Hamiltonian control systems

From modeling to analysis and control

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Outline

Table of Contents for the Week

- Monday: definition of port-Hamiltonian systems and examples
- Tuesday: interconnection of port-Hamiltonian systems; properties of port-Hamiltonian systems
- Wednesday: control by interconnection and passivity-based control
- Thursday: control case studies; relations with other modeling approaches
- Friday: distributed-parameter port-Hamiltonian systems

Today

- Basics of port-based modeling
- Dirac structures
- Definition of port-Hamiltonian systems
- Classes of port-Hamiltonian systems

Overall message:
first principles physical modeling leads to a generalized Hamiltonian system description with lots of useful information for analysis and control; for linear, nonlinear and infinite-dimensional systems.

Outline

The basic elements

Port-based modeling is based on viewing the physical system as the interconnection of ideal energy processing elements, all expressed in (vector) pairs of flow variables $f \in \mathcal{F}$, and effort variables $e \in \mathcal{E}$, where $\mathcal{F}$ and $\mathcal{E}$ are linear spaces of equal dimension.
Furthermore, there is a pairing between $\mathcal{F}$ and $\mathcal{E}$ defining the power

$$< e \mid f >$$

Canonical choice: $\mathcal{E} = \mathcal{F}^*$ with $< e \mid f >= e^T f$.

- Energy-storing elements:

$$\begin{aligned} \dot{x} &= -f \\ e &= \frac{\partial H}{\partial x}(x) \end{aligned}$$

- Energy-dissipating elements:

$$R(f, e) = 0, \quad e^T f \leq 0$$

The basic elements

- Energy-routing elements: generalized transformers, gyrators:

$$f_1 = Mf_2,\ e_2 = -M^T e_1, \quad f = Je,\ J = -J^T$$

They are power-conserving:

$$e^T f = 0$$

- Ideal interconnection constraints

0-junction :
$$e_1 = e_2 = \cdots = e_k, \quad f_1 + f_2 + \cdots + f_k = 0$$
1-junction :
$$f_1 = f_2 = \cdots = f_k, \quad e_1 + e_2 + \cdots + e_k = 0$$
Ideal flow or effort constraints :
$$f = 0, \qquad \text{or } e = 0$$

Also power-conserving:

$$e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = 0$$

From power-conserving elements to Dirac structures

All power-conserving elements/interconnection constraints have the following properties in common.
They are described by linear equations:

$$Ff + Ee = 0, \quad f, e \in \mathbb{R}^\ell$$

satisfying

$$e^T f = e_1 f_1 + e_2 f_2 + \cdots + e_\ell f_\ell = 0,$$

while furthermore

$$\text{rank} \begin{bmatrix} F & E \end{bmatrix} = \ell$$

All power-conserving elements/interconnection constraints will be grouped into one geometric object: the Dirac structure.

Definition of Dirac structures

Definition

A (constant) Dirac structure is a subspace

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$$

such that

(i) $e^T f = 0$ for all $(f, e) \in \mathcal{D}$,

(ii) $\dim \mathcal{D} = \dim \mathcal{F}$.

For any skew-symmetric map $J : \mathcal{E} \to \mathcal{F}$ its graph given as $\{(f, e) \in \mathcal{F} \times \mathcal{E} \mid f = Je\}$ is a Dirac structure !

Alternative definition of Dirac structure

Symmetrized form of power

$$< e \mid f >= e^T f, \quad (f, e) \in \mathcal{F} \times \mathcal{E}.$$

is the indefinite *bilinear form* $\ll, \gg$ on $\mathcal{F} \times \mathcal{E}$:

$$\ll (f^a, e^a), (f^b, e^b) \gg \quad := \quad < e^a \mid f^b > + < e^b \mid f^a >,$$

$$(f^a, e^a), (f^b, e^b) \in \mathcal{F} \times \mathcal{E}.$$

Alternative definition of Dirac structure

Definition

A (constant) Dirac structure is a subspace

$$\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$$

such that

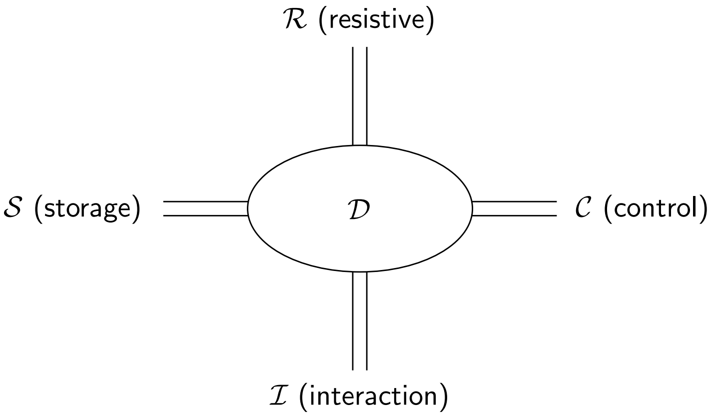
$$\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp},$$

where $\perp\!\!\!\perp$ denotes orthogonal complement with respect to the bilinear form $\ll, \gg$.

Towards port-Hamiltonian systems

Four-port model: Consider a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, where

$$f = (f_S, f_R, f_I, f_C), \quad e = (e_S, e_R, e_I, e_C)$$



Outline

The port-Hamiltonian system is defined by closing the energy-storage and resistive ports by their constitutive relations

$$-\dot{x} = f_S, \frac{\partial H}{\partial x}(x) = e_S$$

respectively

$$R(f_R, e_R) = 0$$

This leads to the DAEs

$$(-\dot{x}(t), f_R(t), f_I(t), f_C(t), \tfrac{\partial H}{\partial x}(x(t)), e_R(t), e_I(t), e_C(t)) \in \mathcal{D}$$

$$t \in \mathbb{R}$$

$$R(f_R(t), e_R(t)) = 0$$

Thus DAEs of a special form, determined by the Dirac structure and energy-conserving and energy-dissipating constitutive relations.
Definition can be also extended to cover infinite-dimensional physical systems.

Energy-balance

Power-conservation

$$e_S^T f_S + e_R^T f_R + e_I^T f_I + e_C^T f_C = 0$$

implies the energy-balance

$$\frac{dH}{dt}(x(t)) = \frac{\partial H}{\partial x}(x(t))\dot{x}(t) =$$

$$e_R^T(t) f_R(t) + e_I^T(t) f_I(t) + e_C^T(t) f_C(t) \leq$$

$$e_I^T(t) f_I(t) + e_C^T(t) f_C(t)$$

This implies that if the Hamiltonian H has a minimum at the equilibrium under study, then it can be used as a Lyapunov function.

Example (The ubiquitous mass-spring system)

Two storage elements:

- **Spring** Hamiltonian $H_s(q) = \frac{1}{2}kq^2$ (potential energy)

$$\begin{array}{rclcl} \dot{q} & = & -f_s & = & \text{velocity} \\ e_s & = & \frac{dH_s}{dq}(q) = kq & = & \text{force} \end{array}$$

- **Mass** Hamiltonian $H_m(p) = \frac{1}{2m}p^2$ (kinetic energy)

$$\begin{array}{rclcl} \dot{p} & = & -f_m & = & \text{force} \\ e_m & = & \frac{dH_m}{dp}(p) = \frac{p}{m} & = & \text{velocity} \end{array}$$

Example

interconnected by the Dirac structure

$$f_s = -e_m = -y, \quad f_m = e_s - u$$

(power-conserving since $f_s e_s + f_m e_m + uy = 0$) yields the port-Hamiltonian system

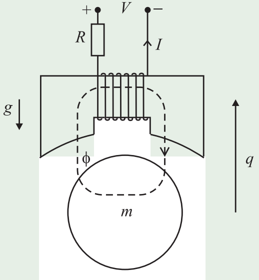
$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}$$

with

$$H(q,p) = H_s(q) + H_m(p)$$

Example (Electro-mechanical systems)



$$\begin{bmatrix} \dot{q} \\ \dot{p} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{R} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p,\phi) \\ \frac{\partial H}{\partial p}(q,p,\phi) \\ \frac{\partial H}{\partial \varphi}(q,p,\phi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} V, \quad I = \frac{\partial H}{\partial \varphi}(q,p,\phi)$$

Coupling electrical/mechanical domain via Hamiltonian $H(q,p,\phi)$

$$H(q,p,\varphi) = mgq + \frac{p^2}{2m} + \frac{\varphi^2}{2k_1(1-\frac{q}{k_2})}$$

Network modeling is prevailing in modeling and simulation of lumped-parameter physical systems (multi-body systems, electrical circuits, electro-mechanical systems, hydraulic systems, robotic systems, etc.), with many advantages:

- Modularity and flexibility. Re-usability ('libraries').
- Multi-physics approach.
- Suited to design/control.

Disadvantage of network modeling: it generally leads to a large set of DAEs, seemingly without any structure.

Port-based modeling and port-Hamiltonian system theory identifies the underlying structure of network models of physical systems, to be used for analysis, simulation and control.

For many systems, especially those with 3-D mechanical components, the interconnection structure will be modulated by the energy or geometric variables.
This leads to the notion of (non-constant) Dirac structures on manifolds.

Definition

Consider a smooth manifold M. A Dirac structure on M is a vector subbundle $\mathcal{D} \subset TM \oplus T^*M$ such that for every $x \in M$ the vector space

$$\mathcal{D}(x) \subset T_xM \times T_x^*M$$

is a Dirac structure as before.

