free system with $u = 0$ but not in general.

What are the difficulties

- In the proof one has to guarantee that the resulting adjoint equation for λ has a unique solution.
- One needs a density argument in the solution space.
- But, the formal adjoint equation may not have a (unique) solution.
- The formal boundary conditions may not be consistent.
- The solution of the optimality systems may not exist or may not be unique.

Consider

$$\mathcal{J}(x,u) = \frac{1}{2}\int_0^1 (x_1^2 + u^2)dt = \min!$$

subject to the differential-algebraic system

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u + \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

A simple calculation yields the optimal solution

$$x_1 = u = \lambda_1 = -\tfrac{1}{2}(f_1 + \dot{f}_2), \quad x_2 = -f_2, \quad \lambda_2 = 0.$$

In the formal optimality system we get

$$x_1 = u = \lambda_1 = -\tfrac{1}{2}(f_1 + \dot{f}_2), \quad x_2 = -f_2, \quad \lambda_2 = -\tfrac{1}{2}(\dot{f}_1 + \ddot{f}_2)$$

without using the initial condition $\lambda_1(1) = 0$.

- Depending on the data, this initial condition may be consistent or not.
- This initial condition should not be present.
- Moreover, λ_2 requires more smoothness of the inhomogeneity than in the optimal solution.

Further examples, see Dissertation Backes 06.

To derive optimality conditions for DAEs, we need the right solution space for x:

$$\mathbb{X} = C^1_{E^+E}(\mathbb{I}, \mathbb{R}^n) = \left\{ x \in C^0(\mathbb{I}, \mathbb{R}^n) \mid E^+Ex \in C^1(\mathbb{I}, \mathbb{R}^n) \right\},$$

where E^+ denotes the Moore-Penrose inverse of the matrix valued function, which is the unique matrix function that satisfies the Penrose axioms.

$$EE^+E = E, \quad E^+EE^+ = E^+, \quad (EE^+)^T = EE^+, \quad (E^+E)^T = E^+E$$

$\mathbb{U}$ is usually a set of piecewise continuous functions or even distributions.

Necessary optimality condition

Theorem Consider the linear quadratic optimal control problem with a consistent initial condition. Suppose that the system has $\mu = 0$ as a behavior system and that $Mx(\bar{t}) \in \text{cokernel}\, E(\bar{t})$.
If $(x, u) \in \mathbb{X} \times \mathbb{U}$ is a solution to this optimal control problem, then there exists a Lagrange multiplier function $\lambda \in C^1_{E^+E}(\mathbb{I}, \mathbb{R}^n)$, such that (x, λ, u) satisfy the optimality boundary value problem

$$
\begin{aligned}
&E\tfrac{d}{dt}(E^+Ex) = (A + E\tfrac{d}{dt}(E^+E))x + Bu + f,\ (E^+Ex)(\underline{t}) = \underline{x},\\
&E^T\tfrac{d}{dt}(EE^+\lambda) = Wx + Su - (A + EE^+\dot{E})^T\lambda,\ (EE^+\lambda)(\bar{t}) = -E^+(\bar{t})^T\\
&0 = S^Tx + Ru - B^T\lambda.
\end{aligned}
$$

For this we need some kind of convexity.

Theorem Consider the optimal control problem with a consistent initial condition and suppose that in the cost functional we have that

$$\begin{bmatrix} W & S \\ S^T & R \end{bmatrix}, \ M$$

are (pointwise) positive semidefinite. If (x^*, u^*, λ) satisfies the formal optimality system then for any (x, u) satisfying the constraint we have

$$\mathcal{J}(x, u) \geq \mathcal{J}(x^*, u^*).$$

Remarks

- If a minimum exists, then it satisfies the optimality system.
- If a unique solution to the formal optimality system exists, then x, u are the same as from the optimality system, λ may be different.
- To get a sufficient condition, we need some extra convexity.
- The optimality DAE may have $\mu > 0$. Then it is numerically difficult to solve and further consistency conditions or smoothness requirements arise.
- The condition that the original system has $\mu = 0$ as a behavior system is not necessary if the cost function is chosen appropriately, so that the resulting optimality system has $\mu = 0$.

Differential-algebraic Riccati equations

If R in the cost functional is invertible, and if the system has $\mu = 0$ as a free system with $u = 0$, then one can (at least in theory) apply the usual Riccati approach to

$$
\begin{aligned}
&E\tfrac{d}{dt}(E^+Ex) = (A + E\tfrac{d}{dt}(E^+E))x + Bu + f, \ (E^+Ex)(\underline{t}) = \underline{x},\\
&E^T\tfrac{d}{dt}(EE^+\lambda) = Wx + Su - (A + EE^+\dot{E})^T\lambda, \ (EE^+\lambda)(\bar{t}) = -E^+(\bar{t})^T\\
&0 = S^Tx + Ru - B^T\lambda.
\end{aligned}
$$

If $\mu > 0$ or R is singular, then the Riccati approach may not work, even if the boundary value problem has a unique solution.

Outline

For nonlinear systems $F(t,x,\dot{x}) = 0$ one considers nonlinear derivative arrays:

$$0 = F_k(t,x,\dot{x},\ldots,x^{(k+1)}) = \begin{bmatrix} F(t,x,\dot{x}) \\ \frac{d}{dt}F(t,x,\dot{x}) \\ \cdots \\ \frac{d^k}{dt^k}F(t,x,\dot{x}) \end{bmatrix}.$$

We set

$$\begin{aligned} M_k(t,x,\dot{x},\ldots,x^{(k+1)}) &= F_{k;\dot{x},\ldots,x^{(k+1)}}(t,x,\dot{x},\ldots,x^{(k+1)}), \\ N_k(t,x,\dot{x},\ldots,x^{(k+1)}) &= -(F_{k;x}(t,x,\dot{x},\ldots,x^{(k+1)}),0,\ldots,0), \\ z_k &= (t,x,\dot{x},\ldots,x^{(k+1)}). \end{aligned}$$

Hypothesis: There exist integers μ, r, a, d, and v such that $\mathbf{L} = F_\mu^{-1}(\{0\}) \neq \emptyset$.

We have $\text{rank}\, F_{\mu;t,x,\dot{x},\dots,x^{(\mu+1)}} = \text{rank}\, F_{\mu;x,\dot{x},\dots,x^{(\mu+1)}} = r$, in a neighborhood of **L** such that there exists an equivalent system $\tilde{F}(z_\mu) = 0$ with a Jacobian of full row rank r. On **L** we have

1. $\text{corank}\, F_{\mu;x,\dot{x},\dots,x^{(\mu+1)}} - \text{corank}\, F_{\mu-1;x,\dot{x},\dots,x^{(\mu+1)}} = v$.
2. $\text{corank}\, \tilde{F}_{x,\dot{x},\dots,x^{(\mu+1)}} = a$ and there exist smooth matrix functions Z_2 (left nullspace of M_μ) and T_2 (right nullspace of $\hat{A}_2 = \tilde{F}_x$) with $Z_2^T \tilde{F}_{x,\dot{x},\dots,x^{(\mu+1)}} = 0$ and $Z_2^T \hat{A}_2 T_2 = 0$.
3. $\text{rank}\, F_{\dot{x}} T_2 = d, \quad d = m - a - v$, and there exists a smooth matrix function Z_1 with $\text{rank}\, Z_1^T F_{\dot{x}} = d$.

Theorem Kunkel/M. 2002 The solution set **L** forms a (smooth) manifold of dimension $(\mu + 2)n + 1 - r$.
The DAE can locally be transformed (by application of the implicit function theorem) to a reduced DAE of the form

$$\begin{aligned} \dot{x}_1 &= G_1(t, x_1, x_3), && (d \text{ differential equations}), \\ x_2 &= G_2(t, x_1, x_3), && (a \text{ algebraic equations}), \\ 0 &= 0 && (v \text{ redundant equations}). \end{aligned}$$

The variables x_3 represent undetermined components (controls).