Example (Mechanical systems with kinematic constraints)

Constraints on the generalized velocities $\dot{q}$:

$$A^T(q)\dot{q} = 0.$$

This leads to *constrained* Hamiltonian equations

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q,p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda + B(q)f \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \\
e &= B^T(q)\frac{\partial H}{\partial p}(q,p)
\end{aligned}$$

with $H(q,p)$ total energy, and $A(q)\lambda$ the constraint forces.
Dirac structure is defined by the symplectic form on T^*Q together with constraints $A^T(q)\dot{q} = 0$ and force matrix $B(q)$.

Can be systematically extended to general multi-body systems.

Example (Rolling coin)

Let x, y be the Cartesian coordinates of the point of contact of the coin with the plane. Furthermore, φ denotes the heading angle, and θ the angle of Queen Beatrix' head (on the Dutch version of the euro). The rolling constraints (rolling without slipping) are

$$\dot{x} = \dot{\theta} \cos \varphi, \quad \dot{y} = \dot{\theta} \sin \varphi$$

The total energy is

$$H = \frac{1}{2} p_x^2 + \frac{1}{2} p_y^2 + \frac{1}{2} p_\theta^2 + \frac{1}{2} p_\varphi^2$$

and the constraints can be rewritten as

$$p_x = p_\theta \cos \varphi, \quad p_y = p_\theta \sin \varphi.$$

Input-state-output port-Hamiltonian systems

Consider a Dirac structure $\mathcal{D}$ given as the graph of the skew-symmetric map

$$\begin{bmatrix} f_S \\ f_C \end{bmatrix} = \begin{bmatrix} -J(x) & -g(x) \\ g^T(x) & 0 \end{bmatrix} \begin{bmatrix} e_S \\ e_C \end{bmatrix},$$

leading ($f_S = -\dot{x}, e_S = \frac{\partial H}{\partial x}(x)$) to a port-Hamiltonian system as before

$$\begin{aligned} \dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + g(x)e_C, \quad x \in \mathcal{X}, e_C \in \mathbb{R}^m \\ f_C &= g^T(x)\frac{\partial H}{\partial x}(x), \qquad\qquad f_C \in \mathbb{R}^m \end{aligned}$$

Energy-dissipation is included by terminating some of the ports by linear resistive elements

$$f_R = -\tilde{R}(e_R), \quad \tilde{R} = \tilde{R}^T \geq 0$$

Writing out

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x) + g_R(x)f_R + g(x)e_C, \quad e_R = g_R^T(x)\frac{\partial H}{\partial x}(x)$$

this leads to an input-state-output port-Hamiltonian system given as

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + g(x)e_C \\ f_C &= g^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

where

$$R(x) = g_R(x)\tilde{R}g_R^T(x) \geq 0$$

Multi-modal physical systems

Example (Bouncing pogo-stick)

Consider a vertically bouncing pogo-stick consisting of a mass m and a massless foot, interconnected by a linear spring (stiffness k and rest-length x_0) and a linear damper d.

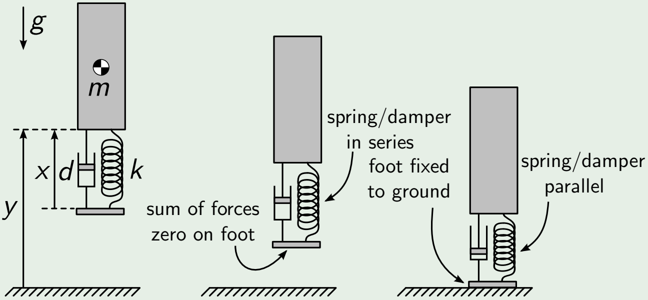


Figure: Model of a bouncing pogo-stick: definition of the variables (left), situation without ground contact (middle), and situation with ground contact (right).

Example

The mass can move vertically under the influence of gravity g until the foot touches the ground. The states of the system are x (length of the spring), y (height of the bottom of the mass), and p (momentum of the mass, defined as $p := m\dot{y}$). Furthermore, the contact situation is described by a variable s with values $s = 0$ (no contact) and $s = 1$ (contact). The Hamiltonian of the system equals

$$H(x, y, p) = \frac{1}{2}k(x - x_0)^2 + mg(y + y_0) + \frac{1}{2m}p^2$$

where y_0 is the distance from the bottom of the mass to its center of mass.

Example

When the foot is not in contact with the ground total force on the foot is zero (since it is massless), which implies that the spring and damper force must be equal but opposite. When the foot is in contact with the ground, the variables x and y remain equal, and hence also $\dot{x} = \dot{y}$.

For $s = 0$ (no contact) the system is described by the port-Hamiltonian system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} y \\ p \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} mg \\ \frac{p}{m} \end{bmatrix} \\ -d\dot{x} &= k(x - x_0) \end{aligned}$$

while for $s = 1$ the port-Hamiltonian description is

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & -d \end{bmatrix} \begin{bmatrix} k(x - x_0) \\ mg \\ \frac{p}{m} \end{bmatrix}$$

Example

The two situations can be taken together into one port-Hamiltonian system with variable Dirac structure:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ p \end{bmatrix} = \begin{bmatrix} \frac{s-1}{d} & 0 & s \\ 0 & 0 & 1 \\ -s & -1 & -sd \end{bmatrix} \begin{bmatrix} k(x-x_0) \\ mg \\ \frac{p}{m} \end{bmatrix}$$

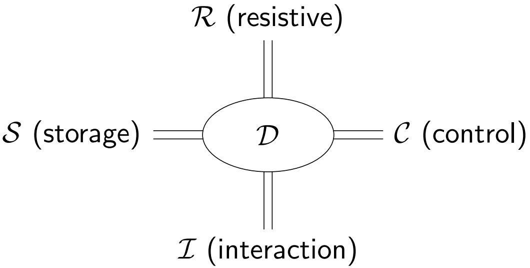
We obtain a *switching* port-Hamiltonian system, specified by a Dirac structure $\mathcal{D}_s$ depending on the switch position $s \in \{0,1\}^n$ (here n denotes the number of independent switches).

Another interesting example is switching electrical circuits and power converters.

Geometric definition of a port-Hamiltonian system

Consider a Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, where

$$f = (f_S, f_R, f_I, f_C), \quad e = (e_S, e_R, e_I, e_C)$$



$$(-\dot{x}(t), f_R(t), f_I(t), f_C(t), \frac{\partial H}{\partial x}(x(t)), e_R(t), e_I(t), e_C(t)) \in \mathcal{D},$$

$$R(f_R(t), e_R(t)) = 0$$

Equational representations

Dirac structures, and therefore port-Hamiltonian systems, admit different equational representations, with different properties for analysis, simulation, and control.
Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$, with $\dim \mathcal{F} = n$, be a Dirac structure.

1. Kernel and Image representation

$$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid Ff + Ee = 0\}$$

for $n \times n$ matrices F and E (possibly depending on x) satisfying

$$(i) \quad EF^T + FE^T = 0,$$

$$(ii) \quad \mathrm{rank}[F \vdots E] = n.$$

It follows that $\mathcal{D}$ can be also written in image representation as
$\mathcal{D} = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* \mid f = E^T\lambda, e = F^T\lambda, \lambda \in \mathbb{R}^n\}$.

In case of the presence of an energy-conserving port S, an energy-dissipating port R and an external port the Dirac structure $\mathcal{D}$ can thus be given in kernel representation as

$$\mathcal{D} = \Big\{(f_S, e_S, f_R, e_R, f, e) \in \mathcal{X} \times \mathcal{X}^* \times \mathcal{F}_R \times \mathcal{F}_R^* \times \mathcal{F} \times \mathcal{F}^* \mid F_S f_S + E_S e_S + F_R f_R + E_R e_R + Ff + Ee = 0\Big\}$$

with

$$(i) \qquad E_S F_S^T + F_S E_S^T + E_R F_R^T + F_R E_R^T + EF^T + FE^T = 0$$

$$(ii) \qquad \operatorname{rank}\big[F_S \mid E_S \mid F_R \mid E_R \mid F \mid E\big] = \dim(\mathcal{X} \times \mathcal{F}_R \times \mathcal{F})$$

Then the port-Hamiltonian system is given by the set of DAEs

$$F_S \dot{x}(t) = E_S \frac{\partial H}{\partial x}(x(t)) + F_R f_R(t) + E e_R(t) + Ff(t) + Ee(t),$$

where the vectors f_R, e_R satisfy at all time-instants the energy-dissipating constitutive relations; e.g., $f_R(t) = Re_R(t)$.

2. Constrained input-output representation

Every Dirac structure $\mathcal{D}$ can be written as

$$\mathcal{D} = \{(f, e) \in \mathcal{F} \times \mathcal{F}^* \mid f = Je + G\lambda, G^T e = 0\}$$

for a skew-symmetric matrix J and a matrix G such that

$$\text{im } G = \{f \mid (f, 0) \in \mathcal{D}\}.$$

Furthermore, $\ker J = \{e \mid (0, e) \in \mathcal{D}\}$.

In the absence of energy-dissipating and external ports it follows that any port-Hamiltonian system can be represented as

$$
\begin{aligned}
\dot{x} &= J(x)\frac{\partial H}{\partial x}(x) + G(x)\lambda, \quad J(x) = -J^T(x) \\
0 &= G^T(x)\frac{\partial H}{\partial x}(x)
\end{aligned}
$$

A typical example are mechanical systems with kinematic constraints

$$
\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p}(q,p) \\
\dot{p} &= -\frac{\partial H}{\partial q}(q,p) + A(q)\lambda \\
0 &= A^T(q)\frac{\partial H}{\partial p}(q,p) \quad (= A^T(q)\dot{q})
\end{aligned}
$$

where $A(q)\lambda$ are the constraint forces.