Optimality conditions

Assume that $\mu = 0$ for the system in behavior form with $z = (x, u)$, then in terms of the reduced DAE, the local optimality system is

$$
\begin{array}{ll}
\text{(a)} & \dot{x}_1 = \mathcal{L}(t, x_1, u), \; x_1(\underline{t}) = \underline{x}_1, \\
\text{(b)} & x_2 = \mathcal{R}(t, x_1, u), \\
\text{(c)} & \dot{\lambda}_1 = \mathcal{K}_{x_1}(t, x_1, x_2, u)^T - \mathcal{L}_{x_1}(t, x_1, x_2, u)^T \lambda_1 - \mathcal{R}_{x_1}(t, x_1, u)^T \lambda_1 \\
 & \qquad \lambda_1(\bar{t}) = -\mathcal{M}_{x_1}(x_1(\bar{t}), x_2(\bar{t}))^T \\
\text{(d)} & 0 = \mathcal{K}_{x_2}(t, x_1, x_2, u)^T + \lambda_2, \\
\text{(e)} & 0 = \mathcal{K}_u(t, x_1, x_2, u)^T - \mathcal{L}_u(t, x_1, u)^T \lambda_1 - \mathcal{R}_u(t, x_1, u)^T \lambda_2, \\
\text{(f)} & \gamma = \lambda_1(\underline{t})
\end{array}
$$

Here λ_1, λ_2 are Lagrange multipliers associated with x_1, x_2 and γ is associated with the initial value constraint.

Remarks

- These are local results.
- All the results can be generalized to general nonsquare nonlinear systems.
- End point conditions for x can be included.
- Input and state constraints can be included to give a maximum principle.

The linear optimality system that we would have to solve numerically has the form

$$
\begin{array}{ll}
\text{(a)} & \hat{E}_1 \dot{x} = \hat{A}_1 x + \hat{B}_1 u + \hat{f}_1, \ (\hat{E}_1^+ \hat{E}_1 x)(\underline{t}) = \underline{x} \\
\text{(b)} & 0 = \hat{A}_2 x + \hat{B}_2 u + \hat{f}_2, \\
\text{(c)} & \frac{d}{dt}(\hat{E}_1^T \lambda_1) = Wx + Su - \hat{A}_1^T \lambda_1 - \hat{A}_2^T \lambda_2, \\
 & \qquad \lambda_1(\bar{t}) = -[\ \hat{E}_1^+(\bar{t})^T \ \ 0\ ] M x(\bar{t}), \\
\text{(d)} & 0 = S^T x + Ru - \hat{B}_1^T \lambda_1 - \hat{B}_2^T \lambda_2.
\end{array}
$$

where E_i, A_i, B_i are obtained by projection with smooth orthogonal projections Z_i from the derivative array.
An analogous structure arises locally in the nonlinear case.

Numerical Problems

- In the implementation of the numerical integration codes we use nonsmooth projectors Z_1^T, Z_2^T.
- It would be too expensive to carry smooth projectors along.
- For numerical forward (in time) simulation, it is enough that we know the existence of smooth projectors.
- Integration methods like Runge-Kutta or BDF don't see the nonsmooth behavior.
- But the adjoint variables (Lagrange multipliers) depend on these projections and their derivatives.

However, even if Z_1^T, Z_2^T are nonsmooth, $Z_1 Z_1^T$ and $Z_2 Z_2^T$ are smooth.

Smooth optimality system

- Choose

$$\hat{E}_1^T \lambda_1 = E^T Z_1 \lambda_1 = E^T Z_1 Z_1^T Z_1 \lambda_1 = E^T Z_1 Z_1^T \hat{\lambda}_1.$$