3. Hybrid input-output representation

Let $\mathcal{D}$ be given by square matrices E and F as in 1. Suppose rank $F = m(\leq n)$. Select m independent columns of F, and group them into a matrix F_1. Write (possibly after permutations) $F = [F_1 \vdots F_2]$, and correspondingly $E = [E_1 \vdots E_2], f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$.

Then the matrix $[F_1 \vdots E_2]$ is invertible, and

$$\mathcal{D} = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \middle| \begin{bmatrix} f_1 \\ e_2 \end{bmatrix} = J \begin{bmatrix} e_1 \\ f_2 \end{bmatrix} \right\}$$

with $J := -[F_1 \vdots E_2]^{-1}[F_2 \vdots E_1]$ skew-symmetric.

From one representation to another

In principle we can freely move from one representation to another. For example, consider a linear port-Hamiltonian system, without energy-dissipating and external ports. This can be always written as

$$\begin{aligned} \dot{x} &= JQx + G\lambda, \quad J = -J^T, Q = Q^T \\ 0 &= G^T Qx, \qquad H(x) = \tfrac{1}{2}x^T Qx \end{aligned}$$

The constraint forces $G\lambda$ can be eliminated by pre-multiplying the first equation by the annihilating matrix $G^\perp$, leading to the DAE system (in kernel representation)

$$\begin{bmatrix} G^\perp \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} G^\perp J \\ G^T \end{bmatrix} Qx$$

The corresponding pencil

$$s \begin{bmatrix} G^\perp \\ 0 \end{bmatrix} - \begin{bmatrix} G^\perp JQ \\ G^T Q \end{bmatrix}$$

is non-singular if $G^T QG$ has full rank, and in fact, the system has index one if and only if $G^T QG$ has full rank.

More explicitly, define the coordinate transformation (assuming w.l.o.g. that G has full rank)

$$z = \begin{bmatrix} G^{\perp} \\ G^T \end{bmatrix} x =: Vx$$

This leads to the transformed system

$$\begin{aligned} \dot{z} &= (VJV^T)(V^{-T}QV^{-1})(Vx) + VG\lambda = \tilde{J}\tilde{Q}z + \begin{bmatrix} 0 \\ G^TG \end{bmatrix} \lambda \\ 0 &= G^TQx = \begin{bmatrix} 0 & G^TG \end{bmatrix} \tilde{Q}z \end{aligned}$$

Since G^TG has full rank, this means that the constraint amounts to $(\tilde{Q}z)_2 = 0$, where $\tilde{Q}z = \begin{bmatrix} (\tilde{Q}z)_1 \\ (\tilde{Q}z)_2 \end{bmatrix}$.
Hence the system reduces to the Hamiltonian differential equations

$$\dot{z}_1 = J_{11}(\tilde{Q}_{11} - \tilde{Q}_{12}\tilde{Q}_{22}^{-1}\tilde{Q}_{21})z_1$$

In general, port-Hamiltonian systems will have generally index one.
Consider the nonlinear port-Hamiltonian system

$$\begin{aligned} \dot{x} &= J(x)\tfrac{\partial H}{\partial x}(x) + G(x)\lambda, \quad J(x) = -J^T(x) \\ 0 &= G^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

Then differentiation of the algebraic constraints yields

$$0 = \frac{d}{dt} G^T(x) \frac{\partial H}{\partial x}(x) = \cdots + G^T(x)\frac{\partial^2 H}{\partial x^2}(x) G(x)\lambda,$$

which can be solved for λ as long as the rank of

$$G^T(x)\frac{\partial^2 H}{\partial x^2}(x) G(x)$$

is equal to the rank of $G(x)$.
In particular if the Hessian $\frac{\partial^2 H}{\partial x^2}(x)$ is invertible, then the system has index one.

Canonical coordinates

Any constant Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ can be represented as follows. There exist linear coordinates $x = (q, p, r, s)$ for $\mathcal{F}$ (with $\dim q = \dim p$), such that in the corresponding bases for (f_q, f_p, f_r, f_s) for $T_x\mathcal{F}$ and (e_q, e_p, e_r, e_s) for $\mathcal{F}^*$, the Dirac structure is given as

$$\begin{cases} f_q &= -e_p \\ f_p &= e_q \\ f_r &= 0 \\ e_s &= 0 \end{cases}$$

In such canonical coordinates a port-Hamiltonian system on $\mathcal{X} = \mathcal{F}$ without energy-dissipating and external ports can be written as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p, r, s) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p, r, s) \\ \dot{r} &= 0 \\ 0 &= \frac{\partial H}{\partial s}(q, p, r, s) \end{aligned}$$

Excursion to integrability

A Dirac structure on a manifold $\mathcal{X}$ is integrable if it is possible to find local coordinates such that, in these coordinates, the Dirac structure becomes a constant Dirac structure, that is, it is not modulated anymore by the state variables. Thus then there also exist canonical coordinates.

First case

Let the modulated Dirac structure $\mathcal{D}$ be given for every $x \in \mathcal{X}$ as the *graph* of a skew-symmetric mapping $J(x)$ from the co-tangent space $T_x^*\mathcal{X}$ to the tangent space $T_x\mathcal{X}$. Integrability in this case means that $J(x)$ satisfies the conditions

$$\sum_{l=1}^{n} \left[J_{lj}(x)\frac{\partial J_{ik}}{\partial x_l}(x) + J_{li}(x)\frac{\partial J_{kj}}{\partial x_l}(x) + J_{lk}(x)\frac{\partial J_{ji}}{\partial x_l}(x) \right] = 0, \quad i, j, k = 1, \dots, n$$

In this case we may find, by Darboux's theorem around any point x_0 where the rank of the matrix $J(x)$ is constant, local canonical coordinates $x = (q, p, r)$ in which the matrix $J(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & -I_k & 0 \\ I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then $J(x)$ defines a Poisson bracket on $\mathcal{X}$, given for every $F, G : \mathcal{X} \to \mathbb{R}$ as

$$\{F, G\} = \frac{\partial^T F}{\partial x} J(x) \frac{\partial G}{\partial x}$$

Indeed, by the integrability condition the Jacobi-identity holds:

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$$

for all functions F, G, K.

Second case

A similar story holds for a Dirac structure given as the graph of a skew-symmetric mapping $\omega(x)$ from the tangent space $T_x\mathcal{X}$ to the co-tangent space $T_x^*\mathcal{X}$. In this case the integrability conditions take the form

$$\frac{\partial \omega_{ij}}{\partial x_k}(x) + \frac{\partial \omega_{ki}}{\partial x_j}(x) + \frac{\partial \omega_{jk}}{\partial x_i}(x) = 0, \quad i, j, k = 1, \ldots, n$$

The skew-symmetric matrix $\omega(x)$ can be regarded as the coordinate representation of a differential two-form ω on $\mathcal{X}$, that is $\omega = \sum_{i=1,j=1}^{n} dx_i \wedge dx_j$, and the integrability condition corresponds to the closedness of this two-form ($d\omega = 0$).

The differential two-form ω is called a pre-symplectic structure, and a symplectic structure if the rank of $\omega(x)$ is equal to the dimension of $\mathcal{X}$. By a version of Darboux's theorem we may find, around any point x_0 where the rank of the matrix $\omega(x)$ is constant, local coordinates $x = (q, p, s)$ in which the matrix $\omega(x)$ becomes the constant skew-symmetric matrix

$$\begin{bmatrix} 0 & I_k & 0 \\ -I_k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

For general Dirac structures, integrability is defined as

Definition

A Dirac structure $\mathcal{D}$ on $\mathcal{X}$ is integrable if for arbitrary pairs of smooth vector fields and differential one-forms $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$ there holds

$$< L_{X_1}\alpha_2 \mid X_3 > + < L_{X_2}\alpha_3 \mid X_1 > + < L_{X_3}\alpha_1 \mid X_2 >= 0$$

with L_{X_i} denoting the Lie-derivative.

The Dirac structure corresponding to mechanical systems with kinematic constraints

$$\mathcal{D} = \{(f_q, f_p, e_q, e_p) \mid f_q = -e_p, f_p = e_q + A(q)\lambda, A^T(q)e_p = 0\}$$

is integrable if and only if the kinematic constraints

$$A^T(q)\dot{q} = 0$$

are holonomic, which means that it is possible to find configuration coordinates $\overline{q} = (\overline{q}_1, \ldots, \overline{q}_n)$ such that the constraints are equivalently expressed as

$$\dot{\overline{q}}_{n-k+1} = \dot{\overline{q}}_{n-k+2} = \cdots = \dot{\overline{q}}_n = 0 \ ,$$

In this case one may eliminate the configuration variables $\overline{q}_{n-k+1}, \ldots, \overline{q}_n$, since the kinematic constraints are equivalent to the geometric constraints

$$\overline{q}_{n-k+1} = c_{n-k+1}, \ldots, \overline{q}_n = c_n \ ,$$

for certain constants $c_{n-k+1}, \ldots, c_n$ determined by the initial conditions.

Outline

Scattering representations

Main idea: Use in the total space of port-variables $\mathcal{F} \times \mathcal{F}^*$ a different splitting than the 'canonical' duality splitting (in flows $f \in \mathcal{F}$ and efforts $e \in \mathcal{F}^*$).
Consider the canonical bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg = < e_1 \mid f_2 > + < e_2 \mid f_1 >, \; (f_i, e_i) \in \mathcal{F} \times \mathcal{F}^*, \quad i = 1, 2,$$

on $\mathcal{F} \times \mathcal{F}^*$. This is an indefinite bilinear form, which has n singular values $+1$ and n singular values -1 ($n = \dim \mathcal{F}$).

A pair of subspaces $\Sigma^+, \Sigma^- \subset \mathcal{F} \times \mathcal{F}^*$ is a pair of scattering subspaces if

(i) $\Sigma^+ \oplus \Sigma^- = \mathcal{F} \times \mathcal{F}^*$

(ii) $\ll \sigma_1^+, \sigma_2^+ \gg > 0$ for all $\sigma_1^+, \sigma_2^+ \in \Sigma^+$ unequal to 0.
$\ll \sigma_1^-, \sigma_2^- \gg < 0$ for all $\sigma_1^-, \sigma_2^- \in \Sigma^-$ unequal to 0.

(iii) $\ll \sigma^+, \sigma^- \gg = 0$ for all $\sigma^+ \in \Sigma^+, \sigma^- \in \Sigma^-$.

It is readily seen that any pair of scattering subspaces (Σ^+, Σ^-) satisfies

$$\dim \Sigma^+ = \dim \Sigma^- = \dim \mathcal{F}$$

The collection of pairs of scattering subspaces can be characterized as follows.

Lemma

Let (Σ^+, Σ^-) be a pair of scattering subspaces. Then there exists an invertible linear map

$$R : \mathcal{F} \to \mathcal{F}^*$$

with

$$< (R + R^*)f | f > \quad > 0, \qquad \textit{for all } 0 \neq f \in \mathcal{F},$$

such that

$$\Sigma^+ \quad := \quad \{(R^{-1}e, e) \in \mathcal{F} \times \mathcal{F}^* \mid e \in \mathcal{F}^*\}$$

$$\Sigma^- \quad := \quad \{(-f, R^* f) \in \mathcal{F} \times \mathcal{F}^* \mid f \in \mathcal{F}\}$$

Conversely, for any invertible linear map $R : \mathcal{F} \to \mathcal{F}^*$ the pair (Σ^+, Σ^-) is a pair of scattering subspaces.

Let (Σ^+, Σ^-) be a pair of scattering subspaces. Then every pair of power vectors $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ can be represented as

$$(f, e) = \sigma^+ + \sigma^-$$

for a uniquely defined $\sigma^+ \in \Sigma^+, \sigma^- \in \Sigma^-$, called the *wave vectors*. Using orthogonality of Σ^+ w.r.t. Σ^- for all $(f_i, e_i) = \sigma_i^+ + \sigma_i^-, i = 1, 2$

$$\ll (f_1, e_1), (f_2, e_2) \gg =$$

$$< \sigma_1^+, \sigma_2^+ >_{\Sigma^+} - < \sigma_1^-, \sigma_2^- >_{\Sigma^-}$$

where $<, >_{\Sigma^+}$ is the restriction of $\ll, \gg$ to Σ^+, and $<, >_{\Sigma^-}$ is minus the restriction of $\ll, \gg$ to Σ^-.

Taking $f_1 = f_2 = f, e_1 = e_2 = e$, and thus $\sigma_1^+ = \sigma_2^+ = \sigma^+, \sigma_1^- = \sigma_2^- = \sigma^-$, leads to

$$< e \mid f >= \frac{1}{2} \ll (f, e), (f, e) \gg =$$

$$\frac{1}{2} < \sigma^+, \sigma^+ >_{\Sigma^+} - \frac{1}{2} < \sigma^-, \sigma^- >_{\Sigma^-}$$

This yields the following interpretation of the wave vectors: The vector σ^+ can be regarded as the incoming wave vector, with half times its norm being the incoming power, and the vector σ^- is the outgoing wave vector, with half times its norm being the outgoing power.

Let $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ be a Dirac structure, that is, $\mathcal{D} = \mathcal{D}^{\perp\!\!\!\perp}$ with respect to $\ll, \gg$. What is its representation in wave vectors ?

Since $\ll, \gg$ is zero restricted to $\mathcal{D}$ it follows that for every pair of scattering subspaces (Σ^+, Σ^-)

$$\mathcal{D} \cap \Sigma^+ = 0, \quad \mathcal{D} \cap \Sigma^- = 0,$$

and hence $\mathcal{D}$ can be represented as the graph of an invertible linear map $\mathcal{O} : \Sigma^+ \to \Sigma^-$, that is,

$$\mathcal{D} = \{(\sigma^+, \sigma^-) \mid \sigma^- = \mathcal{O}\sigma^+, \;\; \sigma^+ \in \Sigma^+\}$$

Furthermore

$$< \sigma_1^+, \sigma_2^+ >_{\Sigma^+} = < \mathcal{O}\sigma_1^+, \mathcal{O}\sigma_2^+ >$$

for every $\sigma_1^+, \sigma_2^+ \in \Sigma^+$, and thus

$$\mathcal{O} : (\Sigma^+, <,>_{\Sigma^+}) \rightarrow (\Sigma^-, <,>_{\Sigma^-})$$

is a unitary map (isometry).

Conversely, every unitary map $\mathcal{O}$ as above defines a Dirac structure. Thus for every pair of scattering subspaces (Σ^+, Σ^-) we have a one-to-one correspondence between unitary maps and Dirac structures $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$.

Inner product scattering representations

A particular useful class of scattering subspaces (Σ^+, Σ^-) are those defined by an invertible map $R : \mathcal{F} \to \mathcal{F}^*$ such that $R = R^*$. In this case R is determined by the inner product on $\mathcal{F}$ defined as

$$< f_1, f_2 >_R := < Rf_1 \mid f_2 > = < Rf_2 \mid f_1 >$$

or equivalently by the inner product on $\mathcal{F}^*$

$$< e_1, e_2 >_{R^{-1}} := < e_2 \mid R^{-1} f_1 > = < e_1 \mid R^{-1} f_2 >$$

Define for every $(f, e) \in \mathcal{F} \times \mathcal{F}^*$ the pair (s^+, s^-) by

$$\begin{aligned} s^+ &:= \frac{1}{\sqrt{2}}(e + Rf) \in \mathcal{F}^* \\ s^- &:= \frac{1}{\sqrt{2}}(e - Rf) \in \mathcal{F}^* \end{aligned}$$

Let (s_i^+, s_i^-) correspond to $(f_i, e_i), i = 1, 2$.

$$\begin{aligned} 2 < s_1^+, s_2^+ >_{R^{-1}} &= < e_1, e_2 >_{R^{-1}} + < f_1, f_2 >_R \\ &\quad + \ll (f_1, e_1), (f_2, e_2) \gg \\ 2 < s_1^-, s_2^- >_{R^{-1}} &= < e_1, e_2 >_{R^{-1}} + < f_1, f_2 >_R \\ &\quad - \ll (f_1, e_1), (f_2, e_2) \gg \end{aligned}$$

Hence, if $(f_i, e_i) \in \Sigma^+$, or equivalently $s_i^- = e_i - Rf_i = 0$, then $2 < s_1^+, s_2^+ >_{R^{-1}} = 2 \ll (f_1, e_1), (f_2, e_2) \gg$, while if $(f_i, e_i) \in \Sigma^-$, or equivalently $s_i^+ = e_i + Rf_i = 0$, then $2 < s_1^-, s_2^- >_{R^{-1}} = -2 \ll (f_1, e_1), (f_2, e_2) \gg$.

Thus the mappings

$$\sigma^+ = (f,e) \in \Sigma^+ \quad \longmapsto \quad s^+ = \frac{1}{\sqrt{2}}(e + Rf) \in \mathcal{F}^*$$

$$\sigma^- = (f,e) \in \Sigma^- \quad \longmapsto \quad s^- = \frac{1}{\sqrt{2}}(e - Rf) \in \mathcal{F}^*$$

are isometries (with respect to the inner products on Σ^+ and Σ^-, and the inner product on $\mathcal{F}^*$). Hence we may identify the wave vectors (σ^+, σ^-) with (s^+, s^-).