- With $\hat{\lambda}_1 = Z_1 \lambda_1$ we obtain smooth coefficients for $\hat{\lambda}_1$.
- However, we have to add the condition that $\hat{\lambda}_1 \in \operatorname{range} Z_1$ to the system.
- If Z_i' completes Z_i to a full orthogonal matrix (we compute these anyway when doing a QR or SVD computation) then these conditions can be expressed as

$$Z_i'^T \hat{\lambda}_i = 0, \ i = 1, 2$$

New linear optimality system

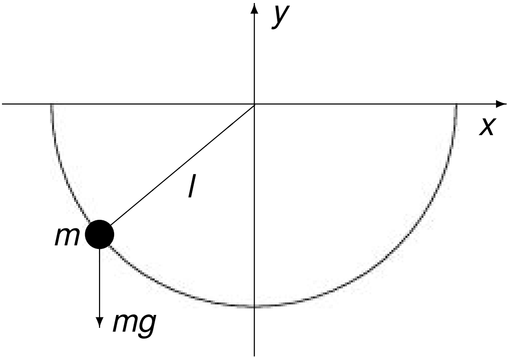
For the numerical solution we use the optimality system.

$$
\begin{array}{ll}
\text{(a)} & \hat{E}_1 \dot{x} = \hat{A}_1 x + \hat{B}_1 u + \hat{f}_1, \ (\hat{E}_1^+ \hat{E}_1 x)(\underline{t}) = \underline{x}, \\
\text{(b)} & 0 = \hat{A}_2 x + \hat{B}_2 u + \hat{f}_2, \\
\text{(c)} & \frac{d}{dt}(E^T Z_1 Z_1^T \hat{\lambda}_1) = Wx + Su - A^T \hat{\lambda}_1 - [\, I_n \ 0 \,|\, 0 \ 0 \,|\, \cdots \,|\, 0 \ 0 \,]N \\
& \qquad (Z_1^T \hat{\lambda}_1)(\bar{t}) = -[\, \hat{E}_1^+(\bar{t})^T \ 0 \,] M x(\bar{t}), \\
\text{(d)} & 0 = S^T x + Ru - B^T \hat{\lambda}_1 - [\, 0 \ I_l \,|\, 0 \ 0 \,|\, \cdots \,|\, 0 \ 0 \,] N_\mu^T \hat{\lambda}_2 \\
\text{(e)} & 0 = {Z_1'}^T \hat{\lambda}_1, \\
\text{(f)} & 0 = {Z_2'}^T \hat{\lambda}_2.
\end{array}
$$

All quantities are available for all time steps.
An analogous system can be derived for each Gauss-Newton step in the nonlinear case.

Numerical Example

A motor controlled pendulum with a motor in the origin shall be driven into its equilibrium with minimal costs, ex. from Büskens/Gerdts 2002.



The model problem for this reads

$$
\begin{array}{llll}
 & J(x,u) = & \int_0^3 u(t)^2\,dt = \min! & \\
\text{s.t.} & \dot{x}_1 = & x_3, & x_1(0) = \frac{1}{2}\sqrt{2}, \\
 & \dot{x}_2 = & x_4, & x_2(0) = -\frac{1}{2}\sqrt{2}, \\
 & \dot{x}_3 = & -2x_1x_5 + x_2u, & x_3(0) = 0, \\
 & \dot{x}_4 = & -g - 2x_2x_5 - x_1u, & x_4(0) = 0, \\
 & 0 = & x_1^2 + x_2^2 - 1, & x_5(0) = -\frac{1}{2}gx_2(0).
\end{array}
$$

The DAE satisfies the Hypothesis with $\mu = 2$, $a = 3$, $d = 2$, and $v = 0$. Hence, only two scalar initial values are sufficient to describe the initial state.
We chose them to be $x_2(0) = -\frac{1}{2}\sqrt{2}$ and $x_3(0) = 0$.
Similarly, $x_1(3) = 0$ and $x_3(0) = 0$ are sufficient to describe the equilibrium at the end point.
As initial trajectory we took

$$\begin{aligned} x_1(t) &= \tfrac{1}{2}\sqrt{2} - \tfrac{1}{6}\sqrt{2}t, \qquad & x_3(t) &= 0, \\ x_2(t) &= -\sqrt{1 - x_1(t)^2}, \qquad & x_4(t) &= 0, \qquad & x_5(t) &= -\tfrac{1}{2}g x_2(t). \end{aligned}$$

with all other unknowns set to zero on a equidistant grid of 60 intervals.

- Tolerance for the Gauß-Newton method was 10^{-7}.
- Let k count the iterations and $\|\Delta w_k\|_2$ denote the Euclidian norm of the corresponding Gauß-Newton correction.

k	$\|\Delta w_k\|_2$
1	0.140D+03
2	0.223D+03
⋮	⋮
17	0.103D+01
18	0.610D-02
19	0.318D-06
20	0.966D-11

- Initial bad convergence is due to a bad initial guess.
- Final value of cost function is $J_{opt} = 3.82$ which is correct up to discretization and roundoff errors.

Outline

Joint work with P. Losse, L. Poppe, T. Reis

- The space $\mathcal{H}_\infty^{p,m}$ consists of all $\mathbb{C}^{p,m}$-valued functions that are analytic and bounded in the complex half plane $\mathbb{C}^+ = \{s \in \mathbb{C} \,:\, \mathbf{R}(s) > 0\}$.
- For $F \in \mathcal{H}_\infty^{p,m}$ the $\mathcal{H}_\infty$-norm is given by

$$\|F\|_\infty = \sup_{s\in\mathbb{C}^+} \sigma_{\max}(F(s)),$$

  where $\sigma_{\max}(F(s))$ denotes the maximal singular value of $F(s)$.
- $\|F\|_\infty$ is used as a measure of the worst case influence of the disturbances w on the output z, where F is the transfer function mapping noise or disturbance inputs to error signals.
- Here only the square case $\ell = n$.

Definition (The optimal $\mathcal{H}_\infty$ control problem)

Determine a controller (dynamic compensator)

$$\mathbf{K}: \quad \begin{aligned} \hat{E}\dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\ u(t) &= \hat{C}\hat{x}(t) + \hat{D}y(t) \end{aligned}$$

with $\hat{E}, \hat{A} \in \mathbb{R}^{N,N}$, $\hat{B} \in \mathbb{R}^{N,p_2}$, $\hat{C} \in \mathbb{R}^{m_2,N}$, $\hat{D} \in \mathbb{R}^{m_2,p_2}$, and transfer function $K(s) = \hat{C}(s\hat{E} - \hat{A})^{-1}\hat{B} + \hat{D}$ such that

$$\begin{aligned} E\dot{x}(t) &= (A + B_2\hat{D}Z_1C_2)x(t) + (B_2Z_2\hat{C})\hat{x}(t) + (B_1 + B_2\hat{D}Z_1D_{21})w(t), \\ \hat{E}\dot{\hat{x}}(t) &= \hat{B}Z_1C_2x(t) + (\hat{A} + \hat{B}Z_1D_{22}\hat{C})\hat{x}(t) + \hat{B}Z_1D_{21}w(t), \\ z(t) &= (C_1 + D_{12}Z_2\hat{D}C_2)x(t) + D_{12}Z_2\hat{C}\hat{x}(t) + (D_{11} + D_{12}\hat{D}Z_1D_{21})w(t) \end{aligned}$$

with $Z_1 = (I_{p_2} - D_{22}\hat{D})^{-1}$ and $Z_2 = (I_{m_2} - \hat{D}D_{22})^{-1}$, satisfies

1.) The closed loop system is internally stable, that is, the solution $\begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$ of the system with $w \equiv 0$ is asymptotically stable, i.e. $\lim_{t\to\infty} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} = 0$.

2.) The closed-loop transfer function $T_{zw}(s)$ from w to z satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ and is minimized in the $\mathcal{H}_\infty$-norm.

Definition (The modified optimal $\mathcal{H}_\infty$ control problem.)

let Γ be the set of positive real numbers γ for which there exists an internally stabilizing dynamic controller of the form so that the transfer function $T_{zw}(s)$ of the closed loop system satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$. Determine $\gamma_{mo} = \inf \Gamma$. If no internally stabilizing dynamic controller exits, we set $\Gamma = \emptyset$ and $\gamma_{mo} = \infty$.

It is possible that there is no internally stabilizing dynamic controller with the property $\|T_{zw}\|_\infty = \gamma_{mo}$. In this case one solves the suboptimal $\mathcal{H}_\infty$ control problem.