Remark Note that the pair of scattering subspaces (Σ^+, Σ^-) corresponding to R may be characterized as

$$\Sigma^+ = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | \ll (f,e),(f,e) \gg = < e,e >_{R^{-1}} + < f,f >_R\}$$

$$\Sigma^- = \{(f,e) \in \mathcal{F} \times \mathcal{F}^* | \ll (f,e),(f,e) \gg = - < e,e >_{R^{-1}} - < f,f >_R\}$$

Representation of a Dirac structure in terms of the wave vectors

. For every Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{F}^*$ there exist linear mappings $F : \mathcal{F} \to \mathcal{V}$ and $E : \mathcal{F}^* \to \mathcal{V}$ satisfying

$$(i) \quad EF^* + FE^* = 0,$$

$$(ii) \quad \text{rank}\,(F + E) = \dim \mathcal{F},$$

where $\mathcal{V}$ is a linear space with the same dimension as $\mathcal{F}$. Thus for any $(f, e) \in \mathcal{D}$ the wave vectors (s^+, s^-) are given as

$$\begin{aligned} s^+ &= \frac{1}{\sqrt{2}}(F^*\lambda + RE^*\lambda) = \frac{1}{\sqrt{2}}(F^* + RE^*)\lambda \\ s^- &= \frac{1}{\sqrt{2}}(F^*\lambda - RE^*\lambda) = \frac{1}{\sqrt{2}}(F^* - RE^*)\lambda, \quad \lambda \in \mathcal{V}^* \end{aligned}$$

It follows that

$$s^- = (F^* - RE^*)(F^* + RE^*)^{-1} s^+$$

Hence the unitary map $\mathcal{O} : \mathcal{F}^* \to \mathcal{F}^*$ is given as

$$\mathcal{O} = (F^* - RE^*)(F^* + RE^*)^{-1} = (FR^{-1} - E)^{-1}(FR^{-1} + E)$$

satisfying

$$\mathcal{O}^* R^{-1} \mathcal{O} = R^{-1}$$

Outline

Passivity

A square nonlinear system

$$\Sigma : \begin{array}{rcll} \dot{x} & = & f(x) + g(x)u, & u \in \mathbb{R}^m \\ y & = & h(x), & y \in \mathbb{R}^m \end{array}$$

where $x \in \mathbb{R}^n$ are coordinates for an n-dimensional state space $\mathcal{X}$, is passive if there exists a storage function $V : \mathcal{X} \to \mathbb{R}$ with $V(x) \geq 0$ for every x, such that

$$V(x(t_2)) - V(x(t_1)) \leq \int_{t_1}^{t_2} y^T(t)u(t)dt$$

for all solutions $(u(\cdot), x(\cdot), y(\cdot))$ and times $t_1 \leq t_2$.
The system is lossless if $\leq$ is replaced by $=$.

If H is differentiable then being passive is equivalent to

$$\frac{d}{dt}V \leq y^T u$$

which reduces to (Willems, Hill-Moylan)

$$\begin{array}{rcl} \frac{\partial^T V}{\partial x}(x) f(x) & \leq & 0 \\ h(x) & = & g^T(x) \frac{\partial V}{\partial x}(x) \end{array}$$

while in the lossless case $\leq$ is replaced by $=$.
In the linear case

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx \end{array}$$

is passive if there exists a quadratic storage function $V(x) = \frac{1}{2}x^T Qx$, with $Q = Q^T \geq 0$ satisfying the LMIs

$$A^T Q + QA \leq 0, \quad C = B^T Q$$

Clearly, any port-Hamiltonian system with Hamiltonian $H \geq 0$ is passive, since

$$\frac{d}{dt}H = -e_R^T f_R + e_C^T f_C \leq e_C^T f_C$$

and thus H is a storage function. Furthermore, if there are no power-dissipating elements R, then a port-Hamiltonian system with $H \geq 0$ is lossless.

From passive to port-Hamiltonian in the linear case

Fixing a particular storage matrix $Q > 0$ and defining J to be the skew-symmetric part of the matrix AQ^{-1} and $-R$ its symmetric part, the system can be written as the port-Hamiltonian system

$$
\begin{aligned}
\dot{x} &= [J - R]\, Qx + Bu \\
y &= B^T Qx
\end{aligned}
$$

where

$$
J = -J^T, \quad R = R^T \geq 0
$$

Thus there are different port-Hamiltonian realizations of the same passive system.

Similarly, most nonlinear passive systems can be written as a port-Hamiltonian system (with dissipation)

$$\begin{aligned} \dot{x} &= [J(x) - R(x)]\tfrac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^T(x)\tfrac{\partial H}{\partial x}(x) \end{aligned}$$

with $R(x) = R^T(x) \geq 0$ specifying the energy dissipation

$$\frac{d}{dt}H = -\frac{\partial^T H}{\partial x}(x)R(x)\frac{\partial H}{\partial x}(x) + u^T y \leq u^T y$$

Outline

Casimirs and Algebraic Constraints

Define the smooth distributions

$$G_0 := \{X \in T\mathcal{X} \mid (X, 0) \in \mathcal{D}\}$$

$$G_1 := \{X \in T\mathcal{X} \mid \exists \alpha \in T^*\mathcal{X} \text{ s.t. } (X, \alpha) \in \mathcal{D}\}$$

and the smooth co-distributions

$$P_0 := \{\alpha \in T^*\mathcal{X} \mid (0, \alpha) \in \mathcal{D}\}$$

$$P_1 := \{\alpha \in T^*\mathcal{X} \mid \exists X \in T\mathcal{X} \text{ s.t. } (X, \alpha) \in \mathcal{D}\}$$

Clearly, $G_0 \subset G_1$, $P_0 \subset P_1$, while by $\mathcal{D} = \mathcal{D}^\perp$ one obtains

$$G_0 = \ker P_1$$

$$P_0 = \mathrm{ann}\, G_1$$

If D is integrable, then the (co-)distributions G_0, G_1, P_0, P_1 are all involutive.

Casimirs and Algebraic Constraints

Definition

Let $\mathcal{X}$ be a manifold with Dirac structure $\mathcal{D}$, and let $H : \mathcal{X} \to \mathbb{R}$ be a smooth function (the Hamiltonian). The Hamiltonian system corresponding to $(\mathcal{X}, \mathcal{D}, H)$ is given as

$$(-\dot{x}, dH(x)) \in \mathcal{D}(x), x \in \mathcal{X}$$

The Casimirs are defined as all functions $C : \mathcal{X} \to \mathbb{R}$ such that $dC \in P_0$. Indeed, this means that

$$< dC \mid f >= 0$$

for all $f \in G_1$; i.e., along al possible evolutions of the system.

Algebraic constraints

In general $(-\dot{x}, dH(x)) \in \mathcal{D}(x)$ induces algebraic constraints on the state variables, leading to the constrained state space

$$\mathcal{X}_c = \{x \in \mathcal{X} \mid \exists f \text{ s.t. } (f, dH(x)) \in \mathcal{D}(x)\}$$

Throughout we assume that this defines a smooth submanifold. This constrained state space is determined by the Hamiltonian H and by the co-distribution P_1 (or, equivalently, the distribution G_0).

Symmetries and conserved quantities

Definition

Let $\mathcal{D}$ be a Dirac structure on $\mathcal{X}$. A vector field f on $\mathcal{X}$ is an infinitesimal symmetry of $\mathcal{D}$ (briefly, a symmetry of $\mathcal{D}$) if

$$(L_f X, L_f \alpha) \in \mathcal{D}, \quad \text{for all} \quad (X, \alpha) \in \mathcal{D}$$

Analogously we say that a diffeomorphism $\varphi : \mathcal{X} \to \mathcal{X}$ is a symmetry of $\mathcal{D}$ if

$$(\varphi_*^{-1} X, \varphi^* \alpha) \in \mathcal{D}, \quad \text{for all} \quad (X, \alpha) \in \mathcal{D}$$

Theorem

Let f be a symmetry of the Dirac structure $\mathcal{D}$, with associated distributions G_0, G_1 and co-distributions P_0, P_1. Then

$$L_f G_i \subset G_i, \quad L_f P_i \subset P_i, \ i = 0, 1$$

We have the following analog of the statement that any Hamiltonian vector field is a symmetry for the symplectic form:

Theorem

Let $\mathcal{D}$ be a **closed** *Dirac structure on $\mathcal{X}$. Let f be a vector field on $\mathcal{X}$ for which there exists a smooth function $F : \mathcal{X} \to \mathbb{R}$ such that $(f, dF) \in \mathcal{D}$. Then f is a symmetry of $\mathcal{D}$.*

Noether type of result on the existence of conserved quantities:

Theorem

Let $(\mathcal{X}, \mathcal{D}, H)$ be a Hamiltonian system. Let f be a vector field on $\mathcal{X}$ for which there exists a smooth function F such that

$$(f(x), dF(x)) \in \mathcal{D}(x), \; x \in \mathcal{X}_c$$

Furthermore, let f be a symmetry for H on $\mathcal{X}_c$, that is

$$L_f H(x) = 0, \; x \in \mathcal{X}_c$$

Then $L_{X_{H_c}}(F) = 0$ on $\mathcal{X}_c$, where H_c is the Hamiltonian H restricted to $\mathcal{X}_c$. Thus F is a conserved quantity for X_{H_c} on $\mathcal{X}_c$.

Proof Because of $\mathcal{D} = \mathcal{D}^\perp$ we have

$$< dH(x) \mid f(x) > + < dF(x) \mid X_{H_c}(x) >= 0, \quad x \in \mathcal{X}_c,$$

since $(f(x), dF(x)) \in \mathcal{D}(x), x \in \mathcal{X}_c$, and $(X_{H_c}(x), dH(x)) \in \mathcal{D}(x), x \in \mathcal{X}_c$, by construction.

Consider a symmetry Lie group G of the Dirac structure $\mathcal{D}$; that is, the Lie group G acts on $\mathcal{X}$ by diffeomorphisms $\Phi_g : \mathcal{X} \to \mathcal{X}, g \in G$, and Φ_g is a symmetry of $\mathcal{D}$ for every $g \in G$. Equivalently, for every $\xi \in \underline{g}$ (the Lie algebra of G) the infinitesimal generator X_ξ of the group action is an (infinitesimal) symmetry of $\mathcal{D}$. Throughout assume that the quotient space $\bar{\mathcal{X}} := \mathcal{X}/G$ of G-orbits on $\mathcal{X}$ is a manifold with smooth projection map $\rho : \mathcal{X} \to \bar{\mathcal{X}}$. Then the Dirac structure $\mathcal{D}$ reduces to $\bar{\mathcal{X}}$ as follows.

Theorem

Let G be a symmetry Lie group of the generalized Dirac structure $\mathcal{D}$ on $\mathcal{X}$, with quotient manifold $\bar{\mathcal{X}}$ and smooth projection $\rho : \mathcal{X} \to \bar{\mathcal{X}}$. Then there exists a reduced Dirac structure $\bar{\mathcal{D}}$ on $\bar{\mathcal{X}}$, defined as

$$(\bar{X}, \bar{\alpha}) \in \bar{\mathcal{D}} \text{ if } \exists X \text{ with } \rho_* X = \bar{X} \text{ s.t. } (X, \alpha) \in \mathcal{D}, \text{ where } \alpha = \rho^* \bar{\alpha}$$

Furthermore, if $\mathcal{D}$ is integrable, then so is $\bar{\mathcal{D}}$.

Theorem

Let $(\mathcal{X}, \mathcal{D}, H)$ be a Hamiltonian system. Let G be a symmetry Lie group of the Dirac structure $\mathcal{D}$ on $\mathcal{X}$, with quotient manifold $\bar{\mathcal{X}}$, smooth projection $\rho : \mathcal{X} \to \bar{\mathcal{X}}$, and reduced Dirac structure $\bar{\mathcal{D}}$ on $\bar{\mathcal{X}}$. Furthermore, suppose the action of G on $\mathcal{X}$ leaves H invariant, leading to a reduced Hamiltonian $\bar{H} : \bar{\mathcal{X}} \to \mathbb{R}$ such that $H = \bar{H} \circ \rho$. Then the Hamiltonian system $(\mathcal{X}, \mathcal{D}, H)$ projects to the Hamiltonian system $(\bar{\mathcal{X}}, \bar{\mathcal{D}}, \bar{H})$.

Interconnection port-Hamiltonian systems, and composition of Dirac structures

The *composition* of two Dirac structures with partially shared variables is *again* a Dirac structure:

$$\mathcal{D}_{12} \subset \mathcal{V}_1 \times \mathcal{V}_1^* \times \mathcal{V}_2 \times \mathcal{V}_2^*$$

$$\mathcal{D}_{23} \subset \mathcal{V}_2 \times \mathcal{V}_2^* \times \mathcal{V}_3 \times \mathcal{V}_3^*$$

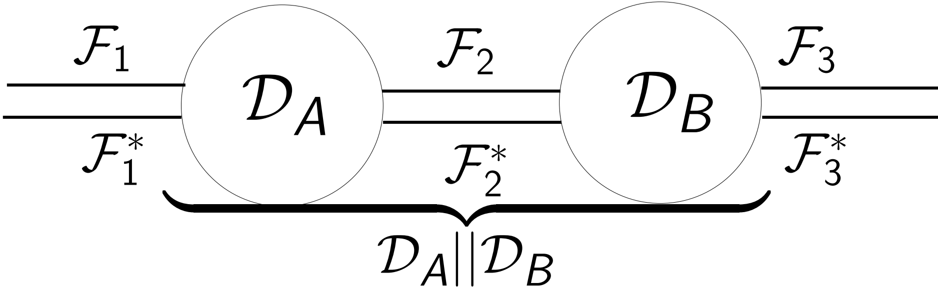


Figure: Composed Dirac structure

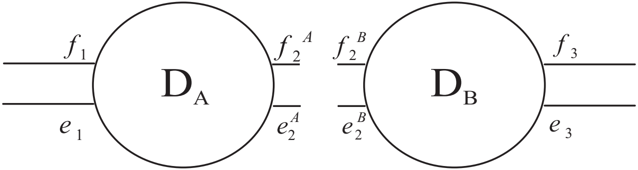


Figure: Standard interconnection

$$
\begin{array}{rcll}
f_2^A & = & -f_2^B & \in \mathcal{F}_2 \\
e_2^A & = & e_2^B & \in \mathcal{F}_2^*
\end{array}
$$

The *gyrating* (or *feedback*) interconnection

$$
\begin{array}{rcl}
f_2^A & = & -e_2^B \\
e_2^B & = & f_2^B
\end{array}
$$

can be easily transformed to this case.

Thus

$$\mathcal{D}_A \| \mathcal{D}_B := \{(f_1, e_1, f_3, e_3) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^* \mid \exists (f_2, e_2) \in \mathcal{F}_2 \times \mathcal{F}_2^*$$

such that

$$(f_1, e_1, f_2, e_2) \in \mathcal{D}_A \text{ and } (-f_2, e_2, f_3, e_3) \in \mathcal{D}_B\}$$

Theorem

Let $\mathcal{D}_A$, $\mathcal{D}_B$ be Dirac structures (defined with respect to $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^*$, respectively $\mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$ and their bilinear forms). Then $\mathcal{D}_A \| \mathcal{D}_B$ is a Dirac structure with respect to the bilinear form on $\mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_3 \times \mathcal{F}_3^*$.

Proof

Consider $\mathcal{D}_A$, $\mathcal{D}_B$ defined in kernel representation by

$$\mathcal{D}_A = \{(f_1, e_1, f_A, e_A) \in \mathcal{F}_1 \times \mathcal{F}_1^* \times \mathcal{F}_2 \times \mathcal{F}_2^* | F_1 f_1 + E_1 e_1 + F_{2A} f_A + E_{2A} e_A = 0\}$$

$$\mathcal{D}_B = \{(f_B, e_B, f_3, e_3) \in \mathcal{F}_2 \times \mathcal{F}_2^* \times \mathcal{F}_3 \times \mathcal{F}_3^* | F_{2B} f_B + E_{2B} e_B + F_3 f_3 + E_3 e_3 = 0\}$$

Make use of the following basic fact from linear algebra:

$$(\exists \lambda \text{ s.t. } A\lambda = b) \Leftrightarrow [\forall \alpha \text{ s.t. } \alpha^T A = 0 \Rightarrow \alpha^T b = 0]$$

Note that $\mathcal{D}_A$, $\mathcal{D}_B$ are alternatively given in matrix image representation as

$$\mathcal{D}_A = \mathrm{im} \begin{bmatrix} E_1^T \\ F_1^T \\ E_{2A}^T \\ F_{2A}^T \\ 0 \\ 0 \end{bmatrix} \qquad \mathcal{D}_B = \mathrm{im} \begin{bmatrix} 0 \\ 0 \\ E_{2B}^T \\ F_{2B}^T \\ E_3^T \\ F_3^T \end{bmatrix}$$

Hence, $(f_1, e_1, f_3, e_3) \in \mathcal{D}_A \parallel \mathcal{D}_B \Leftrightarrow \exists \lambda_A, \lambda_B$ such that

$$\begin{bmatrix} f_1 \\ e_1 \\ 0 \\ 0 \\ f_3 \\ e_3 \end{bmatrix} = \begin{bmatrix} E_1^T & 0 \\ F_1^T & 0 \\ E_{2A}^T & E_{2B}^T \\ F_{2A}^T & -F_{2B}^T \\ 0 & F_3^T \\ 0 & E_3^T \end{bmatrix} \begin{bmatrix} \lambda_A \\ \lambda_B \end{bmatrix} \Leftrightarrow$$

$\Leftrightarrow \forall(\beta_1, \alpha_1, \beta_2, \alpha_2, \beta_3, \alpha_3)$ s.t.

$$(\beta_1^T \alpha_1^T \beta_2^T \alpha_2^T \beta_3^T \alpha_3^T) \begin{bmatrix} E_1^T & 0 \\ F_1^T & 0 \\ E_{2A}^T & E_{2B}^T \\ F_{2A}^T & -F_{2B}^T \\ 0 & F_3^T \\ 0 & E_3^T \end{bmatrix} = 0,$$

$$\beta_1^T f_1 + \alpha_1^T e_1 + \beta_3^T f_3 + \alpha_3^T e_3 = 0$$

Explicit expressions for the composition

The composition of $\mathcal{D}_A$ with $\mathcal{D}_B$ is given by

$$\begin{bmatrix} F_1 & E_1 & F_{2A} & E_{2A} & 0 & 0 \\ 0 & 0 & -F_{2B} & E_{2B} & F_3 & E_3 \end{bmatrix} \begin{bmatrix} f_1 \\ e_1 \\ f_2 \\ e_2 \\ f_3 \\ e_3 \end{bmatrix} = 0,$$

Define

$$M = \begin{bmatrix} F_{2A} & E_{2A} \\ -F_{2B} & E_{2B} \end{bmatrix}$$

and let L_A, L_B be matrices with

$$L = [L_A | L_B], \qquad \ker L = \operatorname{im} M$$

Premultiplication of the equations by the matrix $L := [L_A | L_B]$ results in

$$L_A F_1 f_1 + L_A E_1 e_1 + L_B F_3 f_3 + L_B E_3 e_3 = 0$$

Consequence

The interconnection of a number of port-Hamiltonian systems $(\mathcal{X}_i, \mathcal{D}_i, H_i), i = 1, \cdots, k$, through an interconnection Dirac structure $\mathcal{D}_I$ is a port-Hamiltonian system $(\mathcal{X}, \mathcal{D}, H)$, with

$$H = H_1 + \cdots + H_k,$$

$$\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_k$$

and $\mathcal{D}$ the composition of $\mathcal{D}_1, \cdots, \mathcal{D}_k, \mathcal{D}_I$.