Definition (The suboptimal $\mathcal{H}_\infty$ control problem.)

For the given descriptor system and $\gamma \in \Gamma$ with $\gamma > \gamma_{mo}$, determine an internally stabilizing dynamic controller of the form such that the closed loop transfer function satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$. We call such a controller γ-suboptimal controller.

A1) The triple (E, A, B_2) is strongly stabilizable and the triple (E, A, C_2) is strongly detectable.

A2) $\operatorname{rank}\begin{bmatrix} A - i\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2$ for all $\omega \in \mathbb{R}$.

A3) $\operatorname{rank}\begin{bmatrix} A - i\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2$ for all $\omega \in \mathbb{R}$.

A4) For matrices $T_\infty, S_\infty \in \mathbb{R}^{n,n-r}$ with $\operatorname{Im} S_\infty = \ker E$ and $\operatorname{Im} T_\infty = \ker E^T$ we have

$$\begin{aligned} \operatorname{rank}\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_2 \\ C_1 S_\infty & D_{12} \end{bmatrix} &= n + m_2 - r, \\ \operatorname{rank}\begin{bmatrix} T_\infty^T A S_\infty & T_\infty^T B_1 \\ C_2 S_\infty & D_{21} \end{bmatrix} &= n + p_1 - r. \end{aligned}$$

$$\lambda N_H + M_H(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E^T - A^T & 0 & 0 & -C_1^T \\ \lambda E - A & 0 & -B_1 & -B_2 & 0 \\ \hline 0 & -B_1^T & -\gamma^2 I_{m_1} & 0 & -D_{11}^T \\ 0 & -B_2^T & 0 & 0 & -D_{12}^T \\ -C_1 & 0 & -D_{11} & -D_{12} & -I_{p_1} \end{array}\right]$$

$$\lambda N_J + M_J(\gamma) = \left[\begin{array}{cc|ccc} 0 & -\lambda E - A & 0 & 0 & -B_1 \\ \lambda E^T - A^T & 0 & -C_1^T & -C_2^T & 0 \\ \hline 0 & -C_1 & -\gamma^2 I_{p_1} & 0 & -D_{11} \\ 0 & -C_2 & 0 & 0 & -D_{21} \\ -B_1^T & 0 & -D_{11}^T & -D_{21}^T & -I_{m_1} \end{array}\right].$$

We compute deflating subspaces

$$X_H(\gamma) = \begin{bmatrix} X_{H,1}(\gamma) \\ X_{H,2}(\gamma) \\ X_{H,3}(\gamma) \\ X_{H,4}(\gamma) \\ X_{H,5}(\gamma) \end{bmatrix}, \quad X_J(\gamma) = \begin{bmatrix} X_{J,1}(\gamma) \\ X_{J,2}(\gamma) \\ X_{J,3}(\gamma) \\ X_{J,4}(\gamma) \\ X_{J,5}(\gamma) \end{bmatrix},$$

with

$$X_{H,1}(\gamma),\ X_{H,2}(\gamma), X_{J,1}(\gamma),\ X_{J,2}(\gamma) \in \mathbb{R}^{n,r}, X_{H,4}(\gamma) \in \mathbb{R}^{m_2,r},$$
$$X_{J,4}(\gamma) \in \mathbb{R}^{p_2,r}, \quad X_{H,3}(\gamma), X_{J,5}(\gamma) \in \mathbb{R}^{m_1,r}, X_{H,5}(\gamma), X_{J,3}(\gamma) \in \mathbb{R}^{p_1,r}.$$

Necessary conditions for optimal γ

C1) The index of the two even pencils is at most one.

C2) There exists a matrix $X_H(\gamma)$ such that

C2.a) the space $\mathrm{Im} X_H(\gamma)$ is a semi-stable deflating subspace of $\lambda N_H + M_H(\gamma)$ and $\mathrm{Im} \begin{bmatrix} EX_{H,1} \\ X_{H,2} \end{bmatrix}$ is an r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;

C2.b) $\mathrm{rank}\, EX_{H,1}(\gamma) = r$.