Outline

How to deal with infinite-dimensional components ?

Want to deal with multi-physics systems also incorporating distributed-parameter (infinite-dimensional) components:

1. Power-converter connected to an electrical machine via a transmission line
2. Electrical power-networks including generators and dynamic loads
3. Hydraulic networks with pipes
4. Multi-body systems with flexible components

etc.

Distributed parameter port-Hamiltonian systems

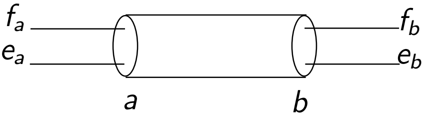


Figure: **Simplest example: Transmission line**

Telegrapher's equations define the boundary control system

$$
\begin{aligned}
\frac{\partial Q}{\partial t}(z,t) &= -\frac{\partial}{\partial z} I(z,t) &&= -\frac{\partial}{\partial z}\frac{\phi(z,t)}{L(z)} \\
\frac{\partial \phi}{\partial t}(z,t) &= -\frac{\partial}{\partial z} V(z,t) &&= -\frac{\partial}{\partial z}\frac{Q(z,t)}{C(z)} \\
f_a(t) &= V(a,t), \quad e_a(t) &&= I(a,t) \\
f_b(t) &= V(b,t), \quad e_b(t) &&= I(b,t)
\end{aligned}
$$

Transmission line as port-Hamiltonian system

Define internal flows $f_S = (f_E, f_M)$ and efforts $e_S = (e_E, e_M)$:

electric flow	$f_E : [a,b] \to \mathbb{R}$
magnetic flow	$f_M : [a,b] \to \mathbb{R}$
electric effort	$e_E : [a,b] \to \mathbb{R}$
magnetic effort	$e_M : [a,b] \to \mathbb{R}$

together with external (boundary) flows $f = (f_a, f_b)$ and boundary efforts $e = (e_a, e_b)$. Define the infinite-dimensional subspace of $(C^\infty[a,b])^2 \times (C^\infty[a,b])^2 \times \mathbb{R}^2 \times \mathbb{R}^2$ by the equations

$$\begin{bmatrix} f_E \\ f_M \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & 0 \end{bmatrix} \begin{bmatrix} e_E \\ e_M \end{bmatrix}$$

$$\begin{bmatrix} f_a \\ e_a \end{bmatrix} = \begin{bmatrix} e_E(a) \\ e_M(a) \end{bmatrix}, \quad \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} e_E(b) \\ e_M(b) \end{bmatrix}$$

This defines a Dirac structure:

Use integration by parts

For any $(f_E, f_M, e_E, e_M, f_a, f_b, e_a, e_b) \in \mathcal{D}$

$$\int_a^b e_E(z)f_E(z) + e_M(z)f_M(z) - e_b f_b + e_a f_a =$$

$$\int_a^b e_E(z)\frac{\partial}{\partial z}e_M(z) + e_M(z)\frac{\partial}{\partial z}e_E(z) - e_b f_b + e_a f_a =$$

$$\int_a^b [-e_M(z)\frac{\partial}{\partial z}e_E(z) + e_b f_b - e_a f_a] + e_M(z)\frac{\partial}{\partial z}e_E(z) - e_b f_b + e_a f_a = 0$$

Thus $e^T f = 0$ for all $(f, e) \in \mathcal{D}$. However, in general this implies that for all $(f_1, e_1), (f_2, e_2) \in \mathcal{D}$

$$0 = (e_1 + e_2)^T(f_1 + f_2) = e_1^T f_1 + e_2^T f_2 + e_1^T f_2 + e_2^T f_1 =$$

$$e_1^T f_2 + e_2^T f_1 = \ll (f_1, e_1), (f_2, e_2) \gg$$

Hence $\mathcal{D} \subset \mathcal{D}^{\perp\!\!\perp}$.

Still need to show that $\mathcal{D}^{\perp\!\!\!\perp} \subset \mathcal{D}$:

Let $(\bar{f}_E, \bar{f}_M, \bar{e}_E, \bar{e}_M, \bar{f}_a, \bar{e}_a, \bar{f}_b, \bar{e}_b) \in \mathcal{D}^{\perp\!\!\!\perp}$, that is

$$\begin{aligned} 0 &= \int_a^b \bar{e}_E f_E + e_E \bar{f}_E + \bar{e}_M f_M + e_M \bar{f}_M + \\ &\quad -\bar{e}_b f_b - e_b \bar{f}_b + \bar{e}_a f_a + e_a \bar{f}_a \end{aligned}$$

for all $(f_E, f_M, e_E, e_M, f_a, e_a, f_b, e_b) \in \mathcal{D}$.
Take first $f_a = e_a = f_b = e_b = 0$. Then

$$0 = \int_a^b \bar{e}_E \frac{\partial}{\partial z} e_M + e_E \bar{f}_E + \bar{e}_M \frac{\partial}{\partial z} e_E + e_M \bar{f}_M$$

for all such (e_E, e_M). This implies

$$\bar{f}_E = \frac{\partial}{\partial z} \bar{e}_M, \quad \bar{f}_M = \frac{\partial}{\partial z} \bar{e}_E$$

Substitution yields

$$0 = \int_a^b \bar{e}_E \frac{\partial}{\partial z} e_M + e_E \frac{\partial}{\partial z} \bar{e}_M + \bar{e}_M \frac{\partial}{\partial z} e_E + e_M \frac{\partial}{\partial z} \bar{e}_E$$
$$-\bar{e}_b f_b - e_b \bar{f}_b + \bar{e}_a f_a + e_a \bar{f}_a$$

which implies

$$e_E(b)\bar{e}_M(b) + e_M(b)\bar{e}_E(b) - e_E(a)\bar{e}_M(a) - e_M(a)\bar{e}_E(a)$$
$$-\bar{e}_b f_b - e_b \bar{f}_b + \bar{e}_a f_a + e_a \bar{f}_a = 0$$

for all $f_a = e_E(a), f_b = e_E(b), e_a = e_M(a), e_b = e_M(b)$. This yields

$$\bar{e}_b = \bar{e}_M(b), \quad \bar{f}_b = \bar{e}_E(b), \quad \bar{e}_a = \bar{e}_M(a), \quad \bar{f}_a = \bar{e}_E(a)$$

Telegrapher's equations as port-Hamiltonian system

Substituting (as in the lumped-parameter case)

$$\left.\begin{array}{rcl} f_E & = & -\frac{\partial Q}{\partial t} \\ f_M & = & -\frac{\partial \varphi}{\partial t} \end{array}\right\} f_S = -\dot{x}$$

$$\left.\begin{array}{rcl} e_E & = & \frac{Q}{C} = \frac{\partial \mathcal{H}}{\partial Q} \\ e_M & = & \frac{\varphi}{L} = \frac{\partial \mathcal{H}}{\partial \varphi} \end{array}\right\} e_S = \frac{\partial H}{\partial x}$$

with, for example, quadratic energy density

$$\mathcal{H}(Q, \varphi) = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\varphi^2}{L}$$

we recover the telegrapher's equations.

Energy-balance

$$\frac{d}{dt}\int_a^b \frac{1}{2}\frac{Q^2(z)}{C} + \frac{1}{2}\frac{\varphi^2(z)}{L}dz = -\int_a^b e_E(z)f_E(z) + e_M(z)f_M(z)dz =$$

$$-e_b f_b + e_a f_a = -I(b,t)V(b,t) + I(a,t)V(b,t)$$

Remark

Of course, the telegrapher's equations can be rewritten as the linear wave equation

$$\frac{\partial^2 Q}{\partial t^2} = -\frac{\partial}{\partial z}\frac{\partial I}{\partial t} = -\frac{\partial}{\partial z}\frac{\partial}{\partial t}\frac{\phi}{L} =$$

$$-\frac{\partial}{\partial z}\frac{1}{L}\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z}\frac{1}{L}\frac{\partial}{\partial z}\frac{Q}{C} = \frac{1}{LC}\frac{\partial^2 Q}{\partial z^2}$$

(provided $L(z), C(z)$ do not depend on z), or similar expressions in ϕ, I or V.

The same equations hold for a vibrating string, or for a compressible gas/fluid in a one-dimensional pipe.

Shallow water equations; distributed-parameter port-Hamiltonian system with non-quadratic Hamiltonian

The dynamics of the water in an open-channel canal can be described by

$$\partial_t \begin{bmatrix} h \\ v \end{bmatrix} + \begin{bmatrix} v & h \\ g & v \end{bmatrix} \partial_z \begin{bmatrix} h \\ v \end{bmatrix} = 0$$

with $h(z,t)$ the height of the water at position z, and $v(z,t)$ the velocity (and g gravitational constant).
This can be written as a port-Hamiltonian system by recognizing the total energy

$$H(h,v) = \frac{1}{2}\int_a^b [hv^2 + gh^2] dz$$

yielding the co-energy functions[1]

$$e_h = \frac{\partial \mathcal{H}}{\partial h} = \frac{1}{2}v^2 + gh \quad \text{Bernoulli function}$$

$$e_v = \frac{\partial \mathcal{H}}{\partial v} = hv \quad \text{mass flow}$$

It follows that the shallow water equations can be written, similarly to the telegrapher's equations, as

$$\frac{\partial h}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial v}$$

$$\frac{\partial v}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial h}$$

with boundary variables $-hv_{|a,b}$ and $(\frac{1}{2}v^2 + gh)_{|a,b}$.