C3) There exists a matrix $X_J(\gamma)$ such that

C3.a) the space $\mathrm{Im} X_J(\gamma)$ is a semi-stable deflating subspace of $\lambda N_J + M_J(\gamma)$ and $\mathrm{Im} \begin{bmatrix} E^T X_{J,1} \\ X_{J,2} \end{bmatrix}$ is an r-dimensional isotropic subspace of $\mathbb{R}^{2n}$;

C3.b) $\mathrm{rank}\, E^T X_{J,1}(\gamma) = r$.

Definition

Consider the even pencils
$\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Define the sets

$$\begin{aligned}\Gamma_H &= \{\gamma \in \mathbb{R}^+ \mid \text{the index of } \lambda N_H + M_H(\gamma) \text{ is greater than one}\},\\ \Gamma_J &= \{\gamma \in \mathbb{R}^+ \mid \text{the index of } \lambda N_J + M_J(\gamma) \text{ is greater than one}\},\end{aligned}$$

and set $\hat{\gamma}_H = \sup \Gamma_H$, $\hat{\gamma}_J = \sup \Gamma_J$ and $\hat{\gamma} = \max\{\hat{\gamma}_H, \hat{\gamma}_J\}$.

- In general the sets Γ_H and Γ_J may be all of $\mathbb{R}^+$, but it follows from the assumptions **A1) – A4)** that $\hat{\gamma}_H$ and $\hat{\gamma}_J$ and therefore also $\hat{\gamma}$ are finite.
- If $\gamma > \hat{\gamma}$ then these pencils have $2r$ finite eigenvalues, where $r = \mathrm{rank}\, E$.

Definition

Consider the even pencils $\lambda N_H \ + \ M_H(\gamma)$ in and $\lambda N_J + M_J(\gamma)$.
Define

$$\begin{aligned}
\Gamma_H^L &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_H + M_H(\gamma) \text{ satisfies } \textbf{C2.a)}\}, \\
\Gamma_J^L &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_J + M_J(\gamma) \text{ satisfies } \textbf{C3.a)}\}, \\
\Gamma^L &= \Gamma_J^L \cap \Gamma_H^L, \\
\Gamma_H^R &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_H + M_H(\gamma) \text{ satisfies condition } \textbf{C2)}\}, \\
\Gamma_J^R &= \{\gamma \geq \hat{\gamma} \mid \text{the pencil } \lambda N_J + M_J(\gamma) \text{ satisfies condition } \textbf{C3)}\}, \\
\Gamma^R &= \Gamma_J^R \cap \Gamma_H^R
\end{aligned}$$

$$\begin{aligned}
\hat{\gamma}_H^L &= \inf \Gamma_H^L, \hat{\gamma}_J^L = \inf \Gamma_J^L, \ \hat{\gamma}^L = \inf \Gamma^L, \\
\hat{\gamma}_H^R &= \inf \Gamma_H^R, \ \hat{\gamma}_J^R = \inf \Gamma_J^R, \hat{\gamma}^R = \inf \Gamma^R.
\end{aligned}$$

Definition

Consider a system satisfying **A1**-**A4** and the associated even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Define the sets

$$\begin{aligned}
\Gamma_H^I &= \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{the pencil } \lambda N_H + M_H(\gamma) \text{ has at least} \\ \text{one finite eigenvalue on the imagi-} \\ \text{nary axis}\end{array} \right\}, \\
\Gamma_J^I &= \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l}\text{the pencil } \lambda N_J + M_J(\gamma) \text{ has at least} \\ \text{one finite eigenvalue on the imagi-} \\ \text{nary axis}\end{array} \right\}, \\
\Gamma^I &= \Gamma_J^I \cap \Gamma_H^I
\end{aligned}$$

and set

$$\hat{\gamma}_H^I = \inf \Gamma_H^I, \quad \hat{\gamma}_J^I = \inf \Gamma_J^I, \quad \hat{\gamma}^I = \inf \Gamma^I.$$

Theorem

Consider a system of the form satisfying assumptions **A1) – A4)**. *Let $X_H(\gamma)$ and $X_J(\gamma)$ be deflating subspace matrices of the form that satisfy conditions* **C2)** *and* **C3)***), respectively. Then there exist parameters $\hat{\gamma}_H^k \geq \hat{\gamma}_H^L$, $\hat{\gamma}_J^k \geq \hat{\gamma}_J^L$ and $\hat{k}_H, \hat{k}_J \in \mathbb{N}$ with the property that for all $\gamma_{H,1}, \gamma_{H,2} > \hat{\gamma}_H^k$, $\gamma_{J,1}, \gamma_{J,2} > \hat{\gamma}_J^k$ the rank conditions hold:*

$$\begin{aligned} \operatorname{rank} E^T X_{H,2}(\gamma_{H,1}) &= \operatorname{rank} E^T X_{H,2}(\gamma_{H,2}) = \hat{k}_H, \\ \operatorname{rank} E X_{J,2}(\gamma_{J,1}) &= \operatorname{rank} E X_{J,2}(\gamma_{J,2}) = \hat{k}_J. \end{aligned}$$

C4) The matrix

$$\mathcal{Y}(\gamma) = \begin{bmatrix} -\gamma X_{H,2}^T(\gamma) E X_{H,1}(\gamma) & X_{H,2}^T(\gamma) E X_{J,2}(\gamma) \\ X_{J,2}^T(\gamma) E^T X_{H,2}(\gamma) & -\gamma X_{J,2}^T(\gamma) E^T X_{J,1}(\gamma) \end{bmatrix} \qquad (1)$$

is symmetric, positive semi-definite and satisfies $\operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J$.

Definition

Consider a system that satisfies assumptions **A1) – A4)**. Then we define

$$\Gamma^\rho = \left\{ \gamma \geq \hat{\gamma} \;\middle|\; \begin{array}{l} \text{the matrix } \mathcal{Y}(\gamma) \text{ is positive semi-} \\ \text{definite} \\ \text{with } \operatorname{rank} \mathcal{Y}(\gamma) = \hat{k}_H + \hat{k}_J \end{array} \right\}$$

and we set $\hat{\gamma}^\rho := \inf \Gamma^\rho$.

Theorem (Losse, M., Poppe, Reis 2008)

Consider a descriptor system the associated even pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$. Suppose that assumptions **A1) – A4)** *hold.*
Then there exists an internally stabilizing controller such that the transfer function from w to z satisfies $T_{zw} \in \mathcal{H}_\infty^{p_1,m_1}$ with $\|T_{zw}\|_\infty < \gamma$ if and only if γ is such that the conditions **C1) – C4)** *hold.*
Furthermore, the set of γ satisfying the conditions **C1) – C4)** *is nonempty.*

Algorithm **Input:** Data of system, value $\gamma \geq 0$.
Output: Decision whether $\gamma < \gamma_{mo}$ or $\gamma \geq \gamma_{mo}$.

1. Form the pencils $\lambda N_H + M_H(\gamma)$ and $\lambda N_J + M_J(\gamma)$.
2. Compute the deflating subspace matrices X_H and X_J associated with the eigenvalues in the closed left half plane.
3. IF the dimension of one/both of these subspaces is less than r, then $\gamma < \gamma_{mo}$,

   ELSE

   IF the rank of $EX_{H,1}$ and/or $E^T X_{J,1}$ is less than r, then $\gamma < \gamma_{mo}$,
   ELSE

   Form the matrix $\hat{\mathcal{Y}}$.
   IF $\hat{\mathcal{Y}}$ is not positive semi-definite and/or $\operatorname{rank} \hat{\mathcal{Y}} < \hat{k}_H + \hat{k}_J$, then $\gamma < \gamma_{mo}$,
   ELSE $\gamma \geq \gamma_{mo}$.

To determine γ_{mo} one then uses a bisection method.

Remarks

- There exist explicit formulas for the optimal controllers, Losse 2010.
- Extensions of H_∞ control to time-varying or nonlinear DAEs are open.
- Other classical control techniques, like stabilization, tracking control, etc.
- Extensions of optimal control to time-varying or nonlinear DAEs, next lecture.