[1]Daniel Bernoulli, born in 1700 in Groningen as son of Johann Bernoulli, professor in mathematics at the University of Groningen and pioneer of the Calculus of Variations (the Brachistochrone problem).

Paying tribute to history:

Figure: Johann Bernoulli, professor in Groningen 1695-1705.

Figure: Daniel Bernoulli, born in Groningen in 1700.

We obtain the energy balance

$$\frac{d}{dt}\int_a^b [hv^2 + gh^2]dz = -(hv)(\frac{1}{2}v^2 + gh)|_a^b$$

which can be rewritten as

$$-v(\tfrac{1}{2}gh^2)|_a^b - v(\tfrac{1}{2}hv^2 + \tfrac{1}{2}gh^2))|_a^b =$$

velocity $\times$ pressure $+$ energy flux through the boundary

Conservation laws

All examples sofar have the same structure

$$\frac{\partial \alpha_1}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial \alpha_2} = -\frac{\partial}{\partial z}\beta_2$$

$$\frac{\partial \alpha_2}{\partial t}(z,t) = -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial \alpha_1} = -\frac{\partial}{\partial z}\beta_1$$

with boundary variables $\beta_1|_{\{a,b\}}, \beta_2|_{\{a,b\}}$, corresponding to two coupled conservation laws:

$$\frac{d}{dt}\int_a^b \alpha_1 = -\int_a^b \frac{\partial}{\partial z}\beta_2 = \beta_2(a) - \beta_2(b)$$

$$\frac{d}{dt}\int_a^b \alpha_2 = -\int_a^b \frac{\partial}{\partial z}\beta_1 = \beta_1(a) - \beta_1(b)$$

(In the transmission line, α_1 and α_2 is charge- and flux-density, and β_1, β_2 voltage V and current I, respectively.)

For some purposes it is illuminating to rewrite the equations in terms of the co-energy variables β_1, β_2:

$$\begin{bmatrix} \frac{\partial \beta_1}{\partial t} \\ \frac{\partial \beta_2}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 H}{\partial \alpha_1^2} & \frac{\partial^2 H}{\partial \alpha_1 \alpha_2} \\ \frac{\partial^2 H}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 H}{\partial \alpha_2^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \alpha_1}{\partial t} \\ \frac{\partial \alpha_2}{\partial t} \end{bmatrix} = - \begin{bmatrix} \frac{\partial^2 H}{\partial \alpha_1^2} & \frac{\partial^2 H}{\partial \alpha_1 \alpha_2} \\ \frac{\partial^2 H}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 H}{\partial \alpha_2^2} \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_2}{\partial z} \\ \frac{\partial \beta_1}{\partial z} \end{bmatrix}$$

For the transmission line this yields

$$\begin{bmatrix} \frac{\partial V}{\partial t} \\ \frac{\partial I}{\partial t} \end{bmatrix} = - \begin{bmatrix} 0 & \frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial z} \\ \frac{\partial I}{\partial z} \end{bmatrix}$$

The matrix is called the characteristic matrix, whose eigenvalues are the characteristic velocities $\frac{1}{\sqrt{LC}}$ and $-\frac{1}{\sqrt{LC}}$ corresponding to the characteristic eigenvectors (and curves).

For the shallow water equations this yields

$$\begin{bmatrix} \frac{\partial \beta_1}{\partial t} \\ \frac{\partial \beta_2}{\partial t} \end{bmatrix} = - \begin{bmatrix} v & g \\ h & v \end{bmatrix} \begin{bmatrix} \frac{\partial \beta_1}{\partial z} \\ \frac{\partial \beta_2}{\partial z} \end{bmatrix}$$

with

$$\beta_1 = \frac{1}{2} v^2 + gh, \quad \beta_2 = hv$$

being the Bernoulli function and mass flow, respectively.
This corresponds to two characteristic velocities $v \pm \sqrt{gh}$, which are, like in the transmission line case, of opposite sign (subcritical or fluvial flow) if

$$v^2 \leq gh$$

Because the Hamiltonian is non-quadratic, and thus the pde's are nonlinear, the characteristic curves may intersect, corresponding to shock waves.

Higher-dimensional spatial domain

Electromagnetic Field: Maxwell's equations

$$
\begin{aligned}
\frac{\partial D}{\partial t} &= \text{curl } H, \quad E = \epsilon^{-1} D \quad \text{Faraday} \\
\frac{\partial B}{\partial t} &= -\text{curl } E, \quad H = \mu^{-1} B \quad \text{Ampère}
\end{aligned}
$$

Differential version of

$$\begin{aligned} \int_{\partial S} E &= -\frac{d}{dt}\int_S B \qquad \text{Faraday} \\ \int_{\partial S} H &= \frac{d}{dt}\int_S D \qquad \text{Ampère} \end{aligned}$$

This means that D and B are differential two-forms, and E and H are differential one-forms!

Similar phenomenon in the telegrapher's equations:

Voltage / current: functions on $[a, b]$
Charge / flux density: one-forms $Qdz, \varphi dz$ on $[a, b]$

General framework

Z is n-dimensional spatial domain with boundary ∂Z.
The exterior derivative $d : \Omega^k(Z) \to \Omega^{k+1}(Z)$ incorporates all vector calculus operations (grad, curl, div).
Define a Dirac structure on the space of flows and efforts:

$$\tilde{f} = (f_E, f_M, f_b) \in \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-q}(\partial Z)$$

$$\tilde{e} = (e_E, e_M, e_b) \in \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-p}(\partial Z)$$

by setting

$$f_E(t,z) = \pm d e_M(t,z), \quad f_M(t,z) = d e_E(t,z),$$

$$f_b(t) = e_E(t, \partial Z), \quad e_b(t) = \pm e_M(t, \partial Z)$$

(Transmission line: $n = p = q = 1$
Maxwell's equations: $n = 3, p = q = 2$)

Regulation of the shallow water equations

Consider again

$$\begin{aligned} \frac{\partial h}{\partial t}(z,t) &= -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial v}(h,v) \\ \frac{\partial v}{\partial t}(z,t) &= -\frac{\partial}{\partial z}\frac{\partial \mathcal{H}}{\partial h}(h,v) \end{aligned}$$

with the 4 boundary variables

$$hv_{|a,b}$$

$$\left(\frac{1}{2}v^2 + gh\right)_{|a,b}$$

(mass flow and Bernoulli function at the boundary points a, b).

Suppose we want to control the water level h to a desired height h^*.
We cannot use the total energy as a Lyapunov function.
An obvious 'physical' controller is to add to one side of the canal, say the right-end b, an infinite water reservoir of height h^*, corresponding to the port-Hamiltonian 'source' system

$$
\begin{aligned}
\dot{\xi} &= u_c \\
y_c &= \frac{\partial H_c}{\partial \xi} (= gh^*)
\end{aligned}
$$

with Hamiltonian $H_c(\xi) = gh^*\xi$, by the feedback interconnection

$$
u_c = y = h(b)v(b), \quad y_c = -u = \frac{1}{2}v^2(b) + gh(b)
$$

This yields a closed-loop port-Hamiltonian system with total Hamiltonian

$$
\int_0^l \frac{1}{2}[hv^2 + gh^2]dz + gh^*\xi
$$

By mass balance,

$$\int_a^b h(z,t)dz + \xi + c$$

is a Casimir for the closed-loop system. Thus we may take as Lyapunov function

$$\begin{aligned} V(h,v,\xi) &:= \tfrac{1}{2}\int_a^b [hv^2 + gh^2]dz + gh^*\xi - gh^*[\int_a^b h(z,t)dz + \xi] \\ &\quad + \tfrac{1}{2}g(b-a)h^{*2} \\ &= \tfrac{1}{2}\int_a^b [hv^2 + g(h-h^*)^2]dz \end{aligned}$$

which has a minimum at the desired set-point $(h^*, v^* = 0, \xi^*)$ (with ξ^* arbitrary).

Remark Note that the source port-Hamiltonian system is not passive, since the Hamiltonian $H_c(\xi) = gh^*\xi$ is not bounded from below.

An alternative, passive, choice of the Hamiltonian controller system is to take

$$H_c(\xi) = \frac{1}{2} g h^* \xi^2$$

leading to the Lyapunov function

$$V(h, v, \xi) = \frac{1}{2} \int_a^b [hv^2 + g(h - h^*)^2] dz + \frac{1}{2} g h^* (\xi - 1)^2$$

Asymptotic stability of the equilibrium $(h^*, v^* = 0, \xi^* = 1)$ can be obtained by adding 'damping', that is, replacing $u_c = y = h(b)v(b)$ by

$$u_c := y - \frac{\partial V}{\partial \xi}(\xi) = h(b)v(b) - g h^* (\xi - 1)$$

leading to (if there is no power flow through the left-end a)

$$\frac{d}{dt} V = -g h^* (\xi - 1)^2$$

(See also the work of Bastin & co-workers for related and more refined results.)

Outline

Conclusions and Outlook

- Port-Hamiltonian systems theory as a modular theory for complex multi-physics systems
- Inclusion of infinite-dimensional component systems: differential-algebraic partial differential equations
- Leads to new control paradigms and strategies
- Not covered here (see literature): structure-preserving model reduction, spatial discretization, network dynamics (port-Hamiltonian systems on graphs), chemical and thermodynamic systems.

THANK YOU !

See www.math.rug.nl/˜arjan for further info.

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